J. Austral. Math. Soc. (*Series A*) 39 (1985), 63–74

SOME COVERING THEOREMS FOR LOCALLY INVERSE SEMIGROUPS

D. B. McALISTER

(Received 3 February 1984)

Communicated by T. E. Hall

Abstract

A regular semigroup S is said to be locally inverse if each local submonoid eSe, with e an idempotent, is an inverse semigroup. In this paper we apply known covering theorems for inverse semigroups and a covering theorem for locally inverse semigroups due to the author to obtain some covering theorems for locally inverse semigroups. The techniques developed here permit us to give an alternative proof for, and slight strengthening of, an important covering theorem for locally inverse semigroups due to F. Pastijn.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 M 10.

1. Introduction

A number of recent papers have dealt with the structure of locally inverse semigroups. This class of semigroups consists of all regular semigroups S in which each local subsemigroup eSe, with e an idempotent, is an inverse semigroup. Locally inverse semigroups are exceedingly prevalent and include, as well as inverse semigroups, all regular semigroups which are subdirect products of completely 0-simple semigroups, and all normal bands. They have been characterized by Nambooripad [13] as those regular semigroupls on which the natural partial order is compatible with multiplication. Thus they inherit many of the pleasant properties of inverse semigroups.

F. J. Pastijn [16] has given an important structure theorem for locally inverse semigroups. He shows that any locally inverse semigroup is a homomorphic image of an order ideal and subsemigroup of a semigroup which, in a sense, generalizes the construction of a P-semigroup, which has proved to be useful in describing inverse semigroups (McAlister [7], [8]), by replacing the group which acts on a semilattice by a completely simple semigroup. The homomorphism which arises in Pastijn's theorem is special in that the inverse image of each idempotent is a completely simple semigroup.

The author [10] has given an alternative covering theorem for locally inverse semigroups. More precisely, he has shown that any locally inverse semigroup is a homomorphic image of the semigroup of regular elements in a Rees matrix semigroup over an inverse semigroup. In this case, the homomorphism is special in that it is one-to-one on each local subsemigroup. It is thus close to being an isomorphism; the inverse image of each idempotent is a rectangular band. In this paper we shall use this covering theorem to obtain a number of division theorems for locally inverse semigroups, which include Pastijn's theorem which was mentioned above. Our proofs depend on known properties of inverse semigroups and on some preliminary results on quasi-ideals, which we shall derive; the arguments are thus directly of a semigroup theoretic nature. Pastijn's, by contrast, depend on a deep analysis of the structure of the biordered set of idempotents of a locally inverse semigroup. This research was partly carried out while the author was visiting Marquette University, Milwaukee in October 1983. He is grateful to the Mathematics Department there for their hospitality.

2. Quasi-ideals and inverse semigroups

A subsemigroup Q of a semigroup S is said to be a *quasi-ideal* of S if $QSQ \subseteq Q$. In this paper, we shall be primarily concerned with quasi-ideals of regular semigroups, which are themselves regular. We shall say that such a subsemigroup is a *regular quasi-ideal.*

The idempotents in any semigroup S admit a natural partial order $\leqslant$ which is defined as follows: $e \leqslant f$ if and only if $e = ef = fe$. A set A of idempotents of S is called an *order ideal* of the set E_S of idempotents of S if $e \leqslant a$, with a in A, implies $e \in A$. We shall say that A is a biorder ideal of E_S if it is an order ideal and, in addition, for $a, b \in A$, we have $S(a, b) \cap A \neq 0$. Here $S(a, b)$ denotes the sandwich set of a and b; thus $S(a, b) = (g \in E_S\colon ab = agb, ga = g = bg)$.

PROPOSITION 2.1. *Let S be a regular semigroup and let Q be a quasi-ideal of S. Then the set E_Q of idempotents of Q is a biorder ideal of E_S. The set* $\operatorname{Reg}(Q)$ *of regular elements of Q is a regular quasi-ideal of S. Indeed*

$$\operatorname{Reg}(Q) = \left\{ a \in S\colon aa', a'a \in E_Q \text{ for some inverse } a' \text{ of } a \right\}.$$

https://doi.org/10.1017/S1446788700022175 Published online by Cambridge University Press

Conversely, suppose that F is a biorder ideal of E_S and set $Q = \{a \in S: aa', a'a \in F$ for some inverse a' of $a\}$. Then Q is a regular quasi-ideal of S and $F = E_Q$. Further $Q = \bigcup\{uSv: u, v \in F\}$.

