REMARKS ON STABILITY CONDITIONS FOR THE DIFFERENTIAL EQUATION $x''+a(t)f(x)=0$ [1]

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Consider the following second order nonlinear differential equation:

(1) $$x''+a(t)f(x) = 0, \qquad t \in [0, \infty),$$

where $a(t) \in C^3[0, \infty)$ and $f(x)$ is a continuous function of x. We are here concerned with establishing sufficient conditions such that all solutions of (1) satisfy

(2) $$\lim_{t\to\infty} x(t) = 0.$$

Since $a(t)$ is differentiable and $f(x)$ is continuous, it is easy to see that all solutions of (1) are continuable throughout the entire non-negative real axis. It will be assumed throughout that the following conditions hold:

(A_1) $$\lim_{t\to\infty} a(t) = \infty,$$

(A_2) $$xf(x) > 0, \qquad x \neq 0,$$

(A_3) $$\lim_{|x|\to\infty} \left| \int_0^x f(u)du \right| = \infty,$$

(A_4) $$xf(x) \geqq 2\gamma \int_0^x f(u)du, \qquad \gamma > 0.$$

Our main results are the following two theorems:

THEOREM 1. *Let* $0 < \alpha < 1$. *If* $a(t)$ *satisfies*

(3) $$\lim_{T\to\infty} \int_{t_0}^{T} \frac{a'_-(t)}{a^\alpha(t)}\, dt < \infty,$$

where $a(t) > 0$, $t \geqq t_0$ *and* $a'_-(t) = \max\left(-a'(t), 0\right)$, *and*

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$$(4) \qquad \int_{t_0}^{T} |(a^{-\alpha}(t))'''|dt = 0\,(a^{1-\alpha}(T)), \qquad (T \to \infty),$$

then every solution of (1) *satisfies* (2).

THEOREM 2. *If* $a(t)$ *satisfies*

$$(5) \qquad \lim_{T\to\infty} \int_{t_0}^{T} \frac{a'_-(t)}{a(t)}\, dt < \infty,$$

where $a(t) > 0$ *for* $t \geqq t_0$, *and*

$$(6) \qquad \int_{t_0}^{T} |(a^{-1}(t))'''|dt = o\,(\log a(T)), \qquad (T \to \infty),$$

then every solution of (1) *satisfies* (2).

Define for each solution $x(t)$ of (1) the following energy function:

$$(7) \qquad V(t, x) = \frac{x'^2}{a(t)} + 2\int_0^x f(u)du,$$

which is clearly non-negative, on account of (A_2). Under assumptions (A_1), (A_2), (A_3) and (5), we can prove the following two propositions concerning solutions of (1).

LEMMA 1. $\lim_{t\to\infty} V(t, x)$ *exists and is finite.*

PROOF. A simple differentiation shows that

$$(8) \qquad V'(t, x) = -\frac{a'(t)}{a^2(t)}\, x'^2 \leqq \frac{a'_-(t)}{a^2(t)}\, x'^2 \leqq \frac{a'_-(t)}{a(t)}\, V(t, x),$$

from which it follows that

$$V(t, x) \leqq V(t_0, x_0) \exp\left(\int_{t_0}^{t} \frac{a'_-(s)}{a(s)}\, ds\right) \leqq M < \infty,$$

where the bound M depends upon $x_0 = x(t_0)$. Integrating the equality in (8), one finds

$$0 \leqq V(t, x) = V(t_0, x_0) + \int_{t_0}^{t} \frac{x'^2(s)}{a^2(s)}\, (a'_-(s) - a'_+(s))ds,$$

and hence

$$(9) \qquad \int_{t_0}^{t} \frac{x'^2(s)}{a(s)} \frac{a'_+(s)}{a(s)}\, ds \leqq V(t_0, x_0) + M \int_{t_0}^{t} \frac{a'_-(s)}{a(s)}\, ds < \infty.$$

From (9), we may conclude that

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$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{x'^2(s)}{a(s)}\frac{a'_+(s)}{a(s)}\,ds$$

exists. Thus,

$$\lim_{t\to\infty}V(t,x)=V(t_0,x_0)+\lim_{t\to\infty}\int_{t_0}^{t}\frac{x'^2(s)}{a^2(s)}\,(a'_+(s)-a'_-(s))\,ds$$

exists, and is finite.

