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ON MULTIPLIERS BETWEEN SOME SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract

We study function multipliers between spaces of holomorphic functions on the unit disc of the complex plane generated by symmetric sequence spaces. In the case of sequence ℓ^p spaces we recover Nikol'skii's results ['Spaces and algebras of Toeplitz matrices operating on ℓ^p', *Sibirsk. Mat. Zh.* **7** (1966), 146–158].

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1. Introduction

Let $H(\mathbb{D})$ denote the space of holomorphic functions on the unit disc $\mathbb{D}$ of the complex plane. For a fixed $p \in [1, \infty]$, consider a space $\mathcal{F}\ell^p$ given by

$$\mathcal{F}\ell^p = \{f \in H(\mathbb{D}) : \{\widehat{f_n}\} \in \ell^p\},$$

where $\{\widehat{f_n}\}$ denotes the sequence of Taylor coefficients of f. Note that the $\mathcal{F}\ell^p$ are Banach spaces when equipped with a norm induced from ℓ^p, that is, $\|f\|_{\mathcal{F}\ell^p} = \|\{\widehat{f_n}\}\|_{\ell^p}$. In 1966, Nikol'skii studied function multipliers between $\mathcal{F}\ell^p$ spaces (see [5]). To be more precise, he investigated spaces $\mathcal{M}(\mathcal{F}\ell^p, \mathcal{F}\ell^q)$ of all functions holomorphic on the unit disc such that their pointwise multiplication with any $\mathcal{F}\ell^p$ function is an element of $\mathcal{F}\ell^q$. Following Nikol'skii's notation, we will write $\mathcal{M}(p, q)$ for $\mathcal{M}(\mathcal{F}\ell^p, \mathcal{F}\ell^q)$. The main result from [5] can be presented as the following theorem.

THEOREM 1.1. *Suppose that $p, q \in [1, \infty]$. Then:*

(i) $\mathcal{M}(p, q) \neq \{0\}$ *is equivalent to* $p \leq q$;
(ii) $\mathcal{M}(p, \infty) = \mathcal{M}(1, p') = \mathcal{F}\ell^{p'}$, *where* $1/p + 1/p' = 1$ *and* $p \in [1, \infty)$;
(iii) $\mathcal{M}(p, p) \subset H^\infty$, *the space of bounded holomorphic functions on* $\mathbb{D}$;
(iv) $\mathcal{M}(p, q) = \mathcal{M}(q', p') \subset \mathcal{F}\ell^q \cap \mathcal{F}\ell^{p'}$.

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The aim of this paper is to prove more general results in the setting of symmetric sequence spaces. This problem of function multipliers can be reduced to the study of convolution multipliers between symmetric sequence spaces. We note that pointwise multipliers between symmetric sequence spaces were used in [1] to investigate eigenvalues of operators between symmetric spaces. Schur multipliers between symmetric sequence spaces were studied in [7].

2. Preliminaries

Throughout the paper, by a Banach sequence space we mean a complex Banach lattice E modelled on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ containing sequences x with $\operatorname{supp} x = \mathbb{N}_0$. A Banach sequence space E is said to be *symmetric* if $\|x\|_E = \|x^*\|_E$ for all $x \in E$, where x^* as usual stands for the nonincreasing rearrangement of x, and E is called *maximal* provided its unit ball B_E is closed in the pointwise convergence topology on the space $\omega := \mathbb{C}^{\mathbb{N}_0}$ of all complex sequences. In what follows, for any sequence $\{x_n\}_{n=0}^{\infty} \in \omega$ we write $\{x_n\}$ to simplify the notation.

The Köthe dual of a Banach sequence space E,

$$E' = \Big\{\{x_n\} \in \omega : \sum_{n=0}^{\infty} |x_n y_n| < \infty \text{ for all } \{y_n\} \in E\Big\},$$

equipped with a norm $\|x\|_{E'} := \sup_{y \in B_E} \sum_{n=0}^{\infty} |x_n y_n|$ is a maximal Banach sequence space which is symmetric provided E is. Recall that if E is separable, then the Banach dual space E^* is order isometrically isomorphic to E' and $E'' = E$ if and only if E is maximal. For more information on Köthe dual spaces of Banach function lattices, we refer the reader to [3].