PROOF. Let Q be a quasi-ideal of S and let $a \in E_Q$, with $f \in E_S$ and $f \leqslant a$. Then $f = af = fa$ so $f = afa \in QSQ$; thus, since Q is a quasi-ideal, $f \in E_Q$ which is therefore an order ideal of E_S. If $a, b \in E_Q$ then, since S is regular, ab has an inverse x in S. Then $g = bxa \in S(a, b) \cap E_Q$. Thus E_Q is a biorder ideal of E_S.

If a is a regular element of Q, with inverse $a' \in Q$, then $aa', a'a \in E_Q$. Conversely, if $aa', a'a \in E_Q$, then $a = aa'.a.a'a \in QSQ$; thus, since Q is a quasi-ideal, a is a regular element of Q. Hence $\mathrm{Reg}(Q)$ is as described in the statement of the proposition; that it is a regular quasi-ideal will follow from the second part of the proof.

Let F be a biorder ideal of E_S and denote by Q the set $\{a \in S: aa', a'a \in F$ for some inverse a' of $a\}$. For $a, b \in Q$ and $x \in S$, axb has an inverse of the form $b'ya'$ for any inverses a' of a and b' of b. Hence, if $aa', a'a, bb', b'b$ are in F, we have $axb.b'ya' \leqslant aa'$ so that $axb.b'ya' \in F$ and similarly $b'ya'.axb \in F$. Hence $QSQ \subseteq Q$ and, since S is regular, this is enough to show that Q is a subsemigroup of S, and hence a quasi-ideal of S. Clearly Q is regular so it is a regular quasi-ideal of S.

Clearly, $F \subseteq E_Q$ so it remains to show the converse inclusion. Let $e \in E_Q$. Then there is an invese e' of e such that both $ee', e'e \in F$. Let $f = ee'$ and $h = e'e$. Then, since F is a biorder ideal, $F \cap S(h, f) \neq \varnothing$. Suppose g belongs to this intersection; then $eg = e(fg) = fg = g$ and also we have $ge = (gh)e = gh = g$ so that $e = ehfe = ehgfe = ege = g$. Thus $e \in F$ so that F is the set of idempotents of Q.

The final statement of the proposition is easily seen to hold since Q is a regular quasi-ideal.

COROLLARY 2.2. *There is an order isomorphism between the partially ordered set of regular quasi-ideals of a regular semigroup S and the set of biorder ideals of E_S.*

COROLLARY 2.3. *Let S be an inverse semigroup. Then the regular quasi-ideals of S form a complete lattice, under intersection, isomorphic to the lattice of order ideals of E_S. The regular quasi-ideal corresponding to the order ideal F of E_S is $\{a \in S: aa^{-1}, a^{-1}a \in F\}$.*

Proposition 2.1 shows that the regular quasi-ideals of a regular semigroup are completely determined by their idempotents. We next turn to a result involving

https://doi.org/10.1017/S1446788700022175 Published online by Cambridge University Press

congruences on quasi-ideals which we shall apply to obtain a useful embedding theorem for inverse semigroups.

PROPOSITION 2.4. *Let Q be a regular quasi-ideal in a regular semigroup S. Then any idempotent separating congruence ρ on Q can be extended to an idempotent separating congruence on S.*

PROOF. Let us denote by $\rho^{\#}$ the congruence on S generated by ρ. Then T. E. Hall, [4; Corollary 6], has shown that $\rho^{\#}$ is idempotent separating. Now, for $s, t \in S$, $(s, t) \in \rho^{\#}$ if and only if there is a chain $s = s_0 \to s_1 \to \cdots \to s_n = t$ of elementary transformations from s to t, where $s_i \to s_{i+1}$ means $s_i = u_i a_i v_i$, $s_{i+1} = u_i b_i v_i$ where $(a_i, b_i) \in \rho$.

