ON ROTOR CALCULUS

II

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Summary

In the development of the rotor calculus presented in a previous paper space-time was taken to be flat. This work is now extended to the case of curved space-times, which, in the first instance, is taken to be Riemannian. (The calculus bears at times a strong formal resemblance to the spinor analysis of Infeld and van der Waerden, but it is in fact developed quite independently of this.) Owing to the fact that all general relations had earlier already been written in a generally covariant form they may be taken over unchanged into the present context. In particular,

$$\tau_{Akm}\tau_{Bl}{}^{m} = \alpha_{AB}g_{kl}-e_{ABC}\tau^{C}{}_{kl}$$

now serves as a defining relation for the connecting quantities τ_{Akl}. A linear rotor connection is introduced, and the covariant derivative of a rotor defined. The covariant constancy of the τ_{Akl} establishes the relation between the linear connections in w-space and in r-space. A rotor curvature tensor is considered alongside a number of other curvature objects. Next, conformal transformations are dealt with, of which duality rotations may be considered as a special case. This leads naturally to a gauge-covariant generalization of the whole calculus. A so-called rotor-derivative is defined, and some general relations involving such derivatives investigated. The relation of the rotor curvature tensor to the spin-curvature tensor is touched upon, after which the introduction of "geodesic frames" is considered. After this general theory some points concerning the Maxwell field are dealt with, which is followed by some work concerning basic quadratic and cubic invariants. Finally, a certain basic symmetric rotor is re-considered in the context of the classification of Weyl tensors, and the idea of a canonical representation suggested.

1. Introduction

In a previous paper[1] (Buchdahl, 1966) I developed the calculus of

[1] This paper will be referred to as I, and likewise its numbered sections and equations will be distinguished by the letter I. It should be remarked that the general introduction to I and the references to earlier work given there are here taken for granted, as are the notaton and terminology of I.

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rotors, i.e. of the vectors and tensors of a three-dimensional complex vector space which can be brought into one-one correspondence with self-dual tensors in space-time. The latter was there supposed to be flat. This assumption notwithstanding, the general formalism was developed so as to be invariant under arbitrary *w*-transformations, and arbitrary *r*-transformations, it being implied that one has the freedom to carry these out independently of one another. This freedom entails the possibility of generalizing the calculus without undue difficulty to the situation when *w*-space is curved. The purpose of this paper is to achieve this generalization proceeding in a way inevitably reminiscent of the work if Infeld and van der Waerden (**1933**) in the context of spinor calculus, not least because no use is made here of orthonormal tetrads.

In the case of flat *w*-space it is possible to arrange the τ_{Akl} to have fixed numerical values; that is to say, corresponding to any Lorentz transformation $L^m{}_n$ one can find a rotor image $\Lambda^B{}_A$ such that the two transformations carried out jointly leave the τ_{Akl} numerically invariant, cf. I Section 3. When *w*-space is curved the τ_{Akl} become functions of the *w*-coordinates x^k, and it is not possible in general to find an *r*-image of an *arbitrary* transformation of the coordinate x^k so as to leave the functional form of the τ_{Akl} invariant. Accordingly *r*-transformations and *w*-transformations are now always to be regarded as being carried out independently of each other. The τ_{Akl} are, as before, the connecting quantities between simple rotors and self-dual *w*-tensors, and they are required only to obey the *defining relation I* (4.10); where the α_{AB} are six arbitrary functions (invariant under *w*-transformations alone), subject only to the condition $\det \alpha_{AB} \neq 0$. The ten components of the metric tensor g_{kl} are now functions of the coordinates. This tensor is required to be non-singular and of signature -2 but is otherwise arbitrary, except in as far as it is supposed to satisfy conditions of continuity and differentiability appropriate to the context. It need not necessarily be that of a Riemann space but might, for instance, be that of a Weyl space.

At this point it is convenient to introduce a terminology relating to the various more or less specialized spaces which may be encountered. A general Riemann space, — the conditions on g_{kl} mentioned above being understood, — will be denoted by V. (The dimensional number 4 usually added as a subscript is omitted since all *w*-spaces here contemplated are 4-dimensional.) Then, in order of increasing specialization, V_0 is a V of zero scalar curvature; V_E is an Einstein space, i.e. a V the trace-free part

$$E_{kl} = R_{kl} - \tfrac{1}{4} g_{kl} R \tag{1.1}$$

of the Ricci tensor of which vanishes; V_R is a Ricci-flat V, i.e. it is such

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that $R_{kl} = 0$; and finally V_C is a V of constant Riemannian curvature, i.e. it has[2]

$$R_{klmn} = \tfrac{1}{6} g_{k[n} g_{m]l} R. \qquad (1.2)$$

A Weyl space will be denoted by W.

In general outline this paper pursues the following course. After a discussion of the defining relations for the τ_{Akl} in Section 2, a linear rotor connection is introduced in Section 3 and its relation to the linear connection of a Riemann space established in Section 4, on the basis of the covariant constancy of the τ_{Akl}. The rotor curvature tensor and various related curvature objects are dealt with in Sections 5 and 6. Conformal transformations form the subject of Section 7, where conformal transformations in r-place alone are also contemplated. This leads naturally to the idea of duality rotations (Section 8). That the whole calculus has a straightforward gauge-covariant generalization is established in Section 9, which deals with rotors on Weyl spaces. From the anti-symmetrized second covariant derivative of a tensor of any kind one can form a tensor whose rotor valence exceeds by unity that of the original tensor: its "rotor derivative". Properties of first and higher rotor derivatives of rotors are considered in Section 10, and the condition is found for rotor differentiation to be commutative. Section 11 briefly refers to the connection between curvature rotors and curvature spinors, after which it is shown in Section 12 how, by suitable transformations, the τ_{Akl} can be arranged to take their standard representative values at any arbitrarily selected point and in a sufficiently small neighbourhood of it. Some formal considerations concerning the Maxwell field appear in Section 13, including a discussion of the consequences of representing any such non-null field by the *constant* rotor whose components are (1, 0, 0). Basic quadratic and cubic invariants are considered in Section 14, and a certain scalar density is exhibited as the divergence of a quantity presented in explicit form as a function of the components of the linear rotor connection and its first derivative. Finally, Section 15 deals with a rotor equivalent K^{AB} of the conformal curvature tensor in somewhat greater detail, with special reference to the classification of Weyl tensors, and the idea of a canonical representation of the τ_{Akl} is suggested.

2. Defining relation for the τ_{Akl}

As already remarked in the Introduction the τ_{Akl} are now any self-dual functions of the x^k which obey the defining relation I (4.10), i.e.

[2] The order of the indices of the Riemann tensor is that adopted by Eisenhart (1949).

$$\tau_{Ak}{}^{m}\tau_{Blm} = g_{kl}\alpha_{AB} - e_{ABC}\tau^{C}{}_{kl}. \tag{2.1}$$

The components of the metric rotor α_{AB} are six (w-invariant) functions which may be prescribed arbitrarily except in as far as the condition

$$\det \alpha_{AB} \neq 0 \tag{2.2}$$

must be satisfied.

All the general relations of I (other than those referring specifically to flat space which occur in Sections 3 and 10) may now be taken over into the present context as they stand. In this respect it may be noted that, superficial appearances notwithstanding, I (4.13) is not stronger than (2.1), for it can be derived from the latter in the following, albeit somewhat tedious, way. Arguing as in Section 2b and 6c of I one must have

$$\tau^{[A}{}_{kl}\tau^{B}{}_{mn}\tau^{C]}{}_{pq} = e^{ABC}f_{klmnpq}, \tag{2.3}$$

where f_{klmnpq} is a w-tensor, to be determined. Transvect the left hand member of (2.3) with $\tau_A{}^{uv}\tau_{Buw}\tau_{Cv}{}^{w}$ and the right hand member with $-4e_{ABC}$; these two quantities being equal, according to I (2.24). On the left there then appears a sum of products of η-tensors, three at a time, which may be reduced by means of[3] S (4.12) and (4.18). It is not necessary to give the details of this reduction, the outcome of which is that

$$f_{klmnpq} = -\tfrac{4}{3}g_{[n[l}\eta_{k]m]pq}.$$

If one now inserts this in (2.3) and transvects the resulting relation throughout with $\tau_C{}^{pq}$ the required relation I (4.13) follows.