Let $\{e_n\}$ denote the standard unit vector basis in $c_0(\mathbb{N}_0)$. We note that if E is a Banach sequence space modelled on $\mathbb{N}_0$, then $e_n \in E$ for each $n \in \mathbb{N}_0$. Observe that if $\|e_n\|_E = 1$ for each $n \in \mathbb{N}_0$, we have $\ell^1 \hookrightarrow E \hookrightarrow \ell^\infty$ and the norms of the continuous inclusions are equal to 1. In what follows, we consider symmetric sequence spaces E for which $\|e_n\|_E = 1$ for each $n \in \mathbb{N}_0$.

The two most classical examples of symmetric spaces of the above type, apart from the ℓ^p spaces, are Lorentz and Orlicz sequence spaces. Let $w = \{w_i\} \in c_0 \backslash \ell^1$ be a decreasing sequence such that $w_1 = 1$. Then, for $p \in [1, \infty)$, the Lorentz sequence space $d(w, p)$ is defined by

$$d(w,p) = \Big\{x = \{x_n\} : \|x\|_{w,p} = \Big(\sum_{n=0}^{\infty} (x_n^*)^p w_n\Big)^{1/p} < \infty\Big\}.$$

Let φ be an Orlicz function, that is, φ is an even, continuous, increasing, convex function defined on $[0, \infty)$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then the Orlicz sequence space ℓ^φ is defined as the space of all sequences $x = \{x_n\}$ such that

$$\|x\|_\varphi = \inf\Big\{\lambda > 0 : \sum_{n=0}^{\infty} \varphi\Big(\frac{x_n}{\lambda}\Big) \le 1\Big\}.$$

For more information on these spaces, see [4].

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Suppose that E is a symmetric sequence space. By $\mathcal{F}E$, we denote the space of holomorphic functions on the unit disc $\mathbb{D}$ given by

$$\mathcal{F}E := \Big\{ f \in H(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} \widehat{f_n} z^n,\ \{\widehat{f_n}\} \in E \Big\}.$$

It can be easily checked that $\mathcal{F}E$ equipped with the norm $\|f\| = \|\{\widehat{f_n}\}\|_E$ is a Banach space. By $\mathcal{M}(\mathcal{F}E, \mathcal{F}F)$, we will denote the set of all functions $f \in H(\mathbb{D})$ called multipliers such that for every $g \in \mathcal{F}E$ we have $fg \in \mathcal{F}F$.

Given two arbitrary sequences $x = \{x_n\}$ and $y = \{y_n\}$ in ω, the convolution of x and y, denoted by $x * y$, is the sequence defined by

$$x * y = \Big\{ \sum_{k=0}^{n} x_k y_{n-k} \Big\}.$$

Suppose that $f \in H(\mathbb{D})$ with $f(z) := \sum_{k=0}^{\infty} \widehat{f_k} z^k$ for all $z \in \mathbb{D}$. It is easy to see that $f \in \mathcal{M}(\mathcal{F}E, \mathcal{F}F)$ if and only if for every $g \in \mathcal{F}E$, $g(z) := \sum_{k=0}^{\infty} \widehat{g}_k z^k$ we have $\{\widehat{f_k}\} * \{\widehat{g}_k\} \in F$. From this observation, describing function multipliers between spaces $\mathcal{F}E$ and $\mathcal{F}F$ is equivalent to describing sequence convolution multipliers for symmetric sequence spaces E and F. We will denote the latter by $\mathcal{M}(E, F)$. Furthermore, let $\mathcal{M}(E) := \mathcal{M}(E, E)$. For every $\mu \in \mathcal{M}(E, F)$, we can construct a bounded operator $T_\mu : E \to F$ by the formula

$$T_\mu(x) = \mu * x \quad \text{for } x \in E.$$

Thus, we can equip $\mathcal{M}(E, F)$ with the operator norm

$$\|\mu\|_{\mathcal{M}(E,F)} = \sup\{\|T_\mu(x)\|_F : \|x\|_E \le 1\}.$$

The following two operators will be of great importance. For a given $n \in \mathbb{N}_0$, we define $P_n : \omega \to \omega$ and $\sigma_n : \operatorname{span}(\{e_0, \dots, e_n\}) \to \operatorname{span}(\{e_0, \dots, e_n\})$ by the formulas

$$P_n x := \sum_{k=0}^{n} x_k e_k \quad \text{for } x = \{x_k\} \in \omega$$

and

$$\sigma_n\Big(\sum_{k=0}^{n} y_k e_k\Big) := \sum_{k=0}^{n} y_{n-k} e_k \quad \text{for } (y_0, \dots, y_n) \in \mathbb{C}^{n+1}.$$

3. Multipliers

We start with some basic properties of convolution multipliers between symmetric sequence spaces.