Suppose that $s_i \in Q$; we show that $s_{i+1} \in Q$ and s_i, s_{i+1} are ρ-related. Since ρ and $\rho^{\#}$ are idempotent separating and $(a_i, b_i) \in \rho$ we have $s_{i+1} = s_i s_i' s_{i+1} s_i' s_i = s_i s_i' u_i a_i a_i' b_i a_i' a_i v_i s_i' s_i$ where s_i' is an inverse of s_i contained in Q. Because Q is a quasi-ideal of S, $w_i = s_i s_i' u_i a_i a_i'$ and $z_i = a_i' a_i v_i s_i s_i'$ are in Q, so s_{i+1} is in Q. Hence, since $(a_i, b_i) \in \rho$, it follows that $(w_i b_i z_i, w_i a_i z_i) \in \rho$. That is $(s_{i+1}, s_i) \in \rho$. Consequently, if $s \in Q$ and $(s, t) \in \rho^{\#}$ then $t \in Q$ and $(s, t) \in \rho$.

REMARKS. (1) T. E. Hall [5] as shown that the set Reg(eS) of regular elements of eS, e an idempotent, forms a subsemigroup in any regular semigroup S. Since eS is a quasi-ideal, this follows immediately from Proposition 1.1. Indeed Reg(eS) is a quasi-ideal of S, Reg(eS) $= \bigcup\{uSv\colon u, v$ are idempotents in $eS\}$.

(2) If $Q = eSe$ for some idempotent $e \in S$ then every congruence on Q extends to a congruence on S (T. E. Hall and P. R. Jones [8; Proposition 4.5]). This need not be the case for larger quasi-ideals. For example, let

$$S = \mathscr{M}(G; \{1,2\}, \{1,2\}; P)$$

be a completely simple semigroup over a group G, with identity 1, and with sandwich matrix $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & a \end{smallmatrix}\right)$, where $a \neq 1$. Then $Q = \{(1, g, j)\colon g \in G, j = 1, 2\}$ is a regular quasi-ideal of S and the relation p defined by

$$(1, g, i)\, p\, (1, h, j) \quad \text{if and only if} \quad g = h$$

is a congruence on Q which cannot be extended to a congruence $p^{\#}$ on S. For $(1,1,1)\, p\, (1,1,2)$ would imply that

$$(1,1,1) = (1,1,1)(2,1,1)\, p^{\#} (1,1,2)(2,1,1) = (1, a, 1)$$

which would contradict the assertion that $p^{\#} \cap (Q \times Q) = p$.

Let G be a group and let X be a semilattice on which G acts by automorphisms, on the left, and let Y be an ideal of X. Then one can construct an inverse semigroup $P(G, X, Y)$ as follows: $P(G, X, Y) = \{(a, g) \in Y \times G\colon g^{-1} \in Y\}$, with $(a, g)(b, h) = (a \wedge gb, gh)$, where $\wedge$ denotes the meet operation in X. It is easy to see that $P(G, X, Y)$ is a quasi-ideal of the semidirect product $P(G, X, X)$ of X by G. Indeed it is the quasi-ideal whose semilattice of idempotents is $\{1\} \times Y$.

N. R. Reilly and W. D. Munn [17; Theorem 5.3] have strengthened the author's division theorem for inverse semigroups [7; Theorem 2.5] to show that every inverse semigroup is an idempotent separating homomorphic image of a semigroup $P(G, X, Y)$ constructed as above. Since $P(G, X, Y)$ is a quasi-ideal of the larger semigroup $P(G, X, X)$, their result can be stated in the form "Every inverse semigroup is an idempotent separating homomorphic image of a quasi-ideal in the semidirect product of a semilattice by a group." Their result, in turn, can be strengthened by adjoining an identity to X, if it does not already contain one. For $P(G, X, Y)$ is still a quasi-ideal of $P(G, X^1, X^1)$, while the latter is *factorizable* in the sense of Chen and Hseih [1]; that is, each element is under a unit. We can use this, together with Proposition 2.4, to obtain the following theorem.

THEOREM 2.5. *Every inverse semigroup can be embedded as a quasi-ideal in a factorizable inverse semigroup.*

PROOF. Let S be an inverse semigroup. Then, by Munn and Reilly's result which was quoted above, S is an idempotent separating homomorphic image of $P(G, X, Y)$ for some group G, semilattice X, and ideal Y of X. By Proposition 1.4, the corresponding idempotent separating congruence ρ on $P(G, X, Y)$ can be extended to a congruence $\rho^{\#}$ on $P(G, X^1, X^1)$. Let T denote the quotient semigroup $P(G, X^1, X^1)/\rho^{\#}$. Then, since $P(G, X, Y)$ is a quasi-ideal of $P(G, X^1, X^1)$, S can be embedded as a quasi-ideal in T. Further, since $P(G, X^1, X^1)$ is factorizable, so is T.

