

A VARIATIONAL-LIKE INEQUALITY PROBLEM

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Given a closed and convex set K in \mathbb{R}^n and two continuous maps $F: K \rightarrow \mathbb{R}^n$ and $\eta: K \times K \rightarrow \mathbb{R}^n$, the problem considered here is to find $\bar{x} \in K$ such that

$$(\forall z \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0.$$

We call it a variational-like inequality problem, and develop a theory for the existence of a solution. We also show the relationship between the variational-like inequality problem and some mathematical programming problems.

1. INTRODUCTION

Given a closed and convex set K in \mathbb{R}^n and a continuous map $F: K \rightarrow \mathbb{R}^n$, the problem of finding $\bar{x} \in K$ such that

$$(1.1) \quad (\forall x \in K) \quad \langle F(\bar{x}), x - \bar{x} \rangle \geq 0,$$

is called a variational inequality problem [8]. In recent years, various extensions of (1.1) have been proposed and analysed [3]. We introduce an extension of (1.1) as follows: Let K be a closed and convex set in \mathbb{R}^n . Given two continuous maps $F: K \rightarrow \mathbb{R}^n$ and $\eta: K \times K \rightarrow \mathbb{R}^n$, find $\bar{x} \in K$ such that

$$(1.2) \quad (\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0.$$

We call it a variational-like inequality problem. If $\eta(x, \bar{x}) = x - \bar{x}$, then (1.2) reduces to (1.1).

In this paper, we establish some existence theorems for (1.2) under various conditions on the maps F and η . We also demonstrate the relationship between the variational-like inequality problem (1.2) and some mathematical programming problems. In a subsequent paper, the results of the present paper will be utilised to establish existence theorems for (1.2) where the underlying map F is η -monotone.

Received 5 May 1988

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2. NOTATION AND DEFINITIONS

The real n -dimensional linear space of column vectors $x = (x_1, \dots, x_n)^T$ is denoted by \mathbb{R}^n . For any $x, y \in \mathbb{R}^n$, we denote the usual inner product in \mathbb{R}^n by $\langle x, y \rangle = y^T x$, while $\|x\|$ is any norm of x . For closed convex cone S of \mathbb{R}^n , S^* denotes the dual cone of S , and is given by

$$S^* = \{y \in \mathbb{R}^n : (\forall x \in S) \langle y, x \rangle \geq 0\}.$$

Given a multivalued map $V : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, V is said to be upper semicontinuous [2, p.111] if $\{x^n\}$ converging to x , and $\{y^n\}$, with $y^n \in V(x^n)$, converging to y , implies $y \in V(x)$.

We introduce a generalisation of monotone functions. The map $F : K \rightarrow \mathbb{R}^n$ is said to be monotone over K if

$$(\forall x, y \in K) \quad \langle F(y) - F(x), y - x \rangle \geq 0.$$

F is said to be η -monotone on K if there exists a continuous map $\eta : K \times K \rightarrow \mathbb{R}^n$ such that

$$(2.1) \quad (\forall x, y \in K) \quad \langle F(y), \eta(x, y) \rangle + \langle F(x), \eta(y, x) \rangle \leq 0.$$

Note that this definition reduces to the definition of monotone functions if $\eta(y, x) = y - x$.

F is said to be strictly η -monotone over K if the equality holds in (2.1) only when $y = x$.

Hanson [4] and Mond and Hanson [9] have defined a certain class of differentiable functions, now known as invex functions [1], which contains as a subclass the class of differentiable convex functions. We follow the nomenclature and definitions given in [1, 4].

Let $\psi : K \rightarrow \mathbb{R}^n$ be differentiable. Then ψ is η -convex on K if there exists a continuous map $\eta : K \times K \rightarrow \mathbb{R}^n$ such that

$$(2.2) \quad (\forall x, y \in K) \quad \psi(y) - \psi(x) \geq \langle \nabla \psi(x), \eta(y, x) \rangle.$$

It is known that if ψ is convex on K , then $\nabla \psi$ is monotone on K . In the same vein, we have here that $\nabla \psi$ is η -monotone whenever ψ is η -convex.