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Proposition 3.1. *Let E, F, G be symmetric sequence spaces. Then:*

(i) $\mathcal{M}(E, F) \subset F$;
(ii) $E \subset F$ *if and only if* $e_k \in \mathcal{M}(E, F)$ *for each* $k \in \mathbb{N}$;
(iii) $E \subset F$ *if and only if* $\ell^1 \subset \mathcal{M}(E, F)$;
(iv) $E \subset \mathcal{M}(F, G)$ *if and only if* $F \subset \mathcal{M}(E, G)$;
(v) $\mathcal{M}(\ell^1, E) = E$.

Proof. Statements (i) and (ii) are easy consequences of the observation that $e_k * y = \{0, 0, \ldots, 0, y_0, y_1, \ldots\}$, where 0 appears k times. Statement (iv) is an immediate consequence of the definition of $\mathcal{M}(E, F)$.

Next, to prove (iii), observe that, by (ii), it is enough to show that $E \subset F$ implies $\ell^1 \subset \mathcal{M}(E, F)$. Suppose that $E \subset F$. Denote by S the right-shift operator given by $Sy = \{0, y_0, y_1, \ldots\}$ for any $y = \{y_n\} \in \omega$. Then $e_k * y = S^k y$ for each $k \in \mathbb{N}_0$ and $y \in \omega$. Since E is symmetric, $S : E \to E$ and $\|S\| = 1$. The same applies to S^k for each k. Then, for $a = \{a_k\} \in \ell^1$,

$$T_a(y) = a * y = \sum_{k=0}^{\infty} a_k S^k y \quad \text{for } y \in E.$$

We need to show that $\|T_a(y)\|_F = \|a * y\|_F < \infty$. Since $E \subset F$, it follows from the closed graph theorem that the inclusion map $i : E \to F$ is bounded. Let $C = \|i\|_{E \to F}$. Then, for all $y \in E$,

$$\sum_{k=0}^{\infty} \|a_k S^k y\|_F \leq \sum_{k=0}^{\infty} |a_k| \, \|S^k y\|_F \leq C \sum_{k=0}^{\infty} |a_k| \, \|S^k y\|_E \leq C \sum_{k=0}^{\infty} |a_k| \, \|y\|_E = C\|y\|_E \|a\|_1,$$

which shows that the series is absolutely convergent. Since E is a Banach space,

$$\|a * y\|_F \leq \left\| \sum_{k=0}^{\infty} a_k S^k y \right\|_F \leq C \sum_{k=0}^{\infty} \|a_k S^k y\|_E \leq C\|y\|_E \|a\|_1$$

and the proof of (iii) is finished.

As to the proof of (v), from (i) we know that $\mathcal{M}(\ell^1, E) \subset E$. We have to show the reverse inequality. Bearing in mind (iv), it suffices to show that $\ell_1 \subset \mathcal{M}(E, E)$, which follows from (iii). □

Now we will focus on the question when $\mathcal{M}(E, F) = \{0\}$. Let us start with the necessary definitions. Let E be a symmetric sequence space. Following [2], we define the upper inclusion index γ_E and the lower inclusion index δ_E by the formulas

$$\gamma_E = \inf\{p \leq \infty : E \hookrightarrow \ell^p\}, \quad \delta_E = \sup\{p \leq \infty : \ell^p \hookrightarrow E\}.$$

The next proposition follows from the definition of multipliers and Theorem 1.1(i).

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PROPOSITION 3.2. *Let E, F be symmetric sequence spaces.*

(i) *If $\mathcal{M}(E,F) \neq \{0\}$, then $\delta_E \leq \gamma_F$.*
(ii) *If $\gamma_E < \delta_F$, then $\mathcal{M}(E,F) \neq \{0\}$.*
(iii) *If $\gamma_E = \delta_F$, $E \hookrightarrow \ell_{\gamma_E}$ and $\ell_{\delta_F} \hookrightarrow F$, then $\mathcal{M}(E,F) \neq \{0\}$.*

For ℓ^p spaces, $\delta_{\ell^p} = \gamma_{\ell^p} = p$, which gives the equivalence of Theorem 1.1(i).

