Stellar Rotation
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On Rotationally Driven Meridional Flows in Stars

P. Garaud

DAMTP, Silver Street, University of Cambridge, CB39EW Cambridge

Abstract. The first entirely self-consistent 2D model of rotational mixing in a stellar radiative zone is presented. This nonlinear problem is solved numerically assuming axisymmetry of the system. The dynamical behaviour of a rotating star is found to be controlled by one parameter only, the ratio of the Eddington-Sweet timescale to the viscous timescale. In the quasi-steady state, the limit of slow rotation recovers Eddington-Sweet theory, whereas in the limit of rapid rotation, the system settles into a centrifugal equilibrium. The evolution of the dynamical structure of the star undergoing spin-down is then studied, and the relevance of these findings to observations of rotational mixing is discussed.

1. Introduction

The overwhelming evidence for the influence of rotation on stellar structure and evolution is highlighted by many authors in these proceedings, both from an observational point of view and from a modeling point of view. A better understanding of all the mixing processes related to rotation is the key to a better understanding of the stars themselves.

There are many difficulties in the construction of a model of a rotating star, some of which are still beyond our understanding. To what extent do rotationally driven instabilities develop non-linearly, and how much mixing of chemical elements and angular momentum do they provide? How is convection affected by rotation? We see evidence of differential rotation in the convective zones of the sun and of other stars (Donati, these proceedings), which is clearly driven by the influence of rotation on the convective eddies, but a quantitative description of this complex interaction is still only in its infancy. More crucially perhaps, there exists to this day no model of the effect of rotation on convective heat transport in a star.

The loss of spherical symmetry of the star is also a significant hurdle. It is not one, however, that cannot be overcome. Zahn (1992) proposed a model for turbulent mixing that effectively reduces the 2D problem to a 1D calculation through a prescription for angular-momentum mixing and for the driving of meridional flows. His angular-momentum mixing model has however been questioned recently (Gough & McIntyre, 1998). Regardless of the outcome of this controversy, the progression of computing power is such that it is now possible to make 2D (even 3D) models of stars. I have made use of this opportunity to construct the first 2D, entirely self-consistent model of laminar rotational mixing.

The generally accepted mechanism for rotational mixing in a stellar radiative zone is the following: rotation acts to deform surfaces of constant pressure

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but has only an indirect influence on surfaces of constant temperature. The resulting baroclinicity is unbalanced and drives large-scale meridional flows (or possibly turbulent mixing). A significant progress in the study of the laminar problem was made by Sweet (1950). Over the years, many refinements of this solution were published but none which could achieve a self-consistent, complete picture of the problem. In this paper I propose such a solution. Section 2 outlines the model and presents the governing equations. Two situations are then explored: in Section 3, I describe numerical solutions for a wide range of parameters in a steady-state situation; in Section 4 the steady-state assumption is dropped and first results of a spin-down calculation are presented. I conclude by discussing the implications of these simulations with regard to observations of Li depletion in stars.

2. The Governing Equations

Standard stellar evolution theory provides the instantaneous chemical and thermodynamical structure of a non-rotating star in hydrostatic equilibrium ($p_{\rm h}, T_{\rm h}, \rho_{\rm h}, \Phi_{\rm h}$), where p is the pressure, T the temperature, ρ the density and Φ is the gravitational potential). Let us assume that the star is rotating, and look at the resulting perturbations that rotation causes on the non-rotating hydrostatic structure. Baroclinic effects drive meridional flows, so that the total velocity field can be written as $\boldsymbol{u} = (u_r, u_\theta, u_\phi)$ in a spherical coordinate system. This flow satisfies the momentum and mass conservation equations. The thermodynamical structure of the star is also perturbed, and by restricting this study to stars far from rotational break-up I can assume that the perturbations (denoted by tildes) are small compared to the corresponding hydrostatic background quantities ($\tilde{p} \ll p_{\rm h}$, $\tilde{T} \ll T_{\rm h}$ ...). The resulting system of equations satisfied by the flow and thermodynamical perturbations is:

$$\rho_{\rm h}\frac{\partial \boldsymbol{u}}{\partial t} + \rho_{\rm h}\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\tilde{p} - \rho_{\rm h}\nabla\tilde{\Phi} - \tilde{\rho}\nabla\tilde{\Phi} + \rho_{\rm h}\ nu\nabla^2\boldsymbol{u} + \frac{1}{3}\rho_{\rm h}\nu\nabla(\nabla\cdot\boldsymbol{u})$$