REMARK. Ideas similar to those involved in Proposition 2.4 and Theorem 2.5 will be found in [9; Section 5]. What was called a heavy subsemigroup in [9] is, by Proposition 2.1 here, exactly a quasi-ideal.

3. Locally inverse semigroups

A regular semigroup S is said to be *locally inverse* if eSe is inverse for each idempotent e in S. Division theorems for locally inverse semigroups have been given by F. J. Pastijn [16] and the author [10]. In this section we shall use the results of Section 1 to derive Pastijn's result, and some other division theorems, from that of McAlister. We shall now describe the latter.

Let T be any regular semigroup and let I, Λ be non-empty sets. Let P be a $\Lambda \times I$ matrix over T. Then the Rees matrix semigroup $\mathcal{M}(T; I, \Lambda; P)$ over T need not be regular. However the set of regular elements forms a subsemigroup $\mathcal{RM} = \mathcal{RM}(T; I, \Lambda; P)$, which we call a *regular Rees matrix semigroup over T*, as in [10]. The triple (i, a, λ) belongs to $\mathcal{RM}$ if and only if a has an inverse in $p_{\lambda j}Tp_{\mu i}$ for some $j \in I$, $\mu \in \Lambda$. If T is inverse then $\mathcal{RM}$ is locally inverse and, in this case, $(i, a, \lambda) \in \mathcal{RM}$ if and only if $a \in p_{\lambda i}^{-1}Tp_{\mu j}^{-1}$ for some $j \in I$, $\mu \in \Lambda$.

Theorem 3.1 shows that regular Rees matrix semigroups over inverse semigroups form a good model for all locally inverse semigroups. To state it we need to introduce the notion of a local isomorphism between semigroups. A homomorphism θ of a regular semigroup S onto a semigroup T is said to be a *local isomorphism* if it is one-to-one on each local submonoid eSe, where e is idempotent.

THEOREM 3.1 [10]. *A regular semigroup is locally inverse if and only if it is a locally isomorphic image of a regular Rees matrix semigroup over an inverse semigroup.*

The regular Rees matrix semigroup produced in the proof of Theorem 3.1 is square; that is, $I = \Lambda$. Further, it is normalized in the sense that p_{ii} is idempotent and $p_{ij} \in p_{ii}Tp_{jj}$ for each $i, j \in I$. Because of this normalization, the elements of $\mathcal{RM}$ are easy to describe. The triple (i, a, j) is there if and only if $a \in p_{ii}Tp_{jj}$.

Suppose now that T is any inverse semigroup and let U be a factorizable inverse semigroup containing T as a quasi-ideal; by Theorem 2.5, such exists. If P is any $\Lambda \times I$ matrix over T then we can form a new $\Lambda \times I$ matrix Q over U by choosing a unit $q_{\lambda i} \geqslant p_{\lambda i}$, for each $\lambda \in \Lambda$, $i \in I$. Note that, if $p_{\lambda i}$ is idempotent then $q_{\lambda i}$ is the identity of U. A natural question would be whether or not $\mathcal{RM}(T; I, \Lambda; P)$ is a quasi-ideal of $\mathcal{M}(T; I, \Lambda; P)$, which is a considerably simpler semigroup. Unfortunately, this need not be the case. However it is true under relatively mild restrictions on P.

LEMMA 3.2. *Suppose that each entry of* $P^{-1}PP^{-1} = \{ p_{\lambda i}p_{\mu j}p_{\nu k}\colon i, j, k \in I; \lambda, \mu, \nu \in \Lambda\}$ *is maximal in the natural partial ordering on T. Then, with the notation introduced above,* $\mathcal{RM}(T; I, \Lambda; P)$ *is a regular quasi-ideal of*

$$\mathcal{RM}(U; I, \Lambda; Q) = \mathcal{M}(U; I, \Lambda; Q).$$

PROOF. First, we note that, since each member of Q is invertible,

$$\mathscr{RM}(U; I; \Lambda; Q) = \mathscr{M}(U; I; \Lambda; Q).$$

Let (i, a, λ), (j, b, μ) be in $\mathscr{RM}(T; I, \Lambda; P)$ and let $(k, c, \nu) \in \mathscr{M}(U; I, \Lambda; Q)$. Then $a \in p_{\rho i}^{-1}T$ and $b \in Tp_{\mu j}^{-1}$ for some $\rho \in \Lambda$ and $h \in I$. But

$$(i, a, \lambda)(k, c, \nu)(j, b, \mu) = (i, w, \mu) \quad \text{where} \quad w = aq_{\lambda k}cq_{\nu j}b \in TUT.$$

