ON PRODUCT k-CHEN SUBMANIFOLDS

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0. Introduction. B. Rouxel [7] and S. J. Li and C. S. Houh [6] have generalised the notion of an $\mathscr{A}$-submanifold (Chen submanifold) to an $\mathscr{A}_k$-submanifold. In [1] we have studied the relation between their definitions for the Euclidean case.

In this work we obtain a k-Chen submanifold by considering the product of 1-Chen submanifolds. Using the definition of k-Chen submanifold given in [6], we show that the product of two submanifolds M_1 (p-Chen) and M_2 (r-Chen) of Riemaniann manifolds N_1 and N_2, respectively is a k-Chen submanifold for some $k \in [\max\{p,r\}, p+r]$. This result for $k \leq p+r$ was obtained by B. Rouxel [7] for the Euclidean case. We also give some examples.

1. Preliminaries. M is an m-dimensional submanifold of an $(m+d)$-dimensional Riemannian manifold N. Let ξ be a normal vector field on M. We choose an orthonormal local basis $n_1, n_2, \ldots, n_d$ normal to M in N such that $n_1 = \dfrac{\xi}{\|\xi\|}$. Then the allied vector field of ξ is defined by $\mathscr{A}(\xi) = \sum_{i=2}^{d} \operatorname{Trace}(A_\xi A_{n_i}) n_i$, where A is the Weingarten map of M in N ([2] p. 203]). In particular $\mathscr{A}(H)$ which is the allied vector field of the mean curvature vector H of M in N is called the allied mean curvature vector. If $\mathscr{A}(H)$ vanishes identically, then the submanifold M was called in [2] an $\mathscr{A}$-submanifold of N, which later became known as a Chen submanifold [4]. It is easily seen that the class of Chen submanifolds contains all minimal and pseudo-umbilical submanifolds, and also all submanifolds for which $\dim N_1 \leq 1$, where N_1 is the first normal space of M in N, in particular it includes all hypersurfaces. These Chen submanifolds are said to be trivial Chen submanifolds. There are many kinds of Chen submanifolds which are neither of the submanifolds we just mentioned. (cf. [3], [5])

In [6] the definition of a k-Chen submanifold is given as follows. Suppose that H does not vanish on M and $n_1 = \dfrac{H}{\|H\|}, n_2, \ldots, n_d$ is a local orthonormal normal basis to M in N. Put $\mathscr{A}_1(H) = \mathscr{A}(H)$. Suppose that $\mathscr{A}(H) \neq 0$. Then the local orthonormal normal basis $n_1, n_2, \ldots, n_d$ is rechosen such that $n_1 = \dfrac{H}{\|H\|}$, $n_2 = \dfrac{\mathscr{A}_1(H)}{\|\mathscr{A}_1(H)\|}$. In general, $\mathscr{A}_2(H)$, $\mathscr{A}_3(H), \ldots$ are defined inductively. Suppose that $\mathscr{A}_0(H) = H$, $\mathscr{A}_1(H), \ldots, \mathscr{A}_{k-1}(H)$ have been defined and are nonzero on M. Thus the local orthonormal normal basis $n_1, n_2, \ldots, n_d$ is rechosen so that $n_i = \dfrac{\mathscr{A}_{i-1}(H)}{\|\mathscr{A}_{i-1}(H)\|}$, $i = 1, 2, \ldots, k$. Then $\mathscr{A}_k(H)$ is defined as $\mathscr{A}_k(H) = \sum_{i=k+1}^{d} \operatorname{Trace}(A_{\mathscr{A}_{k-1}(H)} A_{n_i}) n_i$, and thus $\langle \mathscr{A}_k(H), \mathscr{A}_{i-1}(H) \rangle = 0$, $i = 1, 2, \ldots, k$. As in the definition of Chen submanifold we call $\mathscr{A}_k(H)$ the k-th allied mean curvature

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vector. The submanifold M is said to be an $\mathcal{A}_k$-submanifold if one of $H, \mathcal{A}_1(H), \dots, \mathcal{A}_k(H)$ vanishes on the whole M and is said to be a k-Chen submanifold if M is an $\mathcal{A}_k$-submanifold but not an $\mathcal{A}_{k-1}$-submanifold. All trivial Chen submanifolds with nonzero mean curvature vector would be considered as trivial 1-Chen submanifolds.