3. EXISTENCE THEOREMS

THEOREM 3.1. *Let K be a compact and convex set in \mathbb{R}^n , and let $F : K \rightarrow \mathbb{R}^n$ and $\eta : K \times K \rightarrow \mathbb{R}^n$ be two continuous maps. Suppose that*

$$(3.1) \quad (\forall x \in K) \quad \langle F(x), \eta(x, x) \rangle = 0,$$

(3.2) *for each fixed $x \in K$, the function $\langle F(x), \eta(y, x) \rangle$ is quasiconvex in $y \in K$.*

Then

$$(\exists \bar{x} \in K)(\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0.$$

PROOF: For each $x \in K$, let

$$V(x) = \{s \in K : \langle F(x), \eta(s, x) \rangle = \min_{v \in K} \langle F(x), \eta(v, x) \rangle\}.$$

Since K is compact and $\langle F(x), \eta(v, x) \rangle$ is quasiconvex in v , $V(x)$ is a nonempty, closed and convex subset of K . It is also easy to see that the multivalued map $V: K \rightarrow 2^K$ is upper semicontinuous. Now invoking the Kakutani fixed-point theorem [6], we get $\bar{x} \in V(\bar{x})$. Consequently, for all $x \in K$,

$$\langle F(\bar{x}), \eta(x, \bar{x}) \rangle \geq \langle F(\bar{x}), \eta(\bar{x}, \bar{x}) \rangle = 0.$$

This completes the proof of the theorem. ■

The following result is an extension of Theorem 3.1 to noncompact sets. Consider the set $K_r \subseteq K$ defined by $K_r = \{x \in K \text{ and } \|x\| \leq r\}$ for real $r > 0$. There always exists an $r_0 > 0$ such that K_r is nonempty whenever $r \geq r_0$. From now on we always assume that r satisfies this requirement and K_r is nonempty.

We state this condition:

Condition 3.2. Let K be a closed and convex set in \mathbb{R}^n . Let $F: K \rightarrow \mathbb{R}^n$ and $\eta: K \times K \rightarrow \mathbb{R}^n$ be two continuous maps such that

- (i) $(\forall x \in K) \langle F(x), \eta(x, x) \rangle = 0$, and
- (ii) for each fixed $x \in K$, the function $\langle F(x), \eta(y, x) \rangle$ is convex in $y \in K$.

We notice that K_r is compact and convex, and hence, by the previous theorem there exists at least one $x_r \in K_r$ such that

$$(3.3) \quad (\forall x \in K_r) \quad \langle F(x_r), \eta(x, x_r) \rangle \geq 0$$

whenever Condition 3.2 is satisfied.

THEOREM 3.3. Let K, F and η be such that Condition 3.2 is satisfied. A necessary and sufficient condition that there exists a solution to (1.2) is that there exists an $r > 0$ such that a solution $x_r \in K_r$ of (3.3) satisfies the estimate

$$\|x_r\| < r.$$

PROOF: It is clear that if there exists a solution \bar{x} to (1.2), then \bar{x} is a solution to (3.3) whenever $\|\bar{x}\| < r$. Suppose now that $x_r \in K_r$ is a solution of (3.3) and that $\|x_r\| < r$. Given $x \in K$, we can choose $0 < \lambda < 1$ sufficiently small so that

$w = \lambda x + (1 - \lambda)x_r \in K_r$. Consequently, by the convexity of $\langle F(x_r), \eta(y, x_r) \rangle$, $x_r \in K_r \subseteq K$ satisfies

$$\begin{aligned} 0 &\leq \langle F(x_r), \eta(w, x_r) \rangle \\ &\leq \lambda \langle F(x_r), \eta(x, x_r) \rangle + (1 - \lambda) \langle F(x_r), \eta(x_r, x_r) \rangle \\ &= \lambda \langle F(x_r), \eta(x, x_r) \rangle \end{aligned}$$

for all $x \in K$, which implies that x_r is a solution to (1.2). ■

We now use Theorem 3.3 to give three other important sufficient conditions for the existence of a solution to (1.2).