Thus, since T is a quasi-ideal of U, $w \in T$. Further, since $a \in p_{\rho i}^{-1}T$ and $b \in Tp_{\mu j}^{-1}$ we have $w \in p_{\rho i}^{-1}Tp_{\mu j}^{-1}$. Hence $(i, w, \mu) \in \mathscr{RM}(T; I, \Lambda; P)$.

To complete the proof, it remains to show that the multiplication on $\mathscr{RM}(T; I, \Lambda; P)$ is that inherited from $\mathscr{M}(U; I, \Lambda; Q)$. For this it suffices to show for $i, j \in I$; $\lambda, \mu \in \Lambda$ we have $p_{\lambda j}^{-1}p_{\lambda i}p_{\mu i}^{-1} = p_{\lambda j}^{-1}q_{\lambda i}p_{\mu i}^{-1}$. But, since $p_{\lambda i}$ is under $q_{\lambda i}$ the left side is clearly smaller than the right side of this equation. On the other hand, the right side belongs to S and so, since each element of $P^{-1}PP^{-1}$ is maximal in S, equality must in fact hold.

Our next lemma shows that, by taking an idempotent separating coextension of S, we can ensure that the hypotheses of Lemma 3.2 are satisfied. This will permit us to give an alternative proof of Pastijn's theorem [16; Theorem 2.16].

LEMMA 3.3. *Let S be an inverse semigroup and let X be any non-empty subset of S. Then there is an inverse semigroup T, an idempotent separating homomorphism θ of T onto S and a subset Y of T such that each element of $Y^{-1}YY^{-1}$ is maximal in T and further $X = Y\theta$.*

PROOF. Let G be any non-trivial group. Then direct, if tedious, verification shows that $T = S \times G$ is an inverse semigroup under the multiplication

$$(s, g)(t, h) = \begin{cases} (st, g), & \text{if } s^{-1}s < tt^{-1}, \\ (st, gh), & \text{if } s^{-1}s = tt^{-1}, \\ (st, h), & \text{if } s^{-1}s > tt^{-1}, \\ (st, 1), & \text{otherwise.} \end{cases}$$

The mapping θ which sends (s, g) to g is clearly an idempotent separating homomorphism of T onto S. Further, $(s, g) < (t, h)$ implies $(s, g) = (ss^{-1}, 1)(t, h)$ with $ss^{-1} < tt^{-1}$ so that, from the multiplication above, $(s, g) = (ss^{-1}, 1)$. Thus each (s, g), with g different from 1, is maximal. Hence, to construct Y, we need only choose $g \neq 1$ and set $Y = X \times \{g\}$.

REMARK. The fact that T in the lemma above is an inverse semigroup may be determined directly. Alternatively, it can be regarded as a coextension of a semilattice of groups, in which the linking homomorphisms are trivial, by the

https://doi.org/10.1017/S1446788700022175 Published online by Cambridge University Press

inverse semigroup S. The coextension theories given in [2] or [3] can then be applied to show that T is in fact an inverse semigroup.

For ease of terminology in the statement of the next theorem, and later, we shall say that a Rees matrix semigroup $\mathcal{M}(S; I, \Lambda; P)$ is *classical* if S is a monoid and each entry of P is a unit in S.

A homomorphism θ of a regular semigroup S onto a regular semigroup T is said to be *strictly compatible* if the inverse image of each idempotent of S is a completely simple subsemigroup of T. Clearly each local isomorphism and each idempotent separating homomorphism is strictly compatible, as is any composition of these.