$$\nabla^2\tilde{\Phi} = 4\pi G\tilde{\rho}\ ,\ \frac{\tilde{p}}{p_{\rm h}} = \frac{\tilde{\rho}}{\rho_{\rm h}} + \frac{\tilde{T}}{T_{\rm h}} \tag{2}$$

$$\nabla\cdot(\rho_{\rm h}\boldsymbol{u}) = 0\ ,\ \rho_{\rm h}T_{\rm h}\frac{\partial\tilde{s}}{\partial t} + \rho_{\rm h}T_{\rm h}\boldsymbol{u}\cdot\nabla s_{\rm h} = k\nabla^2\tilde{T} \tag{3}$$

where ν is the viscosity, s the specific entropy and k is the thermal conductivity. Note that this description assumes that the flow remains laminar.

These equations are solved in a spherical shell or a sphere which represents the star's radiative zone. The conditions on the interface with a convection zone are chosen in such a way as to avoid unphysical Ekman layers or thermal boundary layers, and to impose a given (differential) rotation to the radiative region if desired. The numerical method of resolution is presented elsewhere (Garaud, 2001).

Two essential parameters emerge when normalizing the system with the outer radius of the shell/sphere $r_\star$ and with the mean rotation rate $\Omega_\star$: the Ekman number, $E_\nu = \nu/r_\star^2\Omega_\star$ and the Prandtl number $\sigma = \rho_{\rm h}c_{\rm p}\nu/k$. In a stellar radiative zone, typically $E_\nu \ll \sigma \ll 1$. In the sun, $E_\nu \approx 10^{-14}, \sigma \approx 10^{-6}$.

3. Results in the Steady-State Case

The system is mainly described by four timescales: the dynamical (unit) timescale, the thermal timescale $\tau_k = \rho_h c_p r_\star^2/k$, the local Eddington-Sweet timescale $\tau_{ES} = N_h^2 \tau_k/\Omega_\star^2$ (where N_h is the background buoyancy frequency) and the viscous timescale $\tau_\nu = r_\star^2/\nu$. Extensive investigation of the space of parameters reveals that for low enough Ekman number, the behaviour of the system is uniquely determined by the ratio of the local Eddington-Sweet timescale to the viscous timescale, $\lambda = \tau_{ES}/\tau_\nu = \sigma N_h^2/\Omega_\star^2$.

Steady-state analyses are limited by nature when trying to study systems with many timescales – the information of cause and effect being lost. However, a careful study of the scaling of the variables can still provide useful information about the behaviour of the system.

3.1. Limit of Slow Rotation: $\lambda \gg 1$, $\tau_{ES} \gg \tau_\nu \gg \tau_k$

The thermal timescale being the shortest, the thermal balance is expected to be the first one satisfied. The system then quickly adjusts viscously to any perturbation in angular momentum, and the remaining baroclinic imbalance drives Eddington-Sweet currents.

Numerical investigation shows that the angular velocity profile is indeed viscously dominated (see Garaud, 2002), and that the typical velocity of the meridional currents is of the order of the local Eddington-Sweet velocity $|\boldsymbol{u}| \approx E_\nu/\lambda$. The typical temperature fluctuations are found to be of order of the ellipticity ϵ of the system ($\tilde{T} \approx \epsilon T_h$). This shows that the meridional flows (in this steady-state) restore the barotropicity of the star efficiently. However, this balance is quite unrealistic since the Eddington-Sweet timescale is much larger than the stellar lifetime. A proper time-dependent study is therefore required to study this limit.

3.2. Limit of Rapid Rotation: $\lambda \ll 1$, $\tau_k \ll \tau_{ES} \ll \tau_\nu$

In this limit, one might expect that the stellar adjustment to rotation occurs on the Eddington-Sweet timescale whilst any remaining imbalance leads to a slow viscous flow $|\boldsymbol{u}| \approx E_\nu r_\star \Omega_\star$.