THEOREM 3.4. *Let K , F and η be such that Condition 3.2 is satisfied. Then the variational-like inequality problem (1.2) has a solution under each of the following conditions:*

(i) *there is a $u \in K$ and a scalar $r > \|u\|$ such that*

$$(3.4) \quad \langle F(x), \eta(u, x) \rangle \leq 0 \text{ for all } x \text{ with } \|x\| = r;$$

(ii) *for some constant $r > 0$, and for each $x \in K$ with $\|x\| = r$, there is a $u \in K$ with $\|u\| < r$ and $\langle F(x), \eta(u, x) \rangle \leq 0$;*

(iii) *there exists a nonempty, compact and convex subset C of K with the property that, for every $x \in K \setminus C$, there exists a $u \in C$ such that*

$$(3.5) \quad \langle F(x), \eta(u, x) \rangle < 0.$$

PROOF: (i) Suppose that $x_r \in K_r$ is a solution to (3.3) (which certainly has solutions). Then

$$(3.6) \quad (\forall x \in K_r) \quad \langle F(x_r), \eta(x, x_r) \rangle \geq 0.$$

We distinguish two cases. If $\|x_r\| < r$, then Theorem (3.3) yields the desired result. If $\|x_r\| = r$, then it follows from (3.4) and (3.6) that $\langle F(x_r), \eta(u, x_r) \rangle = 0$. Now, let $x \in K$, and choose $0 < \lambda < 1$ small enough so that $w = \lambda x + (1 - \lambda)u$ lies in K_r . Then, by the convexity of $\langle F(x_r), \eta(y, x_r) \rangle$,

$$\begin{aligned} 0 &\leq \langle F(x_r), \eta(w, x_r) \rangle \\ &\leq \lambda \langle F(x_r), \eta(x, x_r) \rangle + (1 - \lambda) \langle F(x_r), \eta(u, x_r) \rangle \\ &= \lambda \langle F(x_r), \eta(x, x_r) \rangle \end{aligned}$$

and consequently, x_r is a solution to (1.2).

(ii) This is little bit more general than the result given in (i) and can be easily proved by arguments similar to that used in the proof of (i).

(iii) Since C is a compact, we can find an $r > 0$ such that $\|x\| < r$ for all $x \in C$. Now, let $x_r \in K_r$ be a solution of (3.3). Then

$$(3.7) \quad \langle F(x_r), \eta(x, x_r) \rangle \geq 0$$

for all $x \in C$ since $C \subseteq K_r$. If $x_r \notin C$, then it follows from (3.5) that there exists $u \in C$ such that $\langle F(x_r), \eta(u, x_r) \rangle < 0$, which contradicts (3.7). Therefore, $x_r \in C$, and consequently, $\|x_r\| < r$. From Theorem 3.3, it follows that x_r solves (1.2). ■

Remark 3.5. From definition of an η -convex function it does not necessarily follow that $\eta(x, x) = 0$, but from examples given in [4, p.547] and the form of $\eta(x, y)$ given in [1, p.2], it follows that $\eta(x, x) = 0$.

In this contest, we mention that the relation $\langle F(x), \eta(x, x) \rangle = 0$ in Theorem 3.1 and Condition 3.2 can be replaced by $\eta(x, x) = 0$ which is, no doubt, a stronger assumption.

Remark 3.6. If we take $\eta(x, y) = x - y$, then the above theorems yield known existence results for the variational inequality (1.1). For example, our Theorems 3.1 and 3.3 reduce to Theorems 3.1 and 4.2, respectively, in [8]. Theorems 3.4 (i) and (ii) reduce to yield the results of Theorems 2.3 and 2.4, respectively, of More' [10].

If $F: K \rightarrow \mathbb{R}^n$ and $\eta: K \times K \rightarrow \mathbb{R}^n$ are continuous on the closed and convex set K , and if, for some $u \in K$,

$$(3.8) \quad \lim_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \frac{\langle F(x), \eta(u, x) \rangle}{\|x\|} = -\infty$$

then it is easily seen that there exists $r > \|u\|$ such that $\langle F(x), \eta(u, x) \rangle < 0$ for all $x \in K$ with $\|x\| = r$. This observation leads us to the following existence theorem.

THEOREM 3.8. *Let K, F and η be such that Condition 3.2 is satisfied. If, for some $u \in K$, (3.8) holds, then there exists a solution to (1.2).*

PROOF: The result follows from Theorem 3.4. ■

Generally, the solution to the variational-like inequality (1.2) may not be unique. There is however a very natural condition which ensures uniqueness.