LEMMA 3.4. *Let S and T be inverse semigroups and let θ be a homomorphism of T onto S. Let I and Λ be non-empty sets and let Q, P be $\Lambda \times I$ matrices over T and S respectively such that $Q\theta = P$. Then the mapping ϕ defined by setting $(i, t, \lambda)\phi = (i, t\theta, \lambda)$ is a homomorphism of $\mathcal{RM}(T; I, \Lambda; Q)$ onto $\mathcal{RM}(S; I, \Lambda; P)$. It is idempotent separating if θ is idempotent separating.*

PROOF. Since $Q\theta = P$, it is immediate that ϕ is a homomorphism of $\mathcal{M}(T; I, \Lambda; Q)$ onto $\mathcal{M}(S; I, \Lambda; P)$ and hence maps $\mathcal{RM}(T; I, \Lambda; Q)$ into $\mathcal{RM}(S; I, \Lambda; P)$. Conversely, suppose that (i, s, λ) belongs to $\mathcal{RM}(S; I, \Lambda; P)$; then $s \in p_{\mu i}^{-1} S p_{\lambda i}^{-1}$ for some $j \in I, \lambda \in \Lambda$. Thus there exists $t \in q_{\mu i}^{-1} T q_{\lambda i}^{-1}$ such that $t\theta = s$. Then $(i, t, \lambda) \in \mathcal{RM}(S; I, \Lambda; P)$ and $(i, t, \lambda)\phi = (i, s, \lambda)$ so that ϕ is onto.

Suppose now that (i, x, λ) and (i, y, λ) are idempotents and that $(i, x, \lambda)\phi = (i, y, \lambda)\phi$. Then $x = xq_{\lambda i}x$, $y = yq_{\lambda i}y$ and $x\theta = y\theta$ together imply $(xq_{\lambda i})\theta = (yq_{\lambda i})\theta$ where these are idempotents, so that since θ is idempotent separating $xq_{\lambda i} = yq_{\lambda i}$. Dually $q_{\lambda i}x = q_{\lambda i}y$ and so we obtain the equalities $x = xq_{\lambda i}x = yq_{\lambda i}x = yq_{\lambda i}y = y$. Hence ϕ is idempotent separating.

THEOREM 3.5. *Let S be a regular semigroup. Then the following statements are equivalent*:

(i) *S is locally inverse*;

(ii) *S is a strictly compatible homomorphic image of a regular quasi-ideal in a classical Rees matrix semigroup over the semidirect product of a semilattice by a group*;

(iii) *S is a strictly compatible homomorphic image of a regular quasi-ideal in a classical Rees matrix semigroup over an E-unitary inverse monoid.*

PROOF. We shall show that (i) implies (iii), which clearly implies (ii). Finally, since each regular subsemigroup and homomorphic image of a locally inverse semigroup is itself a locally inverse semigrou, it is immediate that (ii) implies that (i) holds.

By Theorem 3.1, S is a locally isomorphic image of a regular Rees matrix semigroup $\mathscr{RM}(T; I, \Lambda; P)$ over an inverse semigroup T. By Lemma 3.3, there is an inverse semigroup U, an idempotent separating homomorphism θ of U onto T and a matrix Q over U such that $Q\theta = P$ and each member of $Q^{-1}QQ^{-1}$ is maximal in U. By Lemma 3.4, the homomorphism θ extends to a homomorphism of $\mathscr{RM}(U; I, \Lambda; Q)$ onto $\mathscr{RM}(T; I, \Lambda; P)$.

By Lemma 3.2, $\mathscr{RM}(U; I, \Lambda; Q)$ is a regular quasi-ideal in a classical Rees matrix semigroup $\mathscr{M}(V; I, \Lambda; L)$ over a factorizable inverse semigroup. This factorizable inverse semigroup, in turn, is an idempotent separating homomorphism image of a semidirect product of a semilattice by a group. The latter idempotent separating homomorphism extends, by Lemma 3.4, to give $\mathscr{M}(V; I, \Lambda; L)$ as an idempotent separating homomorphic image of a classical Rees matrix semigroup over a semidirect product of a semilattice by a group. Since the inverse image of a regular quasi-ideal is a regular quasi-ideal, it follows that $\mathscr{RM}(U; I, \Lambda; Q)$ is an idempotent separating homomorphic image of a regular quasi-ideal in a classical Rees matrix semigroup over a semidirect product of a semilattice by a group. Hence, since the composition of idempotent separating homomorphisms is idempotent separating, $\mathscr{RM}(T; I, \Lambda; P)$ is an idempotent separating homomorphic image of such a semigroup.