Numerical simulations reveal that the system adjusts through thermal advection and conduction to a state with negligible temperature fluctuations: $\tilde{T} \approx \lambda\epsilon T_h$ (see Fig. 1, solution for a star with $\epsilon = 10^{-2}$, $\lambda \simeq 10^{-4}$ and $1.5 \times 10^7 \geq T_h \geq 10^6$). Although this is a state of strong baroclinicity, the momentum balance is not dominated by thermal-wind driving but by centrifugal driving. Indeed, the principal terms in the meridional momentum balance are found to be the centrifugal force, the pressure and the perturbation to the gravitational potential ($\rho_h \nabla\tilde{\Phi}$). This can be seen more clearly by rescaling the variables $\boldsymbol{u}$ and $\tilde{T}$, as discussed by Garaud (2002). Note that this balance is only satisfied by a strong radial differential rotation; slow viscous meridional flows are then driven to satisfy angular-momentum conservation.

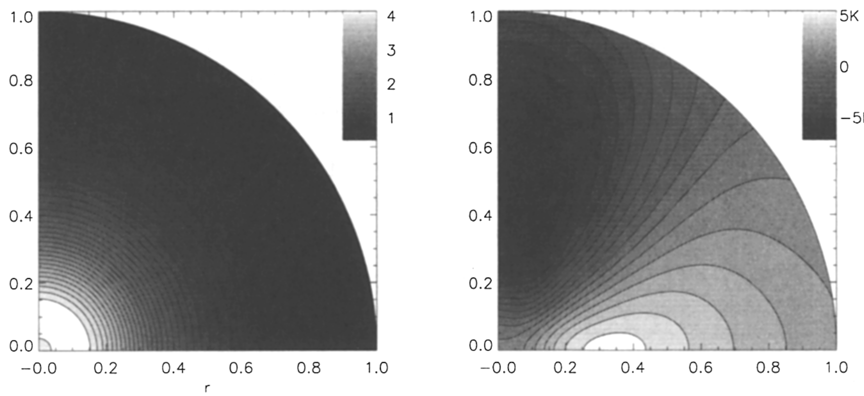

Figure 1. Example of a steady state solution for a rapidly rotating solar-type star ($\epsilon = 10^{-2}$) with $\lambda \simeq 10^{-4}$ (cf. Section 3). The figure on the left shows the angular velocity profile in a quadrant of the radiative zone (where r is the normalized radius). Note the strong radial differential rotation. The figure on the right shows the temperature perturbation profile.

3.3. Discussion

These results unify the various limits studied in the past, and shed light on a long-standing controversy: do Eddington-Sweet circulations exist?

The standard approach to the study of Eddington-Sweet currents usually assumes a given angular-velocity profile, and calculates the meridional motions created by the resulting baroclinicity of the star. It neglects the nonlinear reaction of the meridional motions on angular-momentum transport, which is a good approximation for slowly rotating stars considering the very large turnover timescale of the flow. In this respect, the results of this standard analysis are comparable to those obtained in Section 3.1.

However, standard Eddington-Sweet theory ignores the fact that meridional flows may result from the star's rotation in other ways, namely through centrifugal driving. This phenomenon is important in the limit where the star is more rapidly rotating (e.g. when $\lambda \ll 1$); this was recognized already by Roxburgh (1963). His work assumes (an assumption later proved by Busse, 1982) that the fluid in the radiative zone settles into a state of purely azimuthal flow, with significant differential rotation – as was found in Section 3.2[1].

To conclude this analysis of the steady-state case, it seems that the apparent discrepancy between previous analyses simply corresponds to different limits of the same problem. It is important to see, however, that in all limits the meridional flows remain extremely slow, with a turnover timescale much longer than the stellar evolution timescale. Hence, in the steady-state case, there can be no significant rotational mixing.

[1] As Roxburgh's study neglects the effect of viscosity, it is clear that the results presented in Section 3.2 are entirely compatible with his.

4. First Results for a Spin-Down Calculation

In this section I present some preliminary results of a time-dependent numerical study of the evolution of the thermodynamical and dynamical structure of a star which is spun-down through surface stellar winds. As before, the calculation assumes the knowledge of the hydrostatic equilibrium structure of the non-rotating star, and treats all consequences of rotation or spin-down as perturbations. The boundary conditions are slightly modified to take into account a gradual spin-down of the outer boundary:

$$\begin{aligned}\Omega(r_\star,\theta,t) &= \Omega_{\rm eq}(t)(1-a_2\cos^2\theta - a_4\cos^4\theta) \\ &= \Big((\Omega_{\rm eq}(0)-\Omega_{\rm eq}(\infty))\exp(-t^2/\tau_s^2)+\Omega_{\rm eq}(\infty)\Big)\left(1-a_2\cos^2\theta-a_4\cos^4\theta\right)\end{aligned} \tag{4}$$

where Ω is the angular velocity and τ_s is the spin-down timescale. The parameters a_2 and a_4 set the amount of imposed differential rotation.

