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THE ADDITIVE GROUPS OF SUBDIRECTLY IRREDUCIBLE RINGS II

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The classification of strongly subdirectly irreducible rings, which was begun in a previous paper, is completed by showing that a mixed group G is strongly subdirectly irreducible if and only if $G \simeq Z(p^{\infty}) \oplus H$, H a rank one, p-divisible, torsion free nil group.

I.

In [1] the strongly subdirectly irreducible torsion, and torsion free groups were classified. The following necessary condition was obtained for a mixed group to be strongly subdirectly irreducible [1, Theorem 3.3].

THEOREM A. *Let* G *be a mixed strongly subdirectly irreducible group. Then* $G \simeq Z(p^{\infty}) \oplus H$, H *a rank one, torsion free nil group.*

The object of this short note is to give necessary and sufficient conditions for a mixed group to be strongly subdirectly irreducible, and so complete the classification of the strongly subdirectly irreducible groups.

II.

THEOREM B. *Let* G *be a mixed group.* G *is strongly subdirectly irreducible if and only if* $G \simeq Z(p^{\infty}) \oplus H$, H *a rank one, p-divisible, torsion free nil group.*

Proof. Suppose that G is strongly subdirectly irreducible. By

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Theorem A, $G \simeq Z(p^{\infty}) \oplus H$, H a rank one, torsion free, nil group. It suffices to show that H is p-divisible. If not, there exists $h \in H$, $h \neq 0$, such that the p-height of h is 0 . For $a, b \in G$ define $a \cdot b = 0$ if $a \in Z(p^{\infty})$ or $b \in Z(p^{\infty})$. Choose $a_0 \in Z(p^{\infty})$ with $|a_0| = p$. For $h_1, h_2 \in H$, there exist positive integers n_i , and integers m_i such that $p \nmid n_i$, and $n_i h_i = m_i h$, $i = 1, 2$. There exists a unique element $a \in Z(p^{\infty})$ such that $n_1 n_2 a = m_1 m_2 a_0$. Define $h_1 h_2 = a$. The above products define a ring structure R on G , with ideals $Z(p^{\infty})$ and pH . Clearly $Z(p^{\infty}) \cap pH = 0$, and $h^2 = a_0 \neq 0$. Hence R is a ring satisfying $R^2 \neq 0$, but R is not subdirectly irreducible, a contradiction.

Let $G \simeq Z(p^{\infty}) \oplus H$, H a rank one, p-divisible, torsion free nil group. Clearly G is p-divisible. G is not nil [2, Theorem 120.3]. Let R be a ring with $R^+ = G$, and $R^2 \neq 0$. The quotient ring $R/Z(p^{\infty})$ has a nil additive group. Hence $R^2 \subseteq Z(p^{\infty})$. Let $a_0 \in Z(p^{\infty})$, $|a_0| = p$. Every non-zero subgroup of $Z(p^{\infty})$ contains a_0 . Let I be an ideal in R , $I \neq 0$. Suppose that $a_0 \notin I$. Then $I \cap Z(p^{\infty}) = 0$. However $RI \subseteq R^2 \subseteq Z(p^{\infty})$, and so $RI = 0$, and similarly $IR = 0$. Let $0 \neq x \in I$, $x = a + h$, $a \in Z(p\ )$, $h \in H$. Clearly $h \neq 0$. There exists a positive integer n such that $p^n a = 0$. Hence $p^n x \in H \cap I$. We may therefore assume that there exists $0 \neq h_0 \in H \cap I$. Let $h \in H$. There exist a non-negative integer k , a positive integer r , and an integer s such that $p \nmid r$, and $p^k rh = sh_0$. Therefore $p^k rhR \subseteq IR = 0$. However $p^k rhR = rh(p^k R) = rhR$, and so $rhR = 0$. Now $hR \subseteq Z(p^{\infty})$, and for $0 \neq a \in Z(p^{\infty})$, $ra \neq 0$. Hence $hR = 0$. Therefore

(1) $HR = 0$, and similarly

(2) $RH = 0$.

Let $a \in Z(p^{\infty})$, $x \in R$. There exists a positive integer n such that $p^n a = 0$, and there exists $y \in R$ such that $x = p^n y$. Hence

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$a \cdot x = a \cdot (p^n y) = (p^n a) \cdot y = 0$. Therefore

(3) $Z(p^\infty) \cdot R = 0$, and similarly

(4) $R \cdot Z(p^\infty) = 0$.

Equalities (1), (2), (3), and (4), imply that $R^2 = 0$, a contradiction.

References

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