Finally, since S is a locally isomorphic image of the regular Rees matrix semigroup $\mathscr{RM}(T; I, \Lambda; P)$ and the composition of an idempotent separating homomorphism and a local isomorphism is strictly compatible, S is a strictly compatible homomorphic image of a regular quasi-ideal in a classical Rees matrix semigroup over the semidirect product of a semilattice by a group.

Nambooripad [14] defines a locally inverse semigroup to be *weakly E-unitary* if $x > e = e^2$ implies that x is idempotent. One can check that any classical Rees matrix semigroup over an E-unitary inverse semigroup is weakly E-unitary and has one-to-one structure mappings. That is, it is *proper* in the sense of [14]. Hence Theorem 3.5 shows that any locally inverse semigroup has a proper cover. This result is due to Pastijn [16]. Nambooripad and Veramony [14] have given a short proof of this based on Nambooripad's theory of inductive groupoids. Theorem 3.4 gives an alternative short proof based more directly on semigroup arguments.

A semigroup S is said to be an *elementary rectangular band* of inverse semigroups $S_{i\lambda}$, $i \in I, \lambda \in \Lambda$, if

(i) $S = \bigcup\{S_{i\lambda}: i \in I, \lambda \in \Lambda\}$

(ii) $S_{i\lambda}S_{j\mu} = S_{i\mu}$ for $i \in I, \lambda \in \Lambda$.

A classical Rees matrix semigroup over an inverse semigroup is easily seen to be an elementary rectangular band of inverse semigroups. Furthermore, it is easy to see that any idempotent separating homomorphic image of an elementary rectangular band of inverse semigroups is again an elementary rectangular band of inverse semigroups. In particular, any idempotent separating homomorphic image of an elementary rectangular band of inverse monoids is again an elementary rectangular band of inverse monoids. As such, it is, from [15; Theorem 4.1] or [11; Corollary 4.5] isomorphic to a classical Rees matrix semigroup over an inverse monoid.

The proof of Theorem 3.5 shows that any regular Rees matrix semigroup over an inverse semigroup is an idempotent separating homomorphic image of a regular quasi-ideal in a classical Rees matrix semigroup over an inverse monoid. Hence, Proposition 2.4 gives the following theorem which supplements Theorem 3.1.

THEOREM 3.6. *Let S be a regular semigroup. Then the following are equivalent*:
(i) *S is locally inverse*;
(ii) *S is a locally isomorphic image of a regular quasi-ideal in a classical Rees matrix semigroup over an inverse monoid*;
(iii) *S is a locally isomorphic image of a regular quasi-ideal in an elementary rectangular band of inverse monoids*;
(iv) *S is a locally isomorphic image of a regular quasi-ideal in an elementary rectangular band of inverse semigroups.*

Pastijn's main theorem in [16] gives an explicit form for the covering semigroup of which a given locally inverse semigroup is a strictly compatible image. We can obtain this result too by using the techniques introduced in the proofr of Theorem 3.5 if we take into account the fact that the regular Rees matrix semigroup produced by Theorem 3.1 is somewhat special.

Let us say that a regular Rees matrix semigroup $\mathscr{RM}(T; I, I; P)$ is *diagonally dominant* if p_{ii} is in a subgroup for each $i \in I$, and $p_{ij} \in p_{ii}Tp_{jj}$ for each $i, j \in I$. In this case, (i, x, j) is in $\mathscr{RM}(T; I, I; P)$ if and only if $x \in p_{ii}Sp_{jj}$. The proof of Theorem 3.1 shows that every locally inverse semigroup is a locally isomorphic image of a diagonally dominant regular Rees matrix semigroup over an inverse semigroup; indeed there each diagonal element of P is idempotent. The first part of the proof of Theorem 3.4 then shows that every locally inverse semigroup is a strictly compatible homomorphic image of a diagonally dominant regular Rees matrix semigroup over an E-unitary semigroup, which is a quasi-ideal in a semidirect product of a semilattice by a group, in which each element of the sandwich matrix is maximal. Thus we get the following result.