Let TM and $T^{\perp}M$ denote the tangent and normal bundles of M, respectively. Let $S(M)$ be the bundle whose fiber at each point $p \in M$ is the space of symmetric linear transformations of $T_pM \to T_pM$. Then we consider the Weingarten map A and its transpose ${}^tA$ as a cross-section in $\mathrm{Hom}(T^{\perp}M, S(M))$ and $\mathrm{Hom}(S(M), T^{\perp}M)$, respectively. That is, if $u \in S_p(M)$ and $\xi \in T_p^{\perp}M$, $\langle {}^tA(u), \xi\rangle = \langle u, A_\xi\rangle$, where for any $u, v \in S_p(M)$, $\langle u, v\rangle = \sum_{i=1}^{m} \langle ue_i, ve_i\rangle$, $\{e_1, \dots, e_m\}$ is an orthonormal basis tangent to M. Then Simons' operator $\tilde{A}$ is given by $\tilde{A} = {}^tA \circ A$ [8] and for any normal vector n

$$\tilde{A}(n) = \sum_{j=1}^{d} \langle {}^tA \circ A(n), n_j\rangle n_j = \sum_{j=1}^{d} \mathrm{Trace}(A_n A_{n_j}) n_j. \tag{1.1}$$

However, for positive integers s, $\tilde{A}^s(n) = \tilde{A} \circ \tilde{A}^{s-1}(n)$, where $\tilde{A}^0(n) = n$.

2. Product k-Chen Submanifolds. Let M be an m-dimensional submanifold of an $(m+k)$-dimensional Riemannian manifold N. Assume that $H \neq 0$ on M and $\left\{n_1 = \frac{H}{\|H\|}, n_2, \dots, n_d\right\}$ is a local orthonormal normal basis to M. Let us apply $\tilde{A}$ to H. From (1.1) we have

$$\begin{aligned}\tilde{A}(H) &= \sum_{j=1}^{d} \mathrm{Trace}(A_H A_{n_j}) n_j = \mathrm{Trace}(A_H A_{n_1}) n_1 + \sum_{j=2}^{d} \mathrm{Trace}(A_H A_{n_j}) n_j \\ &= \mathrm{Trace}(A_{n_1} A_{n_1}) H + \mathcal{A}_1(H).\end{aligned} \tag{2.1}$$

Applying Simons' operator k times to H, we get

$$\tilde{A}^k(H) = \sum_{i=0}^{k-1} a_i \tilde{A}^i(H) + \mathcal{A}_k(H), \tag{2.2}$$

where $\tilde{A}^0(H) = H$. For each k, the functions a_i are determined (for details see [6]).

In [6], the following result is given.

THEOREM A [6]. *Let M be an m-dimensional submanifold of an $(m+d)$-dimensional Riemannian manifold N. Then M is a k-Chen submanifold of N for some positive integer $k < d$ if and only if $\tilde{A}^k(H)$ is a linear combination of $H, \tilde{A}^1(H), \dots, \tilde{A}^{k-1}(H)$ which are linearly independent.*

In [3], B.-Y. Chen gave the following proposition about the product of two $\mathcal{A}$-submanifolds.

PROPOSITION B [3]. *Let M_i $(i = 1, 2)$ be m_i-dimensional submanifolds of $(m_i + d_i)$-dimensional Riemannian manifolds N_i with nowhere zero mean curvature vector H_i. The product $M_1 \times M_2$ is an $\mathcal{A}$-submanifold of $N_1 \times N_2$ if and only if M_1 and M_2 are $\mathcal{A}$-submanifolds of N_1 and N_2 respectively, and the second fundamental forms at $n_i = H_i / \|H_i\|$ of M_i in N_i satisfy* $\mathrm{Trace}(A_{n_1} A_{n_1}) = \mathrm{Trace}(A_{n_2} A_{n_2})$.

Also, S. J. Li and C. S. Houh [6] generalised this result to the product of two $\mathcal{A}_2$-submanifolds. In this work we obtain the following results.