3. The linear rotor connection

The analytical part of the calculus rests upon the introduction of a linear connection $\Gamma^{B}{}_{Ak}$ in r-space. Given such a connection, covariant differential equations for rotors can then be written down. Accordingly the covariant derivative of a covariant rotor ϕ_A is defined by the linear relation

$$\phi_{A;k} = \phi_{A,k} - \Gamma^{B}{}_{Ak}\phi_B. \tag{3.1}$$

The $\Gamma^{B}{}_{Ak}$ are functions which in any fixed coordinate system and rotor frame can be assigned arbitrarily but whose transformation properties are fixed by the requirement that the covariant derivative $\phi_{A;k}$ shall in fact transform covariantly under w- and r-transformations. This entails that under w-transformations alone the $\Gamma^{B}{}_{Ak}$ behave as the components of a

[3] The letter S distinguishes references to Buchdahl (1962).

w-vector. Under r-transformations on the other hand the linear connection must behave as follows:

$$\Gamma^{B'}{}_{A'k} = \Lambda^{A}{}_{A'}\Lambda^{B'}{}_{B}\Gamma^{B}{}_{Ak}+\Lambda^{B'}{}_{B}\Lambda^{B}{}_{A',k}. \tag{3.2}$$

The difference between two distinct linear connections is therefore a mixed r-tensor. The covariant derivative of a contravariant rotor is

$$\phi^{A}{}_{;k} = \phi^{A}{}_{,k}+\Gamma^{A}{}_{Bk}\phi^{B}. \tag{3.3}$$

(3.1) and (3.3) then ensure that

$$(\phi_A\theta^A)_{;k} = \phi_{A;k}\theta^A+\phi_A\theta^A{}_{;k}.$$

An arbitrary tensor is to be covariantly differentiated like an outer product of vectors, e.g.

$$\phi_{AB;k} = \phi_{AB,k}-\Gamma^{C}{}_{Ak}\phi_{CB}-\Gamma^{C}{}_{Bk}\phi_{AC}.$$

Of the covariant derivative of the rotor $\phi_{\dot{A}}$ complex conjugate to ϕ_A one requires that it be the complex conjugate of the covariant derivative of ϕ_A, so that

$$\phi_{\dot{A};k} = \phi_{\dot{A},k}-\Gamma^{\dot{B}}{}_{\dot{A}k}\phi_{\dot{B}}, \tag{3.4}$$

with

$$\Gamma^{\dot{B}}{}_{\dot{A}k} = \overline{\Gamma^{B}{}_{Ak}}.$$

4. Covariant derivative of rotors on Riemann spaces

In a V the components of the linear connection $\Gamma^{m}{}_{kl}$ are the Christoffel symbols, i.e.

$$\Gamma^{m}{}_{kl} = g^{mn}(g_{n(k,l)}-\tfrac{1}{2}g_{kl,n}), \tag{4.1}$$

which implies the covariant constancy of the metric tensor,

$$g_{kl;m} = 0. \tag{4.2}$$

Any tensor having both w- and r-indices is to be covariantly differentiated like the outer product of an appropriate number of r-vectors and w-vectors, e.g.

$$\chi_{Ak;l} = \chi_{Ak,l}-\Gamma^{B}{}_{Al}\chi_{Bk}-\Gamma^{m}{}_{kl}\chi_{Am},$$

and the usual product rule of differentiation continues to hold.

A relation between $\Gamma^{m}{}_{kl}$ and $\Gamma^{B}{}_{An}$ is now established by agreeing that the tensor equivalent of the covariant derivative of any rotor ϕ^A be identical with the covariant derivative of its tensor equivalent. (It is understood that the tensor equivalent is to be formed simply by transvection with $\frac{1}{2}\tau_{Akl}$.) Thus it is required that for all ϕ^A

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$$\tau_{Akl}\phi^{A}{}_{;m} = (\tau_{Akl}\phi^{A})_{;m},$$

or

$$\tau_{Akl;m} = 0. \tag{4.2}$$

In view of (4.2) it follows that $\tau_{A}{}^{kl}{}_{;m}$ also vanishes, and then I (2.13) shows that

$$\alpha_{AB;m} = 0, \tag{4.3}$$

i.e. the r-metric is covariant constant. Writting (4.2) out in full,

$$\tau_{Akl,m} - \Gamma^{B}{}_{Am}\tau_{Bkl} - \Gamma^{n}{}_{km}\tau_{Anl} - \Gamma^{n}{}_{lm}\tau_{Akn} = 0,$$

whence

$$\Gamma^{B}{}_{Am} = \tfrac{1}{4}\tau^{Bkl}\tau_{Akl,m} - \tfrac{1}{2}\tau^{Bkl}\tau_{Anl}\Gamma^{n}{}_{km}. \tag{4.4}$$

One can lower the index B on $\Gamma^{B}{}_{Am}$:

$$\Gamma_{BAm} = \alpha_{AC}\Gamma^{C}{}_{Am}, \tag{4.5}$$

and then one can infer from (4.4) that

$$\Gamma_{(AB)m} = \tfrac{1}{2}\alpha_{AB,m}. \tag{4.6}$$

The work involved is tedious, and the result follows more easily at once from (4.3). Transvecting with α^{AB} one gets

$$\Gamma^{B}{}_{Bm} = (ln\sqrt{\alpha})_{,m} \tag{4.7}$$

where $\alpha = \det \alpha_{AB}$; which corresponds exactly to the result

$$\Gamma^{n}{}_{nm} = (ln\sqrt{-g})_{,m}. \tag{4.8}$$

Note that, in view of (2.1), and (4.1), one can write (4.4) as

$$\Gamma^{B}{}_{Am} = \tfrac{1}{4}\tau^{Ckl}(\delta^{B}{}_{C}\tau_{Akl,m} + 2e^{B}{}_{AC}g_{lm,k}) - \tfrac{1}{2}\delta^{B}{}_{A}\Gamma^{n}{}_{nm}. \tag{4.9}$$

In particular, when A and B both have the fixed value J this gives, with (4.6) and (4.8) and I (6.12), (when $\alpha_{JJ} \neq 0$)

$$(\alpha_{JJ})^{-1}\tau_{J}{}^{kl}\tau_{Jkl,m} = (ln \det \tau_{Jpq})_{,m}, \tag{4.10}$$

a result which is obvious by inspection of (2.1), with $A = B = J$; so that the various equations above are consistent with each other.

5. The rotor curvature tensor

(a) By forming the second skew-symmetrized covariant derivative of a rotor ϕ_A one finds that

$$2\phi_{A;[kl]} = P^{B}{}_{Akl}\phi_{B}, \tag{5.1}$$

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where the r-curvature tensor is given by

$$P^B{}_{Akl} = \Gamma^B{}_{Al,k} - \Gamma^B{}_{Ak,l} + \Gamma^C{}_{Al}\Gamma^B{}_{Ck} - \Gamma^C{}_{Ak}\Gamma^B{}_{Cl}. \tag{5.2}$$

The index B may be lowered. With (4.5), and using (4.3), one then gets

$$P_{BAkl} = \Gamma_{BAl,k} - \Gamma_{BAk,l} + \Gamma^C{}_{Ak}\Gamma_{CBl} - \Gamma^C{}_{Al}\Gamma_{CBk}. \tag{5.3}$$

In view of (4.6) it follows at once that

$$P_{(AB)kl} = 0, \tag{5.4}$$

so that, incidentally, P_{ABkl} is trace-free.