THEOREM 3.9. *Let $F: K \rightarrow \mathbb{R}^n$ and $\eta: K \times K \rightarrow \mathbb{R}^n$ be continuous over the closed and convex set K . If F is strictly η -monotone over K , then there can exist at most one solution to the variational-like inequality problem (1.2).*

PROOF: In fact, if \bar{x} and \bar{z} are two distinct solutions to (1.2), then we have

$$(\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0,$$

$$(\forall x \in K) \quad \langle F(\bar{z}), \eta(x, \bar{z}) \rangle \geq 0.$$

Setting $x = \bar{z}$ in the first inequality, $x = \bar{x}$ in the second, and adding the two, we obtain

$$\langle F(\bar{x}), \eta(\bar{z}, \bar{x}) \rangle + \langle F(\bar{z}), \eta(\bar{x}, \bar{z}) \rangle \geq 0$$

which implies that $\bar{x} = \bar{z}$ by the strict η -monotonicity of F . ■

4. SOME PROBLEMS RELATED TO VARIATIONAL-LIKE INEQUALITIES

We touch lightly on some mathematical programming problems that are related to variational-like inequalities.

Convex programming. Consider the minimisation problem:

$$\min f(x) \text{ subject to } x \in K,$$

where K is a closed and convex set in \mathbb{R}^n and $f: K \rightarrow \mathbb{R}^n$ is continuously differentiable.

PROPOSITION 4.1. *Suppose that f is η -convex over K for some continuous map $\eta: K \times K \rightarrow \mathbb{R}^n$. If $\bar{x} \in K$ satisfies*

$$(4.1) \quad (\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0,$$

where

$$F(x) = \nabla f(x),$$

then

$$f(\bar{x}) = \min_{x \in K} f(x).$$

PROOF: By the η -convexity of f , we have

$$(\forall x \in K) \quad f(x) - f(\bar{x}) \geq \langle \nabla f(\bar{x}), \eta(x, \bar{x}) \rangle.$$

Now, setting $\nabla f(\bar{x}) = F(\bar{x})$, and using (4.1), we obtain $f(x) \geq f(\bar{x})$ for all $x \in K$. ■

Complementarity problem. Let K be a closed and convex cone in \mathbb{R}^n , and let $F: K \rightarrow \mathbb{R}^n$ be continuous. Then find $\bar{x} \in K$ such that

$$(4.2) \quad \bar{x} \in K, F(\bar{x}) \in K^*, \langle F(\bar{x}), \bar{x} \rangle = 0.$$

Problem (4.2) is the mathematical form for a variety of problems in optimisation theory, structural mechanics, lubrication theory, elasticity theory, economical equilibrium theory, etcetra, hence its importance. It is known (see, for example [7]) that (4.2) is equivalent to the problem of finding an $\bar{x} \in K$ such that

$$(\forall x \in K) \quad \langle F(\bar{x}), x - \bar{x} \rangle \geq 0.$$

But, if $\eta(x, y) = x - y$, then (1.2) reduces to this problem.

Implicit complementarity problem. Let K be a closed and convex cone in \mathbb{R}^n , and let $F, g: K \rightarrow \mathbb{R}^n$ be continuous. Then find $\bar{x} \in \mathbb{R}^n$ such that

$$(4.3) \quad g(\bar{x}) \in K, F(\bar{x}) \in K^*, \langle F(\bar{x}), g(\bar{x}) \rangle = 0.$$

The implicit complementarity problem arose in some special problems in stochastic control, and in the above form, it is considered by Isac [5]. We associate (4.3) with the following variational-like inequality problem: Find $\bar{x} \in K$ such that

$$(4.4) \quad (\forall x \in K) \quad \langle F(\bar{x}), x - g(\bar{x}) \rangle \geq 0.$$

This problem is a special case of (1.2) when K is a cone and $\eta(x, y) = x - g(y)$. Let y be any vector in K . If \bar{x} is a solution of (4.4), and $g(\bar{x}) \in K$, then replacing x by $y + g(\bar{x})$ in (4.4), we have $\langle F(\bar{x}), y \rangle \geq 0$. Therefore, $F(\bar{x}) \in K^*$. Now setting $x = 0$ in (4.4) we obtain $\langle F(\bar{x}), g(\bar{x}) \rangle \leq 0$, and consequently, \bar{x} is a solution of (4.3). It can also be seen that a solution \bar{x} of (4.3) is a solution of (4.4) if $\bar{x} \in K$.

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