A celebrated result of Schur (see [6]) states that $\mathcal{M}(\ell^2) = \widehat{H}^\infty$, where $\widehat{H}^\infty$ is the space of all sequences of Taylor coefficients of bounded holomorphic functions on $\mathbb{D}$. Based on some ideas from [6], we prove the following result.

THEOREM 3.3. *Let E be a symmetric sequence space. Then $\mathcal{M}(E) \subseteq \widehat{H}^\infty$.*

PROOF. Let $\mu \in \mathcal{M}(E)$ and $C = \|\mu\|_{\mathcal{M}(E)}$. Then

$$\|\mu * x\|_E = \bigg\|\bigg\{\sum_{k=0}^{n} \mu_k x_{n-k}\bigg\}\bigg\|_E \leq C\|x\|_E \quad \text{for } x \in E.$$

First we show that for all $x \in E$ the following inequality holds:

$$\bigg\|\bigg\{\sum_{k=0}^{\infty} \mu_k x_{n+k}\bigg\}\bigg\|_E \leq C\|x\|_E. \tag{3.1}$$

Fix $n \in \mathbb{N}$. Then, since $\|P_n x\|_E \leq \|x\|_E$ for all $x \in E$ and $n \geq 1$,

$$\|P_n(\mu * P_n x)\|_E = \bigg\|\sum_{j=0}^{n}\bigg(\sum_{k=0}^{j} \mu_k x_{j-k}\bigg)e_j\bigg\|_E \leq C\|P_n x\|_E.$$

Now consider the previous inequality with $\sigma_n P_n x$ instead of $P_n x$, that is, interchange x_i for x_{n-i}. This gives

$$\bigg\|\sum_{j=0}^{n}\bigg(\sum_{k=0}^{j} \mu_k x_{n-j+k}\bigg)e_j\bigg\|_E \leq C\|\sigma_n P_n x\|_E.$$

By relabelling $s = n - j$ we obtain for all $x \in E$,

$$\bigg\|\sum_{s=0}^{n}\bigg(\sum_{k=0}^{n-s} \mu_k x_{s+k}\bigg)e_{n-s}\bigg\|_E \leq C\|\sigma_n P_n x\|_E.$$

Since E is a symmetric space, we can rearrange coordinates without changing the norm and so

$$\bigg\|\sum_{s=0}^{n}\bigg(\sum_{k=0}^{n-s} \mu_k x_{s+k}\bigg)e_s\bigg\|_E \leq C\|\sigma_n P_n x\|_E.$$

Fix $m \leq n$ and observe that then

$$\bigg\|\sum_{s=0}^{m}\bigg(\sum_{k=0}^{n-s} \mu_k x_{s+k}\bigg)e_s\bigg\|_E \leq C\|\sigma_n P_n x\|_E.$$

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Combining $\|\sigma_n P_n x\|_E = \|P_n x\|_E \le \|x\|_E$ with the fact that the above inequality is valid for all $n \in \mathbb{N}$, we arrive at

$$\Big\| \sum_{s=0}^{m} \Big(\sum_{k=0}^{\infty} \mu_k x_{s+k} \Big) e_s \Big\|_E \le C\|x\|_E$$

for each positive integer m; thus finally we get the desired inequality (3.1).

For $z \in \mathbb{D}$, set $x := \{z^n\}$. Since $\ell^1 \subset E$, we have $x \in E$. From (3.1),

$$\Big\| \Big\{ \sum_{k=0}^{\infty} \mu_k z^{n+k} \Big\} \Big\|_E \le C\|\{z^n\}\|_E. \tag{3.2}$$

Furthermore, observe that

$$\sum_{k=0}^{\infty} \mu_k z^{n+k} = z^n \sum_{k=0}^{\infty} \mu_k z^k := z^n f_\mu(z).$$

Combining the above with (3.2),

$$|f_\mu(z)|\, \|\{z^n\}\|_E = \|\{z^n f_\mu(z)\}\|_E \le C\|\{z^n\}\|_E,$$

which gives $|f_\mu(z)| \le C$ for all $z \in \mathbb{D}$. Thus, $f \in H^\infty(\mathbb{D})$ and $\|f_\mu\|_\infty \le C = \|\mu\|_{\mathcal{M}(E)}$. □

We need another piece of terminology and a lemma. Let D be a domain of the bilinear form $\langle \cdot\,, \cdot \rangle$ given by

$$\langle x, y \rangle := \sum_{k=0}^{\infty} x_k y_k$$

for all $(x, y) \in D$ with $x = \{x_n\}$ and $y = \{y_n\}$. We will say that $\langle x, y \rangle$ is meaningful provided that it is well defined (that is, $(x, y) \in D$).