(b) Since P_{CDkl} is skew-symmetric in the pair of r-indices it is natural to replace these by a single "dual index", by simply transvecting with e^{ACD}. At the same time the skew-symmetric pair of w-indices may be replaced by one r-index by the usual device of transvecting with $\frac{1}{2}\tau^{Bkl}$. One is thus led naturally to the r-tensor.

$$K^{AB} = \tfrac{1}{2}e^{ACD}\tau^{Bkl}P_{CDkl}. \tag{5.5}$$

Now

$$\begin{aligned} 2F_{kl;[mn]} &= R^p{}_{kmn}F_{pl} + R^p{}_{lmn}F_{kp} \\ &= \tau^A{}_{kl}\phi_{A;[mn]} = \tfrac{1}{2}\tau^A{}_{kl}P^B{}_{Amn}\phi_B \\ &= \tau^B{}_{p[l}R^p{}_{k]mn}\phi_B. \end{aligned}$$

Since ϕ_B is arbitrary one obtains from this after transvection with $\tau_C{}^{kl}$

$$P_{CDmn} = -\tfrac{1}{2}e_{CDE}\tau^{Ekl}R_{klmn}. \tag{5.6}$$

From this and (5.5) it then follows that

$$K^{AB} = -\tfrac{1}{2}\tau^{Akl}\tau^{Bmn}R_{klmn}. \tag{5.7}$$

(The symmetries of the Riemann tensor, viz.

$$R_{(kl)mn} = 0, \quad R_{kl(mn)} = 0, \quad R_{klmn} = R_{mnkl}, \quad R_{k[lmn]} = 0, \tag{5.8—11}$$

should be constantly kept in mind.) Evidently K^{AB} is symmetric,

$$K^{[AB]} = 0. \tag{5.12}$$

The trace of K^{AB} is

$$K = -\eta^{klmn}R_{klmn} = R, \tag{5.13}$$

i.e. the scalar formed from K^{AB} is just the scalar curvature of V.

Eq. (5.6) may be inverted to give

$$\eta_{kl}{}^{st}R_{stmn} = -\tfrac{1}{2}e^{ABC}\tau_{Akl}P_{BCmn}, \tag{5.14}$$

from which R_{klmn} is obtained by drawing upon the complex conjugate relation. If (5.14) be transvected with g^{km} the imaginary part of the left hand member vanishes on account of (5.11) and one is left with

$$R_{kl} = \tfrac{1}{2} e^{ABC} \tau_A{}^m{}_k P_{BCml}. \tag{5.15}$$

The identity S (3.3), i.e.

$$\eta_{kl}{}^{st} \bar{\eta}^{mn}{}_{pq} R_{stmn} = 4\eta_{klr[p} E_{q]}{}^r \tag{5.16}$$

follows easily by multiplying out the first two factors on the left explicitly. Now insert (5.14) on the left and transvect throughout with $e_{ADE}\tau^{Dkl}$, with the result

$$\bar{\eta}^{mn}{}_{pq} P_{BCmn} = -2e_{ABC}\tau^A{}_{r[p} E_{q]}{}^r. \tag{5.17}$$

From this one infers that P_{ABkl} is *self-dual if and only if the* V *is a* V_E. (Cf. the results of Section 7 (ii) of S.) Also, in a V_C

$$P_{ABkl} = \tfrac{1}{12} e_{ABC} \tau^C{}_{kl} R, \qquad (R = \text{const.}). \tag{5.18}$$

Some of the equations above may be given a form which brings out more closely their resemblance to the corresponding spinor equations by using I (4.9). For example, (5.14) becomes

$$\eta_{kl}{}^{st} R_{stmn} = \tfrac{1}{2} T_{kl}{}^{BC} P_{BCmn}, \tag{5.19}$$

which may be compared with S (2.37). (See also Section 11.)

(c) In the heuristic argument leading to the definition (5.5) one might quite well have replaced τ^{Bkl} by $\tau^{\dot{B}kl}$. Consider therefore the alternative quantity

$$\Gamma^{\dot{A}B} = \tfrac{1}{2} e^{BCD} \tau^{\dot{A}kl} P_{CDkl}. \tag{5.20}$$

With (5.6) this becomes

$$\Gamma^{\dot{A}B} = -\tfrac{1}{2} \tau^{\dot{A}kl} \tau^{Bmn} R_{klmn}, \tag{5.21}$$

so that $\Gamma^{\dot{A}B}$ is hermitian,

$$\Gamma^{\dot{A}B} = \Gamma^{B\dot{A}}. \tag{5.22}$$

Using (5.9), (5.21) gives rise to

$$\Gamma^{\dot{A}B} = \mathcal{T}^{mn\dot{A}B} R_{mn}. \tag{5.23}$$

But

$$\mathcal{T}^{kl}{}_{\dot{A}B} \mathcal{T}^{mn\dot{A}B} = 4g^{k(m}g^{n)l} - g^{mn}g^{kl}, \tag{5.24}$$

so that

$$\mathcal{T}^{kl}{}_{\dot{A}B} \Gamma^{\dot{A}B} = 4E^{kl}. \tag{5.25}$$

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It follows that $\Gamma^{\dot{A}B}$ *is a zero rotor if and only if the V is a* V_E. This characterization of Einstein spaces is an alternative to that following eq. (5.17). Finally it may be noted that in view of (5.6) the identity of Bianchi gives rise to

$$P_{CD[mn;r]} = 0. \tag{5.26}$$

6. The conformal curvature tensor

Several of the results of the previous section are effectively already contained in the work of Section 7 of *I*. For example, granted (5.22), it follows from *I* (7.11) that its tensor equivalent is symmetric and trace-free. Since it is moreover an algebraic concomitant of R_{klmn} it can therefore only be a numerical multiple of E_{kl}. At any rate, it seems appropriate to turn now to *I* Section 7a in view of the fact that a symmetric trace-free rotor is at hand, viz.

$$\mathring{K}^{AB} = K^{AB} - \tfrac{1}{3}\alpha^{AB}K. \tag{6.1}$$

Bearing in mind the remark following eq. *I* (7.4) one now knows that the tensor equivalent of $\mathring{K}^{AB}$ has the following properties: (i) it has the symmetries of the Riemann tensor, (ii) it is entirely trace-free, (iii) it is equal to its right and left duals, (iv) it is an algebraic concomitant of R_{klmn}. One concludes that it is a numerical multiple of $\eta_{kl}{}^{st}C_{stmn}$. In fact it is not difficult to confirm that

$$\tfrac{1}{4}\tau_{Akl}\tau_{Bmn}\mathring{K}^{AB} = -\eta_{kl}{}^{st}C_{stmn} \tag{6.2}$$

or conversely,

$$\mathring{K}^{AB} = -\tfrac{1}{2}\tau^{Akl}\tau^{Bmn}C_{klmn}. \tag{6.3}$$

Accordingly the vanishing of $\mathring{K}^{AB}$ is a necessary and sufficient condition for V to be conformally flat.

7. Conformal transformations

It is of interest to study conformal *w*-transformations

$$g_{kl} \to {}'g_{kl} = \lambda g_{kl} \tag{7.1}$$

directly. Here λ is an arbitrary real function of the x^k. The τ_{Akl} are taken to be unaffected under (7.1). No generality would be gained [4] by laying down that it takes a factor λ^β, say, where β is a real number, since the transformation

[4] This is analogous to the situation in spinor analysis (cf. Buchdahl, 1959).