THEOREM 2.1. *Let M_i be m_i-dimensional 1-Chen submanifolds of (m_i+d_i)-dimensional Riemannian manifolds N_i with nowhere zero mean curvature vector H_i, $i=1,2,\ldots,k$ such that $d_1+\ldots+d_k>k$. Then the product manifold $M=M_1\times\ldots\times M_k$ is an l-Chen submanifold of $N=N_1\times\ldots\times N_k$ for some positive integer $l\le k$ if and only if* $\operatorname{rank}(D_l)=\operatorname{rank}(D_l:E_l)=l$ *where* $E_l=\operatorname{col}((b_1)^l,\ (b_2)^l,\ldots,(b_k)^l)$, *$D_l$ is the $k\times l$ matrix with entries $d_{ij}=(b_i)^j$, $b_i=\operatorname{Trace}(A_{n_i}A_{n_i})$, $n_i=\dfrac{H_i}{\|H_i\|}$, $i=1,\ldots,k$, $j=0,1,\ldots,l-1$ and $(D_l:E_l)$ is the augmented matrix.*

Proof. Since the M_i's are 1-Chen submanifolds, then from (2.1) we get $\tilde{A}(H_i)=b_iH_i$, $i=1,\ldots,k$. Applying Simons' operator to $\tilde{A}(H_i)$, we obtain

$$\tilde{A}^j(H_i)=(b_i)^jH_i,\qquad j=1,2,\ldots. \tag{2.3}$$

For the mean curvature vector H of M we have $H=\dfrac{1}{m}(m_1H_1,\ldots,m_kH_k)$, where $m=m_1+\ldots+m_k$. So, for any positive integer r, using (2.3) we get

$$\tilde{A}^r(H)=\frac{1}{m}(m_1\tilde{A}^r(H_1),\ldots,m_k\tilde{A}^r(H_k))=\frac{1}{m}(m_1(b_1)^rH_1,\ldots,m_k(b_k)^rH_k). \tag{2.4}$$

For a positive integer s, suppose that $\tilde{A}^s(H)$ is a linear combination of $H,\tilde{A}(H),\ldots,\tilde{A}^{s-1}(H)$ which are linearly independent, that is, $\tilde{A}^s(H)=\sum_{j=0}^{s-1}x_j\tilde{A}^j(H)$. Thus, from (2.4) for $r=s$ and $r=j$ we have

$$\left(m_1\left[(b_1)^s-\sum_{j=0}^{s-1}x_j(b_1)^j\right]H_1,\ldots,m_k\left[(b_k)^s-\sum_{j=0}^{s-1}x_j(b_k)^j\right]H_k\right)=0.$$

Since $H_i\neq 0$, then we obtain

$$b_i^s=\sum_{j=0}^{s-1}(b_i)^jc_j,\qquad i=1,\ldots,k. \tag{2.5}$$

This is a system of linear equations with respect to variables $x_0,\ldots,x_{s-1}$. We write it as $D_sX_s=E_s$, where $X_s=\operatorname{col}(x_0,\ldots,x_{s-1})$ and, for $l=s$, D_s and E_s are as in the hypothesis. Considering Theorem A,

the product submanifold M is an l-Chen submanifold

$\Leftrightarrow$ $\tilde{A}^l(H)=\sum_{j=0}^{l-1}x_j\tilde{A}^j(H)$

$\Leftrightarrow$ for $s=l$ (2.5) has a unique solution

$\Leftrightarrow$ $\operatorname{rank}(D_l)=\operatorname{rank}(D_l:E_l)=l$. □

Note that for $k=2$ and $l=1$ this theorem reduces to Proposition B. From the above theorem we get the following corollary.

COROLLARY 2.2. *Let M_i be m_i-dimensional 1-Chen submanifolds of (m_i+d_i)-dimensional Riemannian manifolds N_i with nowhere zero mean curvature vector H_i, $i=1,\dots,k$ such that $d_1+\dots+d_k>k$. Then the product manifold $M=M_1\times\dots\times M_k$ is a k-Chen submanifold of $N=N_1\times\dots\times N_k$ if and only if $b_i\neq b_j$ for $i\neq j$, $i,j=1,2,\dots,k$, where* $b_i=\mathrm{Trace}(A_{n_i}A_{n_i})$, $n_i=\dfrac{H_i}{\|H_i\|}$.