Lemma 3.4. *For sequences $x, y, \mu \in \omega$, the equality*

$$\langle \mu * x, y \rangle = \langle \mu * y, x \rangle$$

holds whenever the terms in the equation are meaningful.

Proof. We have

$$\begin{aligned}
\langle \mu * x, y \rangle &= \sum_{k=0}^{\infty} (\mu * x)_n y_n = \lim_{n\to\infty} ((\mu * x) * \sigma_n P_n y)_n \\
&= \lim_{n\to\infty} (\mu * (x * \sigma_n P_n y))_n = \lim_{n\to\infty} (\mu * (\sigma_n P_n x * y))_n \\
&= \lim_{n\to\infty} (\mu * y * (\sigma_n P_n x))_n = \lim_{n\to\infty} ((\mu * y) * (\sigma_n P_n x))_n \\
&= \lim_{n\to\infty} (\sigma_n P_n (\mu * y) * x)_n = \sum_{k=0}^{\infty} (\mu * y)_n x_n = \langle \mu * y, x \rangle.
\end{aligned}$$ □

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Theorem 3.5. *If E, F are maximal symmetric sequence spaces, then $\mathcal{M}(E,F) = \mathcal{M}(F', E')$.*

Proof. Suppose that $\mu \in \mathcal{M}(E,F)$. Since $F'' = F$ isometrically, for any $x \in E$,

$$\|\mu * x\|_F = \sup_{y \in B_{F'}} |\langle \mu * x, y\rangle| \le C\|x\|_E.$$

Hence, for all $x \in B_E$ and $y \in F'$,

$$|\langle \mu * x, y\rangle| \le C\|y\|_{F'},$$

which yields

$$\sup_{x \in B_E} |\langle \mu * x, y\rangle| \le C\|y\|_{F'}.$$

Using Lemma 3.4,

$$\sup_{x \in B_E} |\langle \mu * y, x\rangle| \le C\|y\|_{F'}.$$

Thus, $\|\mu * y\|_{E'} \le C\|y\|_{F'}$, which completes the proof. □

The following corollary is an immediate consequence of Proposition 3.1(v) and Theorem 3.5.

Corollary 3.6. *If E is a maximal symmetric sequence space, then $E' = \mathcal{M}(E, \ell^\infty)$ and $E = \mathcal{M}(E', \ell^\infty)$.*

Following the method of [5, Theorem 6], we will prove the following theorem.

Theorem 3.7. *The Banach space $\mathcal{M}(E)$ contains an isomorphic copy of ℓ^1 for any symmetric sequence space E.*

Proof. Given $\lambda > 1$, let $\{n_k\}$ be a sequence of positive integers such that $n_{k+1}/n_k \ge \lambda$ for all $k \in \mathbb{N}_0$. Define

$$M = \{\mu = \{\mu_n\} \in \mathcal{M}(E) : \mu_i = 0 \text{ for } i \ne n_k,\ k \in \mathbb{N}_0\}.$$

Since $E \hookrightarrow \ell^\infty$, it follows that $E \hookrightarrow \omega$. This fact easily yields that M is a closed linear subspace of $\mathcal{M}(E)$. Then, from Theorem 3.3 and Sidon's theorem (see [8, Theorem 6.1 in Ch. VI]), we see that $M \subset \ell^1$. Having in mind that $\|x\|_1 \ge \|x\|_{\mathcal{M}(E)}$ for all $x \in \ell^1$ and using the closed graph theorem, we obtain the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_{\mathcal{M}(E)}$ on M. Consequently, we conclude that M is isomorphic to ℓ^1. □

It is well known that every closed subspace of a reflexive Banach space must be reflexive. Hence, from Theorem 3.7, we get the following corollary.

Corollary 3.8. *The Banach space $\mathcal{M}(E)$ is not reflexive for any symmetric sequence space E.*

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