(7.2) $$\tau_{Akl} \to {}'\tau_{Akl} = \lambda^{\beta}\tau_{Akl}$$

is tantamount to a mere r-transformation

(7.3) $$\Lambda^{A}{}_{A'} = \lambda^{\beta}\delta^{A}{}_{A'},$$

which one is always at liberty to carry out. (See, however, (b) below.) One confirms easily on the basis of (2.1) that α_{AB} must take a factor λ^{-2}. The behaviour of other quantities is likewise easily determined, e.g. e_{ABC} and τ^{Akl} take factors λ^{-3} and 1 respectively. Write

(7.4) $$\lambda = e^{2q}.$$

Then it is well known that

(7.5) $$'\Gamma^{m}{}_{kl} = \Gamma^{m}{}_{kl} + 2\delta^{m}{}_{(k}q_{;\,l)} - g_{kl}q^{;m}.$$

From (4.4) one then infers that

(7.6) $$\Delta^{B}{}_{Am} = {}'\Gamma^{B}{}_{Am} - \Gamma^{B}{}_{Am} = -2\delta^{B}{}_{A}q_{;m} + e_{A}{}^{BC}\tau_{Cmn}q^{;n}.$$

Then

$$\Delta^{B}{}_{Akl} = {}'P^{B}{}_{Akl} - P^{B}{}_{Akl} = 2\Delta^{B}{}_{A[l;k]} + \Delta^{C}{}_{A[l}\Delta^{B}{}_{Ck]},$$

into which (7.6) is to be inserted. Of the terms which remain only one is rather complicated, but this may be reduced by means of *I* (4.13). One finally arrives at

(7.7) $$\Delta^{B}{}_{Akl} = e_{A}{}^{BC}\{2\tau_{C}{}^{m}{}_{[k}(q_{;\,l]m} - q_{;\,l]}q_{;m}) - \tau_{Ckl}q_{;m}q^{;m}\}.$$

$'K^{AB}$ is now easily constructed, and one finds that

(7.8) $$'K^{AB} = e^{2q}\{K^{AB} + 2\alpha^{AB}(q_{;m}{}^{m} + q_{;m}q^{;m})\},$$

whence

(7.9) $$'\mathring{K}^{AB} = e^{2q}\mathring{K}^{AB},$$

as must of course be the case, in view of (6.3). A subsequent r-transformation of the form (7.3), with $\beta = \frac{1}{2}$, will remove the factor e^{2q} on the right of (7.9). In other words $\mathring{K}^{AB}$ will be invariant under the transformation in which g_{kl}, τ_{Akl}, α_{AB} take factor λ, $\lambda^{\frac{1}{2}}$, λ^{-1} respectively.

(b) It has already been remarked that one might take a conformal transformation to mean the transition to new basic tensors $'g_{kl}$, $'\tau_{Akl}$, $'\alpha_{AB}$ which arise from the old by providing them with factors λ^{α}, λ^{β}, λ^{γ} respectively, subject to the condition that

(7.10) $$2\alpha - 2\beta + \gamma = 0;$$

but that taking $\beta \neq 0$ represents only an apparent gain of generality on account of the freedom one has at any stage in making an r-transformation

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of the form (7.3). This remark must now be somewhat modified. It is true that the results previously derived in this section provide the corresponding results in the more general case. On the other hand it has to be realized that in any r-transformation of any equation *every* rotor contained in it has to be transformed in the appropriate manner. A conceptually different situation arises if one lays down that, — under prescribed circumstances, — the transformation (7.3) be carried out only on the *basic* rotors τ_{Akl}, α_{AB} and their concomitants. This amounts to defining a *conformal transformation in r-space*:

$$\tau_{Akl} \to {}'\tau_{Akl} = \lambda^{\frac{1}{2}}\tau_{Akl}, \quad \alpha_{AB} \to {}'\alpha_{AB} = \lambda\alpha_{AB}, \tag{7.11}$$

a special example of which will be met in the next section. Under (7.11)

$$\Delta^{B}{}_{Am} = \delta^{B}{}_{B} q_{;m},$$

(which is in harmony with (3.2)), but, of course,

$$\Delta^{B}{}_{Akl} = 0. \tag{7.12}$$

8. Duality rotations

A duality rotation, as understood by Misner and Wheeler (1957) is a formal rotation of a skew-symmetric tensor and its dual [5] $f^*_{kl} = \frac{1}{2}e_{kl}{}^{mn}f_{mn}$:

$$'f_{kl} = f_{kl}\cos\alpha + f^*_{kl}\sin\alpha, \quad 'f^*_{kl} = -f_{kl}\sin\alpha + f^*_{kl}\cos\alpha, \tag{8.1}$$

where α is an arbitrary real angle. f_{kl} defines a self-dual tensor according to I (2.4), and in terms of this (8.1) reduces to

$$'F_{kl} = e^{i\alpha}F_{kl}. \tag{8.2}$$

If ϕ_A is the rotor equivalent of F_{kl} the duality rotation (8.1) represents merely the multiplication of ϕ_A by a phase factor

$$'\phi_A = e^{i\alpha}\phi_A. \tag{8.3}$$

(It should, however, be kept in mind that the "square of the length" of ϕ_A, i.e. $\phi_A\phi^A$, is not invariant under (8.3) since the metric is symmetric rather than hermitian. On the other hand the hermitian tensor $\phi_{\dot{A}}\phi_B$ *is* obviously invariant under (8.3).)

Now in passing from (8.2) to (8.3) the basic rotors have evidently been regarded as fixed quantities. One may, however, think of a duality rotation from the alternative point of view presented in Section 7b. In other words, one may look upon the duality relation of the field f_{kl} not as

[5] This is the dual as defined by these authors. In terms of the present notation $f^*_{kl} = if^{\dagger}_{kl}$.

induced by the phase transformation of its r-equivalent, but rather as induced by a *unimodular conformal transformation in r-space* of the form (7.11) with $q = i\alpha$, ϕ_A itself being kept fixed. One then sees particularly clearly how the effect of a duality rotation on f_{kl} differs *essentially* from that of a Lorentz transformation.

9. Gauge covariant theory

Just as spinor analysis can be readily extended to Weyl spaces W (Buchdahl, 1958) by introducing the gauge-invariant covariant derivative of spinors, so the same can be achieved for rotors in an essentially similar way. In this section (and in it alone) the gauge-covariant version of rotor analysis will be considered in outline. Accordingly any kind of tensor T (indices suppressed) will be said to be of gauge-weight η if it takes a factor λ^η when g_{kl} takes a factor λ. (λ must evidently be required to be real.) Then the formation of the gauge-covariant derivative is adequately exemplified by that of a tensor ξ_{Ak},

$$\xi_{Ak;\,l} = \xi_{Ak,\,l} - \Gamma^B{}_{Al}\xi_{Bk} - \Gamma^m{}_{kl}\xi_{Am} - \eta k_l \xi_{Ak}, \tag{9.1}$$

where η is the gauge-weight of ξ_{Ak}. $\Gamma^m{}_{kl}$ is the usual symmetric linear connection in W, $\Gamma^B{}_{Al}$ the linear rotor connection, and k_l is the Weyl vector. The gauge-invariant covariant derivative of g_{kl} vanishes, so that, since $\eta = 1$,

$$\Gamma^m{}_{kl} = \overset{\circ}{\Gamma}{}^m{}_{kl} - (\delta^m{}_{(k}k_{l)} - \tfrac{1}{2}g_{kl}k_m), \tag{9.2}$$

where $\overset{\circ}{\Gamma}{}^m{}_{kl}$ is a Christoffel symbol generated by g_{st}.

To τ_{Akl} the gauge-weight β will be assigned so that the gauge-weight of α_{AB} must be $2(\beta-1) = \gamma$, say; cf. Section 7b. The requirement that τ_{Akl} be covariant constant then gives, after the fashion of Section 4,

$$\Gamma^B{}_{Am} = \overset{\circ}{\Gamma}{}^B{}_{Am} - \tfrac{1}{2}(\gamma\delta^B{}_A k_m + e_A{}^{BC}\tau_{Cmn}k^n), \tag{9.3}$$

where $\overset{\circ}{\Gamma}{}^B{}_{Am}$ is the part of $\Gamma^B{}_{Am}$ independent of k_s. Proceeding as in Section 7 one obtains an expression for the curvature rotor of the form

$$P^B{}_{Akl} = \overset{\circ}{P}{}^B{}_{Akl} + \gamma\delta^B{}_A k_{[k;\,l]} - e_A{}^{BC}\{\tau_{Cn[k}(k^n{}_{;\,l]} + \tfrac{1}{2}k^n k_{l]}) + \tfrac{1}{4}\tau_{Ckl}k_n k^n\}, \tag{9.4}$$

where the first term on the right does not contain k_s. In place of (5.4) one now has

$$P_{(AB)kl} = \gamma\alpha_{AB}k_{[k;\,l]}. \tag{9.5}$$

This result is somewhat remarkable in as far as it shows that there is a choice of gauge-weights for which the curvature rotor is skew-symmetric in its rotor indices. It seems hardly worth while to investigate the relation

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of $P^B{}_{Akl}$ to the Weyl curvature tensor, and Weyl spaces will therefore now be set aside.