Proof. For $l=k$ considering Theorem 2.1, D_k is the $k\times k$ matrix and $\mathrm{rank}(D_k)=\mathrm{rank}(D_k:E_k)=k$ if and only if $\det(D_k)\neq 0$, where D_k and E_k are as in Theorem 2.1. According to the entries $d_{ij}=(b_i)^j$, $i=1,\dots,k$, $j=0,1,\dots,k-1$, of D_k, its determinant is obtained as

$$\det(D_k)=\prod_{i,j(i>j)=1}^{k}(b_i-b_j).$$

Therefore, $\det(D_k)\neq 0$ if and only if $b_i\neq b_j$ for $i\neq j$, $i,j=1,\dots,k$. □

We now construct an example which is a k-Chen submanifold as follows.

EXAMPLE 2.3. Let $\mathbb{S}^{p_i}(a_i)$ be a p_i-dimensional hypersphere of $\mathbb{R}^{p_i+1}$ with radius a_i. Put $q_i=\dfrac{p_i}{a_i^2}$. Let

$$M_i=\underbrace{\mathbb{S}^{p_i}(a_i)\times\dots\times\mathbb{S}^{p_i}(a_i)}_{d_i\text{ times}}\subset\mathbb{R}^{p_i+1}\times\dots\times\mathbb{R}^{p_i+1}\equiv\mathbb{R}^{m_i+d_i},\qquad i=1,\dots,k$$

where $m_i=p_id_i$. Then the product submanifold $M_1\times\dots\times M_k$ of $\mathbb{R}^{m_1+d_1}\times\dots\times\mathbb{R}^{m_k+d_k}$ is a k-Chen if and only if $q_i\neq q_j$ for $i\neq j$, $i,j=1,\dots,k$.

Proof. Since the M_i's are m_i-dimensional pseudo-umbilical submanifolds of $\mathbb{R}^{m_i+d_i}$, they are trivial 1-Chen submanifolds. Calculating $b_i=\mathrm{Trace}(A_{n_i}A_{n_i})$, $n_i=\dfrac{H_i}{\|H_i\|}$, we obtain $b_i=q_i$. Thus $b_1,\dots,b_k$ are all different. Therefore, the proof is an immediate result of Corollary 2.2.

THEOREM 2.4. *Let M_1 and M_2 be p-Chen and r-Chen submanifolds of Riemannian manifolds N_1 and N_2, respectively. Then for some k, $\max\{p,r\}\le k\le p+r$, the product $M_1\times M_2$ is a k-Chen submanifold of $N_1\times N_2$.*

Proof. First, we will show that $k\ge\max\{p,r\}$. Let $r\le p$. Suppose that $k<\max\{p,r\}=p$. Since $M_1\times M_2$ is a k-Chen submanifold we can write $\tilde{A}^k(H)=\sum_{i=0}^{k-1}x_i\tilde{A}^i(H)$, where H is the mean curvature vector of the product $M_1\times M_2$. Since

$$H=\frac{1}{m}(m_1H_1,m_2H_2),\qquad m=m_1+m_2$$

we have

$$\frac{1}{m}[m_1\tilde{A}^k(H_1),m_2\tilde{A}^k(H_2)]=\frac{1}{m}\sum_{i=0}^{k-1}x_i[m_1\tilde{A}^i(H_1),m_2\tilde{A}^i(H_2)]$$

and then

$$\tilde{A}^k(H_1) = \sum_{i=0}^{k-1} x_i \tilde{A}^i(H_1).$$

However, as M_1 is p-Chen and $k < p$,

$$\mathcal{A}_k(H_1) + \mathcal{L}_k(H_1, \tilde{A}^1(H_1), \tilde{A}^2(H_1), \ldots, \tilde{A}^{k-1}(H_1)) = \sum_{i=0}^{k-1} x_i \tilde{A}^i(H_1),$$

where $\mathcal{L}_k$ is a linear combination of $H_1, \tilde{A}^1(H_1), \tilde{A}^2(H_1), \ldots, \tilde{A}^{k-1}(H_1)$. Since $\mathcal{A}_k(H_1)$ is independent of H_1, $\tilde{A}^1(H_1)$, $\tilde{A}^2(H_1), \ldots, \tilde{A}^{k-1}(H_1)$, we get $\mathcal{A}_k(H_1) = 0$. By hypothesis, $\mathcal{A}_{p-1}(H_1) \neq 0$. Then this is a contradiction to our assumption $k < p$. Therefore, $k \geq \max\{p, r\}$.

