

A THEOREM ON CONNECTED GRAPHS IN WHICH EVERY EDGE BELONGS TO A 1-FACTOR

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1. Introduction

In this paper, we consider factor covered graphs, which are defined basically as connected graphs in which every edge belongs to a 1-factor. The main theorem is that for any two edges e and e' of a factor covered graph, there is a cycle C passing through e and e' such that the edge set of C is the symmetric difference of two 1-factors.

2. Definitions

We adopt the definitions and notation of [1] with the following exceptions and additions. All graphs considered are connected. Let G be a graph, and let $e = \{v, w\}$ where $e \in E(G)$ and $v, w \in V(G)$. It follows that $G - e = G - \{v, w\}$. The contradictory definition of $G - e$ given in [1] is not used in this paper. We call v and w the *ends* of e . Cycles, paths and 1-factors are defined as in [1] except that they are each regarded as sets of edges rather than sequences or subgraphs. If C is a cycle, $V(C)$ is defined as the union of the edges of C . If P is a $v-w$ path, we call v and w the *terminal vertices* of P .

The symmetric difference $F_1 \Delta F_2$ of distinct 1-factors F_1 and F_2 is a set of disjoint cycles; these we call *alternating cycles*. Thus an alternating cycle can also be defined as a cycle A of even length such that $G - V(A)$ has a 1-factor.

A graph with at least four vertices is called *factor covered* if for any edge there is a 1-factor that contains that edge. Now let e be an arbitrary edge of the factor covered graph G . Since G is connected and $|V(G)| \geq 4$, there is another edge incident on one of the ends of e , and this edge must belong to some 1-factor F_1 . Therefore $e \notin F_1$. On the other hand e belongs to some 1-factor F_2 . Therefore $e \in F_1 \Delta F_2$, and e thus belongs to some alternating cycle of G . On the other hand, any edge that belongs to an alternating cycle certainly belongs to a 1-factor, and so we see that factor covered graphs are those connected graphs in which for any edge there is an alternating cycle containing that edge.

If A and B are distinct subsets of $E(G)$, we define an AB -path to be a path P of maximal length such that $P \subseteq A \cap B$. An $\bar{A}B$ -path is defined as a path P of maximal length such that $P \subseteq B - A$.

3. The theorem

We now prove that for any pair of edges of a factor covered graph, there is an alternating cycle containing those edges.

The proof of the theorem hinges on the following lemma. The author is indebted to P. W. Kasteleyn for greatly simplifying the original proof of this lemma.

LEMMA. *Let G be a connected graph with distinct 1-factors F_0, F_1, F_2 and alternating cycles A_1, A_2 such that $A_1 = F_0 \Delta F_1$, $A_2 = F_0 \Delta F_2$ and $E(G) = A_1 \cup A_2$.*

Then for any pair e, e' of edges of G there is an alternating cycle containing both e and e' .

PROOF. The proof is by induction on the number n of $\bar{A}_1 A_2$ -paths in G .

Suppose $n = 1$. If e and e' are both in A_1 or both in A_2 , the lemma is immediate; suppose therefore that $e \in A_2 - A_1$ and $e' \in A_1 - A_2$. Then $F_1 \Delta F_2$ is clearly an alternating cycle containing e and e' .

Now let the lemma be true whenever the number of $\bar{A}_1 A_2$ -paths is less than n . Again, if e and e' are both in A_1 or both in A_2 , the lemma is immediate; hence without loss of generality, let $e \in A_2 - A_1$ and $e' \in A_1 - A_2$.

Clearly $A_1 \Delta A_2 = F_1 \Delta F_2$, and $F_1 \Delta F_2$ is the union of a collection of disjoint alternating cycles; let C be the cycle in this collection which contains e . If $e' \in C$, the proof is complete. Suppose therefore $e' \notin C$.