10. The rotor derivative

(a) Given any tensor T (indices suppressed) its anti-symmetrized second covariant derivative $2T_{;[kl]}$ may be transvected with $\tau_B{}^{kl}$ so that one ends up with a tensor whose r-valence exceeds that of T by unity. Accordingly it is of formal interest to define a *rotor derivative*

(10.1) $$T_{;B} = 2\tau_B{}^{kl}T_{;kl}.$$

In what follows T will be taken to have only r-indices, for this is the more interesting case

The rotor derivative of a scalar vanishes identically. Next consider a simple rotor ϕ_A. One has

(10.2) $$\phi_{A;B} = 2\tau_B{}^{kl}\phi_{A;kl} = \tau_B{}^{kl}P^C{}_{Akl}\phi_C,$$

in view of (5.1). Drawing upon (5.5) one then has

(10.3) $$\phi_{A;B} = e_{ACD}K_B{}^C\phi^D.$$

The divergence-like expression $\alpha^{AB}\phi_{A;B}$ vanishes identically:

(10.4) $$\phi_{A;}{}^A = 0.$$

One can also contemplate a sort of curl of ϕ_A:

(10.5) $$\tilde{\phi}^C \equiv e^{ABC}\phi_{A;B} = (K\delta^C{}_D - K^C{}_D)\phi^D.$$

For rotors of higher valence one has, for instance,

(10.6) $$\phi_{AB;C} = K_C{}^D(e_{ADE}\phi^E{}_B + e_{BDE}\phi_A{}^E).$$

The divergence-like expression $\phi_{AB;}{}^B$ does not vanish in general. However,

(10.7) $$K_{AB;}{}^B = 0.$$

(b) Next one may go on to form higher rotor derivatives; and here the most interesting case is the antisymmetrized second derivative. It suffices to consider the rotor $2\phi_{A;[BC]}$. One has to differentiate (10.3) again and then remove the derivative of ϕ^D by means of (10.3). The details of the work need not be reproduced; the result is that

(10.8) $$2\phi_{A;[BC]} = L^D{}_{ABC}\phi_D,$$

where

(10.9) $$L_{DABC} = 4\alpha_{[A[B}k_{C]D]} + 6K_{A[B}K_{C]D},$$

with

(10.10) $$k_{AB} = K_{AC}K_B{}^C - KK_{AB}.$$

L_{DABC} is remarkable in that it has all the symmetries of the Riemann tensor, eqs. (5.8—11). Therefore, r-space being 3-dimensional,

(10.11) $$L_{DABC} = 4\alpha_{[A[B}(L_{C]D]} - \tfrac{1}{4}\alpha_{C]D]}L),$$

where

(10.12) $$L_{AB} = L^C{}_{ABC}, \quad L = L_A{}^A.$$

However, from (10.9)

(10.13) $$L_{AB} = -2k_{AB} + \alpha_{AB}k, \quad L = k\ (= k^A{}_A),$$

and then (10.11) may also be written as

(10.14) $$L_{DABC} = -8\alpha_{[A[B}(k_{C]D]} - \tfrac{3}{8}\alpha_{C]D]}k).$$

It may be noted that the relation

(10.15) $$K_{A[B}K_{C]D} = -2\alpha_{[A[B}(k_{C]D]} - \tfrac{1}{4}\alpha_{C]D]}k)$$

is implied by (10.9), (10.14). It may be verified by reference to the definition of K_{AB} in terms of the Riemann tensor, but the work involved is very tedious.

The necessary and sufficient condition for rotor differentiation to be commutative is evidently that k_{AB} be a zero rotor, i.e.

(10.16) $$K_A{}^B K_B{}^C = KK_A{}^C.$$

In a V_0 the scalar R vanishes, so that *in a* V_0 *the condition for rotor differentiation to the commutative is that the square of the matrix* $\mathbf{K}$ *be zero* [6]. (See Section 15.)

11. Relation to the spin curvature

Since the relations of the Riemann tensor to P_{ABkl} on the one hand and to the curvature spinor $P_{\mu\nu kl}$ on the other are already known, the direct relation between P_{ABkl} and $P_{\mu\nu kl}$ is at hand. Taking the contraction $\Gamma^\lambda{}_{\lambda k}$ of the linear spinor connection to be zero (as in S) one has by inspection of (5.14) and S (2.37)

$$S_{kl\mu\nu}P^{\mu\nu}{}_{mn} = -\tfrac{1}{4}e^{ABC}\tau_{Akl}P_{BCmn}.$$

Transvecting throughout with $e_{DGF}\tau^{Dkl}$ it follows that

(11.1) $$P_{ABmn} = -\tfrac{1}{2}e_{ABC}\lambda^{C\mu\nu}P_{\mu\nu mn},$$

[6] Whenever the array of the components of a mixed tensor are thought of as the elements of a square matrix it may be symbolized simply by its kernel symbol in bold face type.

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where I (9.5) has been used. (11.1) may be inverted to give

(11.2) $$P_{\mu\nu mn} = -\tfrac{1}{4}e^{ABC}\lambda_{A\mu\nu}P_{BCmn}.$$

Many of the relations derived in Section 5 may then be shown to be more or less trivially equivalent to relations which appear in Section 5 of S. For example (5.15) is equivalent to S (5.2), (5.17) is equivalent to S (5.5), and so on. Finally, it may be noted that

(11.3) $$K^{AB} = -\tfrac{1}{2}\lambda^{A\mu\nu}\tau^{Bkl}P_{\mu\nu kl}.$$

and

(11.4) $$\mathring{K}^{AB} = -\lambda^{A\mu\nu}\lambda^{B\rho\sigma}\Gamma_{\mu\nu\rho\sigma},$$

where

(11.5) $$\Gamma_{\mu\nu\rho\sigma} = \tfrac{1}{2}S^{kl}{}_{\mu\nu}S^{mn}{}_{\rho\sigma}R_{klmn}.$$

$\Gamma_{\mu\nu\rho\sigma}$ is easily shown to be completely symmetric, and it is this spinor which forms the starting point of one method of classifying Weyl tensors (e.g. Jordan, Ehlers and Sachs, 1961).

12. Geodesic frames

At any point P_0 of V one can introduce coordinates such that the g_{kl} take their Minkowski values $\mathring{g}_{kl}$. If at the same time one reduces, — by means of a suitable r-transformation, — the α_{AB} to their special forms $\mathring{\alpha}_{AB} = \text{diag}\,(1, 1, 1)$, then the τ_{Akl} can be chosen to have at P_0 the values $\mathring{\tau}_{Akl}$ provided by the standard representation of I, Section 10. Let the components of the linear connections now have the values $\mathring{\Gamma}^m{}_{kl}$ and $\mathring{\Gamma}^B{}_{Ak}$. A transformation of coordinates

(12.1) $$x^{s'} = \delta^{s'}{}_s(x^s + \mathring{\Gamma}^s{}_{mn}x^m x^n + \cdots)$$

leaves the values of the components of the metric tensor unchanged at P_0, but it causes all the Christoffel symbols to take the value 0 at P_0. In other words, geodesic coordinates have been introduced in w-space. In an analogous way one may carry out the r-transformation

(12.2) $$\Lambda^{A'}{}_A = \delta^{A'}{}_A + \delta^{A'}{}_B\mathring{\Gamma}^B{}_{Ak}x^k + \cdots.$$

At P_0 the α_{AB} then retain their values $\mathring{\alpha}_{AB}$, but the components of the linear r-connection are reduced to zero. The situation is now that the g_{kl} and τ_{Akl} (and their algebraic concomitants) have their standard values not only at P_0 but also throughout a sufficiently small neighbourhood of P_0. A coordinate system and r-frame which are such that these conditions obtain will be said together to constitute a *geodesic frame*.