We will now show that $k \leq p + r$. For this, we have to prove that if $\mathcal{A}_{p+r-1}(H) \neq 0$, then $\mathcal{A}_{p+r}(H) = 0$. Let $r \leq p$ and put $q = p - r$. Since M_1 and M_2 are, respectively, p-Chen and r-Chen and considering (2.2), we have

$$\tilde{A}^p(H_1) = \sum_{i=0}^{p-1} b_i \tilde{A}^i(H_1), \qquad \tilde{A}^r(H_2) = \sum_{i=0}^{r-1} c_i \tilde{A}^i(H_2). \tag{2.6}$$

Applying Simons' operator to $\tilde{A}^p(H_1)$, we get

$$\tilde{A}(\tilde{A}^p(H_1)) = \tilde{A}^{p+1}(H_1) = \sum_{i=0}^{p-1} b_i \tilde{A}^{i+1}(H_1) = \sum_{i=0}^{p-2} b_i \tilde{A}^{i+1}(H_1) + b_{p-1} \tilde{A}^p(H_1).$$

Using (2.6), we have $\tilde{A}^{p+1}(H_1) = b_0 b_{p-1} H_1 + \sum_{i=1}^{p-1} (b_{i-1} + b_i b_{p-1}) \tilde{A}^i(H_1)$. Put $B_0^1 = b_0 b_{p-1}$ and $B_i^1 = b_{i-1} + b_i b_{p-1}$, $i = 1, \ldots, p-1$. Thus, $\tilde{A}^{p+1}(H_1) = \sum_{i=0}^{p-1} B_i^1 \tilde{A}^i(H_1)$. If we apply Simons' operator again, we obtain $\tilde{A}^{p+2}(H_1) = \sum_{i=0}^{p-1} B_i^2 \tilde{A}^i(H_1)$, where $B_0^2 = B_{p-1}^1 b_0$ and $B_i^2 = B_{i-1}^1 + B_{p-1}^1 b_i$, $i = 1, \ldots, p-1$. When we keep on applying Simons' operator, we get

$$\tilde{A}^{p+s}(H_1) = \sum_{i=0}^{p-1} B_i^s \tilde{A}^i(H_1), \qquad s = 1, 2, \ldots, r. \tag{2.7}$$

For each s, we can define the functions B_i^s inductively as $B_0^s = B_{p-1}^{s-1} b_0$, $B_i^s = B_{i-1}^{s-1} + B_{p-1}^{s-1} b_i$, $i = 1, \ldots, p-1$ (for $s = 1$, $B_i^0 = b_i$).

Similarly, for the submanifold M_2 we obtain

$$\tilde{A}^{r+s}(H_2) = \sum_{i=0}^{r-1} C_i^s \tilde{A}^i(H_2), \qquad s = 1, 2, \ldots, p. \tag{2.8}$$

The functions C_i^s are determined as B_i^s.

Suppose that for some j, $j = 1, 2, \ldots, r$, the product manifold is a $(p+j)$-Chen

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submanifold of $N_1 \times N_2$, that is, $\mathscr{A}_{p+j-1}(H) \neq 0$ and $\mathscr{A}_{p+j}(H) = 0$. Therefore, $\tilde{A}^{p+j}(H) = \sum_{i=0}^{p+j-1} y_i \tilde{A}^i(H)$, and then we have $\tilde{A}^{p+j}(H_1) = \sum_{i=0}^{p+j-1} y_i \tilde{A}^i(H_1)$ and, $\tilde{A}^{p+j}(H_2) = \sum_{i=0}^{p+j-1} y_i \tilde{A}^i(H_2)$. So, for $\tilde{A}^{p+j}(H_1)$, we obtain

$$\begin{aligned}\tilde{A}^{p+j}(H_1) &= \sum_{i=0}^{p-1} B_i^j \tilde{A}^i(H_1) \\ &= \sum_{i=0}^{p-1} y_i \tilde{A}^i(H_1) + y_p \tilde{A}^p(H_1) + y_{p+1}\tilde{A}^{p+1}(H_1) + \dots + y_{p+j-1}\tilde{A}^{p+j-1}(H_1).\end{aligned}$$