Since $e' \in A_1 \Delta A_2$, we have $e' \in F_1 \Delta F_2$. Hence e' belongs to some alternating cycle C' which is a subset of $F_1 \Delta F_2$. If P is the $\bar{A}_2 A_1$ -path that contains e' , there must clearly be edges of $A_2 \cap A_1$ (and therefore edges of $A_2 - A_1$) incident on the terminal vertices of P because of the maximality property of $\bar{A}_2 A_1$ -paths. The edges of $A_2 - A_1$ incident on a terminal vertex of P must belong to C' and therefore are not in C since C and C' are disjoint; hence $A_2 - A_1 - C \neq \emptyset$. It clearly follows that there exist $\bar{A}_1 A_2$ -paths whose edge sets are contained in $E(G) - C$. Let G' be the graph obtained from G by deleting all non-terminal vertices of every such path.

Clearly A_1 and C are alternating cycles of G' since $F_1 \cap (C - A_1)$ and $F_1 \cap (A_1 - C)$ are 1-factors of $G' - V(A_1)$ and $G' - V(C)$ respectively. Furthermore, if for $i = 0, 1$ we define $F_i' = F_i \cap E(G')$, then F_i' is clearly a 1-factor of G' . In G' , we have $A_1 = F_1' \Delta F_0'$, and since $C = F_1' \Delta F_2'$ for some 1-factor F_2' of G' , the hypotheses of the lemma hold for alternating cycles A_1 and C in

G' . Since the number of A_1C -paths in G' is less than n , the lemma must hold for G' by the induction hypothesis. Therefore there is an alternating cycle A of G' that contains both e and e' .

Since A is an alternating cycle, $G' - V(A)$ has a 1-factor, F say. It follows that $F \cup (F_0 \cap (E(G) - E(G')))$ is a 1-factor of $G - V(A)$. Therefore A is an alternating cycle of G containing e and e' , and the lemma is proved.

We are now ready to prove the theorem.

THEOREM. *A graph G is factor covered if and only if for every $e, e' \in E(G)$ there exists an alternating cycle containing both e and e' .*

PROOF. The sufficiency of the given condition is clear, since any edge that belongs to an alternating cycle belongs to a 1-factor.

To prove necessity, let P be the shortest path in G containing both e and e' . P exists because G is connected. The proof is by induction on $|P|$. If $|P| = 1$, then $e = e'$ and the theorem follows from the fact that G is factor covered.

If $|P| = 2$, then e and e' are both incident on some vertex v . Let F and F' be 1-factors such that $e \in F$ and $e' \in F'$. Hence $e \notin F'$ and $e' \notin F$, so that $F \Delta F'$ contains an alternating cycle which includes both e and e' .

Suppose now that the theorem is true whenever $|P| \leq k$ for some integer $k > 1$, and let $|P| = k + 1$. Let e_k be the edge of $P - \{e'\}$ that has an end in common with e' . Let A_1 be an alternating cycle containing e and e_k and let A_2 be an alternating cycle containing e_k and e' ; A_1 and A_2 exist by the induction hypothesis. Let F_0 and F_1 be the 1-factors satisfying $A_1 = F_0 \Delta F_1$, where $e_k \in F_0$. If $e' \in F_1$, then $e' \in A_1$ and we are done; consequently suppose $e' \notin F_1$. Thus there is a 1-factor F_2 of G such that $F_2 \neq F_0, F_2 \neq F_1$ and $e' \in F_2$. Consequently we may choose $A_2 = F_0 \Delta F_2$.

Now let H be the graph with $V(H) = V(A_1) \cup V(A_2)$ and $E(H) = A_1 \cup A_2$. Thus $F_0 \cap E(H)$ is clearly a 1-factor of H , and so $F_0 \cap (E(G) - E(H))$ is a 1-factor of $G - V(H)$. A_1 is clearly an alternating cycle of H since it is of even length and $F_0 \cap (A_2 - A_1)$ is a 1-factor of $H - V(A_1)$. Similarly A_2 is an alternating cycle of H . Since $E(H) = A_1 \cup A_2$ and H is connected, by the lemma there is an alternating cycle A of H containing e and e' . Hence $H - V(A)$ has a 1-factor F . Thus $F \cup (F_0 \cap (E(G) - E(H)))$ is a 1-factor of $G - V(A)$, so that A is an alternating cycle of G . The theorem is proved.

Reference

- [1] M. Behzad, and G. Chartrand; *Introduction to the Theory of Graphs* (Allyn & Bacon, Boston, 1971).

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