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13. The Maxwell field

(a) The "simplest" object one can form by differentiation from a given rotor field ϕ_A is a w-vector, j^k say, according to

$$\tfrac{1}{2}\tau^{Akl}\phi_{A;l} = j^k. \tag{13.1}$$

If j^k is a given vector field one may therefore look upon (13.1) as the simplest differential equation involving a rotor which one can write down. If F^{kl} is the (self-dual) tensor equivalent of ϕ_A, (13.1) reads

$$F^{kl}{}_{;l} = j^k, \tag{13.2}$$

on account of I (2.7). If j^k is prescribed to be real and F^{kl} is generated by a real bi-vector according to (2.4), (13.2) splits up into the pair of real equations

$$f^{kl}{}_{;l} = j^k, \quad f_{[kl,m]} = 0, \tag{13.3}$$

and these are just Maxwell's equations in vacuo.

If one introduces a geodesic frame at a point then the relation there between ϕ_A and the field intensities $\boldsymbol{E}$, $\boldsymbol{H}$ is

$$(\phi_1, \phi_2, \phi_3) \rightarrow 2(\boldsymbol{H}-i\boldsymbol{E}). \tag{13.4}$$

(b) In (13.2) the use of Lorentz Heaviside units was understood. In what follows, however, units are chosen such that the numerical values of the velocity of light c and Newton's constant G are

$$c = 1, \quad G = 1, \tag{13.5}$$

whilst Gaussian units are used for electromagnetic quantities. Then the Einstein-Maxwell equations are

$$R_{kl} = -2T_{kl}, \tag{13.6}$$

where

$$T_{kl} = \tfrac{1}{4}g_{kl}f_{st}f^{st} - f_{ks}f_l{}^s. \tag{13.7}$$

Because of I (2.4), (2.7) and (5.4), (13.7) becomes

$$T_{kl} = -\tfrac{1}{8}\hat{T}_{kl\dot{A}B}\phi^{\dot{A}}\phi^B. \tag{13.8}$$

On the other hand one has from (5.25) that

$$R_{kl} = \tfrac{1}{4}\hat{T}_{kl\dot{A}B}\Gamma^{\dot{A}B}, \tag{13.9}$$

since the scalar curvature vanishes here. With (13.8) and (13.9), (13.6) now becomes

$$\Gamma^{\dot{A}B} = \phi^{\dot{A}}\phi^B, \tag{13.10}$$

where (6.8) has been used.

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(c) One can now go on to discuss the Rainich Problem, (e.g. Misner and Wheeler 1957) in the present setting. However, the details of this would be rather similar to what has been done for instance by Witten (1962) and Peres (1962) and will therefore be omitted, apart from a brief mention of a point concerning the Rainich uector r_k which is defined as

$$r_k = (R_{pq}R^{pq})^{-1}e_{klst}R^{sm;t}R^l{}_m. \tag{13.11}$$

Here R_{kl} may be expressed in terms of ϕ_A by means of (13.9). The resulting expression may be greatly simplified by means of I (6.7), (6.8) and (2.10). One ends up with

$$r_k = \text{Im}\,(e^{ABC}\tau_{Akl}\phi_B\phi_C{}^{;l}/\phi_D\phi^D) \tag{13.12}$$

as an expression for r_k in terms of the electromagnetic rotor, where the field has been assumed throughout not to be null, i.e. $\phi_A\phi^A \neq 0$.

The point is now that in an expression such as that on the right of (13.12) one still has the freedom of making an arbitrary (non-singular) r-transformation. If one is willing to sacrifice manifest covariance one may for instance choose $\Lambda^A{}_{A'}$ in such a way that $\phi_{J'} = \phi_{K'} = 0$, where I', J', K' are *fixed* indices and are a cyclic permutation of 1, 2, 3 (i.e. strictly speaking of I′, II′, III′). Since the invariant $\phi_D\phi^D = \alpha^{I'I'}\phi_{I'}\phi_{I'} \neq 0$, $\alpha^{I'I'}$ cannot be zero and one can further impose the condition on $\Lambda^A{}_{A'}$ that it be such as to make $\alpha^{I'I'} = 1$. Then, omitting primes, (13.12) becomes

$$r_k = 2\,\text{Im}\,\tau_{[Kk}{}^l\Gamma^I{}_{J]l}. \tag{13.13}$$

In this the rotor ϕ_A no longer appears explicitly. The particular given *field* ϕ_A has given way, as it were, to a particular representation of the τ_{Akl}.

(d) This last remark may be illustrated by a specific example. Since ϕ_A is non-null one may arrange the $\phi_{A'}$ to have the values (1, 0, 0). Taking $\phi_1 \neq 0$ without loss of generality, this can be achieved for instance by the transformation

$$\Lambda^{A'}{}_A = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13.14}$$

It suffices here to take the case of flat w-space, and to adopt as initial representation the standard representation of I Section 10. Then

$$\tau^{1'}{}_{kl} \overset{*}{=} 2F_{kl}, \quad \tau^{2'}{}_{kl} \overset{*}{=} \tau^2{}_{kl}, \quad \tau^{3'}{}_{kl} \overset{*}{=} \tau^3{}_{kl}, \tag{13.15}$$

the asterisk serving as a reminder that the equations refer to specific representations. The electromangetic field rotor $\phi_{A'}$ is everywhere constant, being simply the unit rotor (1, 0, 0), which by itself says no more than that the field is non-null. The full information concerning the *physical* field has simply been transferred here to $\tau^{1'}{}_{kl}$, that is to say, to the basic rotors

τ_{Akl} and their algebraic and differential concomitants. Thus (13.1) becomes

(13.16) $$-2j^k \overset{*}{=} \tau^{A'kl}\Gamma^{1'}{}_{A'l}.$$

Using (3.2) the circle of the argument may be completed by verifying *directly* that (13.16) is equivalent to (13.1).

In the case of a null-field one may proceed along analogous lines, but it is not necessary to spell out the details. It may, however, be noted that the remarks above are intimately related to those of *I* Section 10d.

14. Concerning some basic invariants

(a) From P_{ABkl} one can form two quadratic invariants. The first of these is

(14.1) $$P_{ABkl}P^{ABkl} = K_2 - \tfrac{1}{2}iK_4,$$

in the notation of Buchdahl (1960), where (5.6) and *I* (2.17) have been used. The second, more interesting, invariant S is

$$S = e^{klmn}P_{ABkl}P^{AB}{}_{mn} = e^{klmn}\eta^{stpq}R_{klst}R_{mnpq}.$$

Using (5.16) or S (3.1) this becomes

(14.2) $$S = K_4 + 2i(K_1 - 4K_2 + K_3).$$

Consider the invariant integral

(14.2) $$\int \mathbf{S}d^4x = \int (-g)^{\frac{1}{2}}Sd^4x = -\int \varepsilon^{klmn}P^A{}_{Bkl}P^B{}_{Amn}d^4x,$$

extended over some finite region. Under an infinitesimal variation of the linear connection which vanishes on the boundary of the region of integration one then has

$$\begin{aligned}\delta\mathbf{S} &= -2\int \varepsilon^{klmn}P^A{}_{Bkl}(\delta\Gamma^B{}_{An})_{;m}d^4x \\ &= 2\int \varepsilon^{klmn}P^A{}_{Bkl;m}\delta\Gamma^B{}_{An}d^4x,\end{aligned}$$

on integrating by parts, and because of (5.26) it then follows that

(14.3) $$\delta\int \mathbf{S}d^4x = 0.$$

This provides an elegant proof that the functional derivatives with respect to g_{kl} of the invariants K_4 and $K_1 - 4K_2 + K_3$ vanish identically. The result (14.3) will lead one to expect that $\mathbf{S}$ can be written as a divergence. In fact it turns out that

$$\mathbf{S} = \mathbf{U}^n{}_{,n},$$

where

(14.4) $$\mathbf{U}^n = 2\varepsilon^{klmn}\Gamma^A{}_{Bm}(P^B{}_{Akl} - \tfrac{2}{3}\Gamma^C{}_{Ak}\Gamma^B{}_{Cl}).$$