Considering (2.7) and, since $H_1, \tilde{A}^1(H_1), \tilde{A}^2(H_1), \dots, \tilde{A}^{p-1}(H_1)$ are linearly independent, we get

$$B_i^j = y_i + y_p b_i + y_{p+1} B_i^1 + \dots + y_{p+j-1} B_i^{j-1}, \qquad i = 0, 1, \dots, p-1. \tag{2.9}$$

Similarly, for the submanifold M_2 we obtain

$$C_i^{q+j} = y_i + y_r c_i + y_{r+1} C_i^1 + \dots + y_{r+q+j-1} C_i^{q+j-1}, \qquad i = 0, 1, \dots, r-1. \tag{2.10}$$

Considering (2.9) and (2.10), we have a system of linear equations $D_{p+j} Y_{p+j} = E_{p+j}$, where $Y_{p+j} = \mathrm{col}(y_0, y_1, \dots, y_{p+j-1})$, $E_{p+j} = \mathrm{col}(B_0^j, B_1^j, \dots, B_{p-1}^j, C_0^{q+j}, C_1^{q+j}, \dots, C_{r-1}^{q+j})$ and D_{p+j} is the $(p+r) \times (p+j)$ matrix as $D_{p+j} = \begin{pmatrix} I_1 & B_{p+j} \\ I_2 & C_{p+j} \end{pmatrix}$, where I_1 and I_2 are, respectively, the $p \times p$ and $r \times r$ identity matrices and, B_{p+j} and C_{p+j} are, respectively, the $p \times j$ and $r \times (q+j)$ matrices as following

$$B_{p+j} = \begin{pmatrix} b_0 & B_0^1 & B_0^2 & \dots & B_0^{j-1} \\ b_1 & B_1^1 & B_1^2 & \dots & B_1^{j-1} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{p-1} & B_{p-1}^1 & B_{p-1}^2 & \dots & B_{p-1}^{j-1} \end{pmatrix}, \qquad C_{p+j} = \begin{pmatrix} c_0 & C_0^1 & C_0^2 & \dots & C_0^{q+j-1} \\ c_1 & C_1^1 & C_1^2 & \dots & C_1^{q+j-1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{r-1} & C_{r-1}^1 & C_{r-1}^2 & \dots & C_{r-1}^{q+j-1} \end{pmatrix}.$$

From (2.9) and (2.10) it is easily seen that $D_{p+j+1} = (D_{k+j} : E_{p+j})$.

Considering Theorem A,

the product manifold is a $(p+j)$-Chen submanifold of $N_1 \times N_2$
$\Leftrightarrow D_{p+j} Y_{p+j} = E_{p+j}$ has a unique solution for Y_{p+j}
$\Leftrightarrow \mathrm{rank}(D_{p+j}) = \mathrm{rank}(D_{p+j} : E_{p+j}) = \mathrm{rank}(D_{p+j+1}) = p + j$.
(Since $\mathscr{A}_{p+j-1}(H) \neq 0$, $\mathrm{rank}(D_{p+j}) = p + j$.)

Conversely, we can say $\mathscr{A}_{p+j}(H) \neq 0 \Leftrightarrow \mathrm{rank}(D_{p+j}) = p + j \neq \mathrm{rank}(D_{p+j} : E_{p+j}) = \mathrm{rank}(D_{p+j+1}) = p + j + 1$. So, for $j = r - 1$, let $\mathscr{A}_{p+r-1}(H) \neq 0$. Then $\mathrm{rank}(D_{p+r-1} : E_{p+r-1}) = \mathrm{rank}(D_{p+r}) = p + r$. Therefore $\det(D_{p+r}) \neq 0$.