Inspection of the angular-momentum equation shows that the typical velocity of the meridional flow required to advect momentum out of the star is of the order of $\dot{\Omega}/\Omega$. Two situations arise: either the turnover timescale of this flow is extremely slow compared to the thermal diffusion timescale ($\tau_s \gg \tau_k$), or both timescales are of comparable order ($\tau_s \geq \tau_k$).

In the first case, numerical simulations suggest that the whole radiative zone adjusts continuously to the spin-down through a series of the quasi-static equilibria described in Section 3.2. Indeed, when the fluid is slow enough it is at all times in thermal equilibrium with its surroundings. The background stratification offers little resistance to the flow, which redistributes angular momentum freely within the star. As a result, the angular velocity profile has little "memory" of past configurations, and is seen to follow roughly[2] a similarity solution with $\Omega(r,\theta,t) = \Omega(r,\theta,0)\Omega_{\rm eq}(t)/\Omega_{\rm eq}(0)$. This is illustrated in Fig. 2.

In the second case, the fluid flow becomes too fast to be able to remain in thermal equilibrium with its surroundings; the buoyancy force acts to slow its progression into the radiative zone. As a result, the angular momentum L is not advected outwards as rapidly, and an inversion in the profile (i.e. $\mathrm{d}L/\mathrm{d}r \leq 0$) is seen to appear (see Fig. 2) near the spun-down surface. This situation is intrinsically unstable, and would result in the generation of turbulent motions. The numerical simulations are however unable to follow the evolution of this region beyond the onset of instability.

5. Discussion and Conclusion

This analysis of the effects of rotation on the chemical and dynamical structure of a star (see Section 3) has revealed that, in a steady state, no significant meridional mixing may occur. However, the time-dependent numerical analysis presented in Section 4 reveals that meridional motions are strongly coupled to angular-momentum losses in a star. To a first approximation, when the spin-down timescale is much larger than the thermal-diffusion timescale, a laminar

[2]This is true to zeroth order only. Meridional flows also drive a small degree of differential rotation, which is dissipated on a longer timescale.

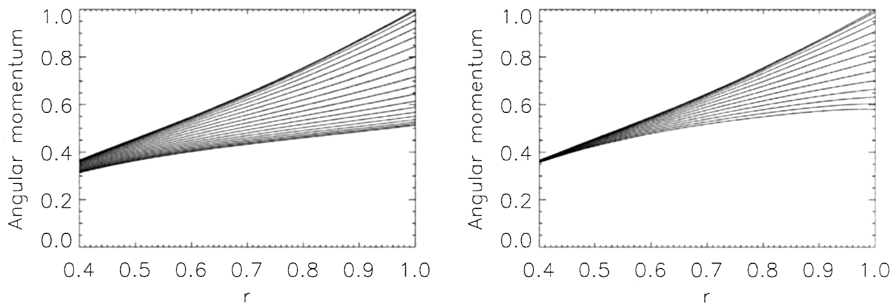

Figure 2. Evolution with time of the angular-momentum profile at co-latitude $\theta = 0.6$ in the radiative zone of a solar-type star. The star was initially rotating 100 times the solar rotation rate, and was spun-down by 50% on a timescale of $\tau_s = 10\tau_k$ in the left panel, and $\tau_s = 2\tau_k$ in the right panel. Snapshots of the angular momentum profile are shown at intervals of $\tau_s/10$.

solution exists with $|\boldsymbol{u}| \simeq \dot{\Omega}/\Omega$. When the spin-down timescale is reduced, the associated flows are hindered by the stratification and the removal of angular momentum cannot be propagated efficiently throughout the whole star. This leads to an unstable angular-momentum profile, and the laminar solution breaks down near the surface.

These new results can be compared with the latest observations of the correlations between rotation and Li depletion. Tschäpe & Rüdiger (2001) found that in the Pleiades, more rapidly rotating stars have lower Li depletion. Mallik's (these proceedings) extensive study of F, G and K, subgiants also reveals that rapidly rotating stars have little Li depletion whereas slowly rotating stars show a significant spread in abundances. A theory in which Li depletion is associated, as suggested by the numerical simulations shown here, not to rotation but to angular-momentum loss, fits this data qualitatively well. It is left to show through more extensive numerical calculation, that this theory can also provide a quantitative description of the observations.

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