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(b) Invariants formed from the rotor $\overset{0}{K}{}^{AB}$ are of considerable interest. Since this alone, rather than K^{AB}, is to be investigated the superscript 0 will henceforth be omitted, i.e. from (6.3)

$$K^{AB} = -\tfrac{1}{2}\tau^{Akl}\tau^{Bmn}C_{klmn}. \tag{14.5}$$

First, there is the quadratic invariant

$$K^{AB}K_{AB} = \eta^{kl}{}_{pq}\eta^{mn}{}_{st}C_{klmn}C^{pqst}.$$

Recalling that C_{klmn} is entirely trace-free one easily obtains the result

$$K^{AB}K_{AB} = \operatorname{tr}\boldsymbol{K}^2 = 2(A+iA^*) = 2\tilde{A}, \tag{14.6}$$

say, where

$$A = C_{klmn}C^{klmn}, \quad A^* = -\tfrac{1}{2}e^{pqst}C_{klpq}C^{kl}{}_{st}. \tag{14.7}$$

In an entirely similar fashion one next gets the cubic invariant

$$K^A{}_BK^B{}_CK^C{}_A = \operatorname{tr}\boldsymbol{K}^3 = -4(B+iB^*) = -4\tilde{B}, \tag{14.8}$$

say, where

$$B = C_{klmn}C^{klpq}C^{mn}{}_{pq}, \quad B^* = -\tfrac{1}{2}e^{pqst}e^{klmn}C_{klpq}C_{mnst}. \tag{14.9}$$

In a V_R there is no distinction between R_{klmn} and C_{klmn}, and A, A^*, B, B^*, are the only algebraically independent invariants of the Riemann tensor, i.e. any other invariant is a function of these four basic invariants. (In special cases even these need not all be distinct from one another.)

The determinant of $K^A{}_B$ is simply

$$\det K^A{}_B = \delta^{ABC}_{DEF}K^D{}_AK^E{}_BK^F{}_C = -\tfrac{4}{3}\tilde{B}. \tag{14.10}$$

With scarcely more labour one infers that the secular determinant of $K^A{}_B$ is given by

$$-\det(K^A{}_B-\lambda\delta^A{}_B) = \lambda^3-\tilde{A}\lambda+\tfrac{4}{3}\tilde{B}. \tag{14.11}$$

Its zeros are the values of λ for which one or more eigenrotors U^A of $K^B{}_A$ exist, i.e. rotor fields such that

$$K^B{}_AU_B = \lambda U_A. \tag{14.12}$$

The "eigenvalues" of λ of $K^B{}_A$ are of course scalar fields.

15. Remarks concerning the rotor K^{AB}

At the end of Section 10 it was found that commutativity of rotor differentiation in a V_0, and therefore a fortiori in a V_R, required that the square of the matrix $\boldsymbol{K}^B{}_A$ be zero. More generally one may enquire into the properties of the matrix [7] $\boldsymbol{K}$, viz. one may investigate canonical forms into which it may be cast. Essentially such an investigation would form a basis for the Petrov classification (e.g. Petrov 1962) of spacetimes, and since this has been fully covered elsewhere (see, for example, the detailed account of Kundt (1958)) it would be out of place to consider the problem here in any detail. A few remarks, appropriate to the present context, will therefore suffice.

To begin with, since α_{AB} need not have the "standard form" diag (1, 1, 1) one has to distinguish clearly between the rotor $K^A{}_B$ and its concomitant K^{AB}. It is true that in any frame each of these can be regarded as defining a square matrix. But if one does so $K^A{}_B$ will not in general be symmetric. Further, reduction to the canonical form is here to be achieved by suitable r-transformations. In the case of K^{AB} this does not amount to a similarity transformation, whereas in the case of $K^A{}_B$ it does. (That the matrices usually considered are symmetric, yet are subjected to similarity transformations implies, — when translated into the present context, — an initial choice of the standard form of the r-metric). As a matter of fact, there is nothing to prevent one from diagonalizing K^{AB}, indeed reducing it to the form diag (1, 1, 1), save in the exceptional case when det $K^{AB} = 0$, i.e. when $\tilde{B} = 0$, in view of (14.10). This amounts to shifting the burden of the complexities of K^{AB} to the metric α_{AB}, or what comes to the same thing, to the τ_{Akl}, as in the analogous situation of Section 13b, c. (See the example below.) In short, one has more freedom here to manipulate since one can always consider arbitrary non-singular r-transformations, rather than merely orthogonal transformations.

Perhaps the most straightforward procedure here is to classify the matrix $\boldsymbol{K}$ according to the form of its minimal polynomial. Indeed, suppose $\boldsymbol{K}$ to be represented, if possible, in terms of two mutually orthogonal rotors U^A and V^A,

$$U_A V^A = 0, \tag{15.1}$$

in the following way:

$$K^{AB} = U^A U^B + V^A V^B - \tfrac{1}{3}\alpha^{AB} s, \tag{15.2}$$

where

$$s = u + v, \quad u = U_A U^A, \quad v = V_A V^A. \tag{15.3}$$

$K^B{}_A$ is evidently trace-free, and K^{AB} is symmetric. One confirms at once that U_A and V_A, for the moment supposed linearly independent and non-null, are eigenrotors of $K^B{}_A$, belonging to the eigenvalues

[7] Recall the remark at the beginning of Section 14b.

(15.4) $$\lambda_u = \tfrac{1}{3}(2u-v), \quad \lambda_v = \tfrac{1}{3}(2v-u),$$

respectively. If one defines a rotor

(15.5) $$W^A = e^{ABC}U_B V_C$$

then, keeping (15.1) in mind,

(15.6) $$w = W_A W^A = uv,$$

and W^A is seen also to be an eigenrotor of $K^B{}_A$ belonging to the eigenvalue

(15.7) $$\lambda_w = -\tfrac{1}{3}s.$$

Of course

(15.8) $$\lambda_u+\lambda_v+\lambda_w = 0.$$

From (15.2) one infers quite generally that

(15.9) $$(\boldsymbol{K}^2)^B{}_A = -\lambda_v U^B U_A - \lambda_u V^B V_A + \tfrac{1}{9}s^2\delta^B{}_A,$$

and

(15.10) $$(\boldsymbol{K}^3)^B{}_A = (\lambda_u{}^2+\lambda_u\lambda_v+\lambda_v{}^2)(U^B U_A+V^B V_A)-\tfrac{1}{27}s^3\delta^B{}_A.$$

One can now distinguish various cases as follows:

(i) U_A, V_A *general.*

By simple elimination it follows from (15.2), (15.9) and (15.10) that

(15.11) $$(\boldsymbol{K}-\lambda_u)(\boldsymbol{K}-\lambda_v)(\boldsymbol{K}-\lambda_w) = 0,$$

where the unit matrices multiplying λ_u etc. are left understood. (15.10) is an example of the operation of the Cayley-Hamilton theorem, i.e. $\boldsymbol{K}$ satisfies its own characteristic equation (14.11)

(15.12) $$\lambda^3-\tfrac{1}{3}(u^2-uv+v^2)\lambda-\tfrac{1}{27}(u+v)(2u-v)(u-2v) = 0.$$

Comparison of (15.11) with (14.11) gives the relations between u and v on the one hand, and $\tilde{A}$ and $\tilde{B}$ on the other. At any rate, (15.11) shows that for general rotors U_A, V_A (i.e. excluding the special cases to be enumerated below) $\boldsymbol{K}$ is of Type I. The eigenvalues of $\boldsymbol{K}$ are distinct, and its eigenrotors are non-null and mutually orthogonal.

(ii) U_A, V_A *have equal non-zero norms.*

The equality of $u = v$ entails that $\lambda_u = \lambda_v$ and (15.9) shows that the minimal equation is

(15.13) $$(\boldsymbol{K}-\lambda_w)(\boldsymbol{K}+\tfrac{1}{2}\lambda_w) = 0,$$

i.e. $\boldsymbol{K}$ is of Type $\mathrm{I_d}$.

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(iii) *U_A, V_A are zero rotors.*
Here $\boldsymbol{K}$ vanishes and so is of Type $\mathrm{I_{dd}}$.