THEOREM 3.7. *Let S be a semidirect product of a semilattice with identity by a group and let I be a non-empty set. Let P be an $I \times I$ matrix over the group of units of S and for each $i \in I$ let e_i be an idempotent of S. Then the set of all triples*

https://doi.org/10.1017/S1446788700022175 Published online by Cambridge University Press

(i, s, j) in $I \times S \times I$ such that $s \in e_i S e_j$ is a locally inverse semigroup under the multiplication

$$(i, s, j)(u, t, v) = (i, sp_{ju}t, v).$$

Conversely, any locally inverse semigroup is a strictly compatible image of a semigroup constructed as above.

If we write S as a set of pairs (e, g) with e in the semilattice and g in the group then the multiplication in Theorem 3.7 becomes precisely that in Pastijn's theorem. Namely we have the following corollary which represents a slight strengthening and simplification of Pastijn's Theorem 2.16 in [16].

COROLLARY 3.8. *Let X be a semilattice with identity and let G be a group which acts on X by automorphisms, on the left. Let I be a non-empty set and let P be an $I \times I$ matrix over G and for each $i \in I$ let e_i be in X. Then he set of all quadruples $(i, a, g, j) \in I \times X \times G \times I$ with $a \leqslant e_i \wedge g^{-1}e_j$ forms a locally inverse semigroup under the multiplication*

$$(i, a, g, j)(u, b, h, v) = (i, a \wedge gp_{ju}b, gp_{ju}h, v).$$

Conversely each locally inverse semigroup is a strictly compatible homomorphic image of a semigroup constructed as above.

Corollary 3.9 which follows is a strengthened version of [16; Theorem 2.17]. The terminology is that of [16].

COROLLARY 3.9. *Every locally inverse semigroup is a strictly compatible homomorphic image of a regular quasi-ideal in the semidirect product of a semilattice by a completely simple semigroup.*

References

[1] S. Y. Chen and S. C. Hseih, 'Factorizable inverse semigroups', *Semigroup Forum* **8** (1974), 283–297.

[2] A. Coudron, 'Sur les extensions de demigroupes reciproques', *Bull. Soc. Roy. Sci. Liege* **37** (1968), 409–419.

[3] H. D'Alarcao, 'Idempotent separating extensions of inverse semigroups', *J. Austral. Math. Soc.* **9** (1969), 211–217.

[4] T. E. Hall, "On regular semigroups", *J. Algebra* **24** (1973), 1–24.

[5] T. E. Hall, "Some properties of local subsemigroups inherited by larger subsemigroups", *Semigroup Forum* **25** (1982), 35–48.

[6] T. E. Hall and P. R. Jones, "On the lattice of varieties of bands of groups", *Pacific J. Math.* **91** (1980), 327–337.

[7] D. B. McAlister, "Groups, semilattices and inverse semigroups, I", *Trans. Amer. Math. Soc.* **192** (1974), 227–244.
[8] D. B. McAlister, "Groups, semilattices and inverse semigroups, II", *Trans. Amer. Math. Soc.* **196** (1974), 351–370.
[9] D. B. McAlister, "Some covering and embedding theorems for inverse semigroups", *J. Austral. Math. Soc.* **22A** (1976), 188–211.
[10] D. B. McAlister, "Rees matrix covers for locally inverse semigroups", *Trans. Amer. Math. Soc.* **277** (1983), 727–738.
[11] D. B. McAlister and R. B. McFadden, "Regular semigroups with an inverse transversal as matrix semigroups", submitted to *Quart. J. Math. Oxford*.
[12] K. S. S. Nambooripad, "The structure of regular semigroups, I", *Memoirs of the Amer. Math. Soc.* **224** (1979).
[13] K. S. S. Nambooripad, "The natural partial order on a regular semigroup", *Proc. Edinburgh Math. Soc.* (2) **23** (1980), 249–260.
[14] K. S. S. Namboorpiad and R. Veramony, "Subdirect products of regular semigroups", *Semigroup Forum* **27** (1983), 265–308.
[15] F. Pastijn, "Rectangular bands of inverse semigroups", *Simon Stevin* **56** (1982), 3–95.
[16] F. Pastijn, "The structure of pseudo-inverse semigroups", *Trans. Amer. Math. Soc.* **273** (1982), 631–655.
[17] N. R. Reilly and W. D. Munn, "*E*-unitary congruences on inverse semigroups", *Glasgow Math. J.* **17** (1976), 57–75.

Department of Mathematical Sciences
Northern Illinois University
DeKalb, Illinois 60115
U.S.A.