Finally, suppose that $\mathscr{A}_{p+r-1}(H) \neq 0$. Then, for $j = r$, the system of linear equations $D_{p+r} Y_{p+r} = E_{p+r}$ has a unique solution because of $\det(D_{p+r}) \neq 0$. Therefore the product manifold $M_1 \times M_2$ is a $(p+r)$-Chen submanifold of $N_1 \times N_2$, that is, $k = p + r$. Thus the proof is completed. □

Using Example 2.3 we give the following example for Theorem 2.4.

EXAMPLE 2.5. Let $M_i = \underbrace{\mathbb{S}^{p_i}(a_i) \times \ldots \times \mathbb{S}^{p_i}(a_i)}_{d_i \text{ times}}$, $q_i = \frac{p_i}{a_i^2}$, $i = 1, \ldots, p$ $\Big($and $\tilde{M}_i = \underbrace{\mathbb{S}^{\tilde{p}_i}(\tilde{a}_i) \times \ldots \times \mathbb{S}^{\tilde{p}_i}(\tilde{a}_i)}_{\tilde{d}_i \text{ times}}$, $\tilde{q}_i = \frac{\tilde{p}_i}{\tilde{a}_i^2}$, $i = 1, \ldots, r\Big)$ be as in Example 2.3. Let $M = M_1 \times \ldots \times M_p$ and $\tilde{M} = \tilde{M}_1 \times \ldots \times \tilde{M}_k$ be p-Chen and r-Chen submanifolds of $\mathbb{R}^{m+d} \equiv \mathbb{R}^{m_1+d_1} \times \ldots \times \mathbb{R}^{m_p+d_p}$ and $\mathbb{R}^{\tilde{m}+\tilde{d}} \equiv \mathbb{R}^{\tilde{m}_1+\tilde{d}_1} \times \ldots \times \mathbb{R}^{\tilde{m}_r+\tilde{d}_r}$ with codimension $d = d_1 + \ldots + d_p \geq p$ and $\tilde{d} = \tilde{d}_1 + \ldots + \tilde{d}_r \geq r$, respectively, where $m_i = p_i d_i$, $\tilde{m}_i = \tilde{p}_i \tilde{d}_i$, $m = m_1 + \ldots + m_p$ and $\tilde{m} = \tilde{m}_1 + \ldots + \tilde{m}_r$. Let $q_i = \tilde{q}_i$, $i = 1, \ldots, s$, such that $s < \min\{p, r\}$. Then the product manifold $M \times \tilde{M}$ is a $(p + r - s)$-Chen submanifold of $\mathbb{R}^{m+d} \times \mathbb{R}^{\tilde{m}+\tilde{d}}$ and $\max\{p, r\} < p + r - s < p + r$.

Proof. Since M (resp. $\tilde{M}$) is p-Chen (resp. r-Chen) then $q_1, \ldots, q_p$ (resp. $\tilde{M}$, $\tilde{q}_1, \ldots, \tilde{q}_r$) are all different, Since, for $i = 1, 2, \ldots, s$, $q_i = \tilde{q}_i$, then the products $M_i \times \tilde{M}_i$ are pseudo-umbilical, namely, they are trivial 1-Chen submanifolds. Also, the functions $b_i = \text{trace}(A_{\xi_i} A_{\xi_i}) = q_i = \tilde{q}_i$, where $\xi_i = \frac{\bar{H}_i}{|\bar{H}_i|}$, $\bar{H}_i$'s are mean curvature vectors of $M_i \times \tilde{M}_i$ $i = 1, 2, \ldots, s$.

Therefore, for each i belonging to $\{1, \ldots, s\}$, we consider $M_i \times \tilde{M}_i$ as one factor in the product $M \times \tilde{M}$. Since $q_1(=\tilde{q}_1), \ldots, q_s(=\tilde{q}_s)$, $q_{s+1}, \ldots, q_p$, $\tilde{q}_{s+1}, \ldots, \tilde{q}_r$ are all different, then, according to Example 2.3, the product $M \times \tilde{M}$ is a $(p + r - s)$-Chen. Thus $\max\{p, r\} < p + r - s < p + r$, since $s < \min\{p, r\}$.

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