(iv) *V_A is a null rotor.*
Now $v = 0$ and so $\lambda_v = \lambda_w$, and at the same time V_A and W_A are linearly dependent. The minimal equation is in this case

$$(15.14) \qquad (\boldsymbol{K}-2\lambda_v)(\boldsymbol{K}+\lambda_v)^2 = 0,$$

so that $\boldsymbol{K}$ is of Type II.

(v) *U_A is a null rotor and V_A a zero rotor.*
Here

$$(15.15) \qquad \boldsymbol{K}^2 = 0,$$

so that $\boldsymbol{K}$ is of Type $\mathrm{II_d}$. This case is the one to which the remark at the beginning of this section refers.

(vi) *A special case, with $U_A V^A \neq 0$.*
So long as U_A and V^A are mutually orthogonal $K^B{}_A$, in the form (15.2), cannot have the minimal equation

$$(15.16) \qquad \boldsymbol{K}^3 = 0.$$

Let the condition of orthogonality be relaxed and lay down that $v = -u$ and $U_A V^A = iu$. Then (15.16) obtains, and $K^B{}_A$ is of Type III. U_A and V_A are no longer eigenrotors of $K^B{}_A$. On the other hand one has the null eigenrotor $U_A + iV_A$.

As pointed out earlier, although the initial choice $\alpha_{AB} = \mathrm{diag}\ (1, 1, 1)$ of the λ-metric erases the initial distinction between K^{AB} on the one hand and $K^A{}_B$ on the other, so that both may then be exhibited as a symmetric matrix $\boldsymbol{K}$, an arbitrary r-transformation $\Lambda^{A'}{}_A$ ($\equiv \boldsymbol{S}$, say) gives rise to the transforms $\boldsymbol{SKS}^{tr}$ and $\boldsymbol{SKS}^{-1}$ respectively, and these are in general distinct. They coincide if and only if $\boldsymbol{S}$ is an orthogonal transformation $\boldsymbol{O}$. One can choose $\boldsymbol{O}$ such that $\boldsymbol{OKO}^{-1}$ has some particularly simple standard form $'\boldsymbol{K}_P$, which is of course still symmetric. In fact $'\boldsymbol{K}_P$ may be arranged to take the Petrov form $M+iN$ as given by Petrov (1962). Alternatively $\boldsymbol{S}$ may be chosen such that $\boldsymbol{SKS}^{-1}$ becomes a Jordan normal matrix $'\boldsymbol{K}_J$

$$\begin{pmatrix} \lambda_u & \varepsilon' & 0 \\ 0 & \lambda_v & \varepsilon'' \\ 0 & 0 & \lambda_w \end{pmatrix},$$

where λ_u, λ_v, λ_w are the eigenvalues of $\boldsymbol{K}$ as enumerated in the six cases above, whilst ε', ε'' have the values given in the following Table:

Type	ε'	ε''
I, I_d, I_{dd}	0	0
II, II_d	0	1
III	1	1.

(b) Here, as in Section 13d it may be apposite to give one, albeit rather trivial example. If a V_R is static and spherically symmetric the metric may be taken as

$$\begin{aligned} ds^2 &= -e^{\lambda(r)}dr^2 - e^{\mu(r)}(d\theta^2+\sin^2\theta d\phi^2)+e^{\nu(r)}dr^2 \\ &= -\alpha^2 dr^2 - \beta^2(d^2\theta+\sin^2\theta d\phi^2)+\gamma^2 dt^2, \end{aligned} \tag{15.17}$$

say. By explicit calculation one finds that every component of C_{klmn} has

$$\chi = \tfrac{1}{3}\{e^{-\lambda}[(\nu''-\mu'')+\tfrac{1}{2}(\nu'-\lambda')(\nu'-\mu')]-2e^{-\mu}\} \tag{15.18}$$

as a factor; e.g. $C_{1212} = -\frac{1}{4}e^{\lambda+\mu}\chi$. The basic rotors

$$\tau_1 = \begin{pmatrix} 0 & 0 & -\alpha\beta\sin\theta & 0 \\ 0 & 0 & 0 & i\beta\gamma \\ \alpha\beta\sin\theta & 0 & 0 & 0 \\ 0 & 0 & -i\beta\gamma & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & \alpha\beta & 0 & 0 \\ -\alpha\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & i\beta\gamma\sin\theta \\ 0 & 0 & -i\beta\gamma\sin\theta & 0 \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} 0 & 0 & 0 & i\alpha\gamma \\ 0 & 0 & \beta^2\sin\theta & 0 \\ 0 & -\beta^2\sin\theta & 0 & 0 \\ -i\alpha\gamma & 0 & 0 & 0 \end{pmatrix} \tag{15.19}$$

satisfy (2.1), if $\alpha_{AB} = \text{diag}\,(1, 1, 1)$. They reduce to the standard representatives I (10.6) when $\alpha = \beta = \gamma = 1$, $\theta = \pi/2$, provided the rotor indices be cyclically permuted. The K^{AB} may now be calculated, and one finds

$$K^{AB} = \chi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{15.20}$$

As a check, direct calculation of the invariants $\tilde{A}$ and $\tilde{B}$ yields

$$A = 3\chi^2, \quad A^* = 0, \quad B = -\tfrac{3}{2}\chi^3, \quad B^* = 0. \tag{15.21}$$

(14.11) then gives the secular equation

$$(\lambda-\chi)^2(\lambda+2\chi) = 0,$$

in harmony with (15.10). The metric is of Type I_d, and

$$U^A = (3\chi)^{\frac{1}{2}}(1, 0, 0), \quad V^A = (3\chi)^{\frac{1}{2}}(0, 1, 0), \quad W^A = 3\chi(0, 0, 1) \tag{15.22}$$

is a set of mutually orthogonal eigenrotors of K_{AB}. Of course any linear sum of the first two of these is also an eigenrotor.

As observed previously, it is possible to diagonalize K^{AB} whenever $\det K^{AB} \neq 0$, i.e. the diagonalization can be carried out when K^{AB} is of Type I, $\mathrm{I_d}$, or II. Here K^{AB} is already diagonal. However, it may be reduced to the form

$$K^{A'B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{15.23}$$

by means of the r-transformation

$$\Lambda^{A'}{}_{A} = \chi^{-\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i/\sqrt{2} \end{pmatrix}. \tag{15.24}$$

Then

$$\tau_{1'} = \chi^{\frac{1}{2}}\tau_1, \quad \tau_{2'} = \chi^{\frac{1}{2}}\tau_2, \quad \tau_{3'} = -i(2\chi)^{\frac{1}{2}}\tau_3, \tag{15.25}$$

whilst $\alpha_{A'B'}$ is of course just given by the right hand member of (15.20). This, then, illustrates the procedure of choosing a representation of the τ_{Akl} such that the symmetric contravariant rotor K^{AB} takes the form of the right hand member of (15.23). Any representation having this property might be called canonical, though this terminology is then restricted to situations in which $\det K^{AB} \neq 0$. To obviate this difficulty one might therefore more generally call a representation canonical if, in this, the components of the rotors in (15.2) have certain prescribed values, consistent with the generic form of these rotors enumerated previously. If K^{AB} is of Type $\mathrm{II_d}$, for instance, one might prescribe $U^A = (0, 1, i)$, $V^A = (0, 0, 0)$. In certain cases this definition implies little or no restriction on the representation; e.g. if K^{AB} is of Type $\mathrm{I_{dd}}$ every representation is canonical.

16. Concluding remark

The present investigation may be broken off at this point, for it has been adequately demonstrated how, starting from the correspondence between self-dual w-tensors and the elements of a three-dimensional complex linear vector space, a general calculus can be built up which is covariant under arbitrary independent transformations in every space contemplated within the calculus. There is, as might be expected, much formal analogy between this work and the spinor calculus of Infeld and van der Waerden, though each can be developed entirely independently of the other. At any rate, the machinery which is now available allows one to deal in a unified and straightforward manner with theories which are characterized by the implicit or explicit occurrence of bi-vector fields as basic elements.

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