LEMMA 2. *Every solution $x(t)$ of* (1) *is oscillatory,* i.e. *there exists a sequence $\{t_k\}$ such that $x(t_k)=0$, $k=0,1,2,3,\cdots$ and $t_k\to\infty$ as $k\to\infty$.*

PROOF. Let $x(t)$ be a non-oscillatory solution of (1). On account of (A_2), we may assume without loss of generality that $x(t)>0$ for $t\geqq t_0$. From (1), it follows that $x'(t)$ is non-increasing, and hence has a limit. If the limit is negative or $-\infty$, then $x(t)$ must eventually be negative which has been ruled out at the beginning. Thus we may assume that $x'(t)$ is eventually non-negative and so $x(t)$ is non-decreasing and has a limit c. If c is finite, then we may choose $T\geqq t_0$ such that $c/2\leqq x(t)\leqq c$ for $t\geqq T$. Denote

$$k=\inf_{c/2\leqq x\leqq c}f(x),\qquad 0<k<\infty.$$

Integrating (1), we have

$$x'(t)+\int_T^t a(s)f(x(s))ds=x'(T),$$

from which the desired contradiction follows. On the other hand, if $c=+\infty$, we multiply (1) through by $x'(t)$ and integrate to obtain:

$$\frac{x'^2(t)}{2}+\int_T^t a(s)f(x(s))x'(s)ds\leqq\frac{x'^2(T)}{2}. \tag{10}$$

We may assume that T is so chosen such that $a(t)\geqq 1$ for $t\geqq T$. Thus, (10) becomes

$$\frac{x'^2(t)}{2}+\int_{x(T)}^{x(t)}f(u)du\leqq\frac{x'^2(T)}{2}. \tag{11}$$

Letting t tend to infinity in (11), one easily obtains a contradiction to (A_2).

PROOF OF THEOREM 1. Let $x(t)$ be any non-trivial solution of (1) and $V(t,x)$ be defined by (7). Clearly (3) implies (5), so by Lemma 1, $\lim_{t\to\infty}V(t,x)=L$ exists. If $L=0$, then (2) clearly follows on account of (A_2). Now assume that $L>0$ for some solution $x(t)$ of (1). By Lemma 2, there exists an increasing sequence $\{t_k\}$ such that $t_k\to\infty$ as $k\to\infty$ and $x'(t_k)=0$, $k=0,1,2,3,\cdots$. Let $\varepsilon>0$, we choose $t_0\geqq 0$ such that

$$(12) \qquad a(t) > 0 \text{ and } (1-\varepsilon)L \leqq V(t, x) \leqq (1+\varepsilon)L,$$

for $t \geqq t_0$. Write $V(t) = V(t, x)$ for short and denote $\varphi = a^{-\alpha}$. A simple computation using (1) yields the following identity:

$$(13) \qquad \frac{d}{dt}\{\varphi a V + \tfrac{1}{2}\varphi'' x^2 - \varphi' x x'\} = \tfrac{1}{2}\varphi''' x^2 + 2(1-\alpha)a^{-\alpha}a' F(x) - \alpha a^{-\alpha} a' x f(x),$$

where

$$F(x) = \int_0^x f(u)du.$$

Integrating (13) from t_0 to t_k, we obtain

$$(14) \qquad \begin{aligned} a^{1-\alpha}(t_k)V(t_k) = {} & c_0 - \tfrac{1}{2}\varphi''(t_k)x^2(t_k) + \tfrac{1}{2}\int_{t_0}^{t_k} \varphi''' x^2 dt \\ & + 2(1-\alpha)\int_{t_0}^{t_k} a^{-\alpha} a' F(x) dt - \alpha \int_{t_0}^{t_k} a^{-\alpha} a' x f(x) dt, \end{aligned}$$

where $c_0 = a^{1-\alpha}(t_0)v(t_0) + \frac{1}{2}\varphi''(t_0)x^2(t_0)$. By Lemma 1 and (A_3), we conclude that every solution $x(t)$ is bounded, say $|x(t)| \leqq B$. Note that

$$|\varphi''(t_k)| \leqq |\varphi''(t_0)| + \int_{t_0}^{t_k} |\varphi'''| dt.$$

Denoting $\beta = \sup_{|x| \leqq B} x f(x)$ and using (A_4) and (12) in (14), we get

$$(15) \qquad \begin{aligned} a^{1-\alpha}(t_k)(1-\varepsilon)L \leqq {} & |c_1| + B^2 \int_{t_0}^{t_k} |\varphi'''| dt \\ & + a^{1-\alpha}(t_k)(1+\varepsilon)L \operatorname{Max}\left(1 - \frac{\alpha\gamma}{1-\alpha}, 0\right) \\ & + (2(1-\alpha)(1+\varepsilon)L - \alpha\beta) \int_{t_0}^{t_k} \frac{a'_-}{a^\alpha} dt, \end{aligned}$$

where c_1 is some appropriate constant. Using (3) and (4), we obtain from (15)

$$(1-\varepsilon) \leqq (1+\varepsilon) \operatorname{Max}\left(1 - \frac{\alpha\gamma}{1-\alpha}, 0\right) + o(1),$$

which produces the desired contradiction with any $\varepsilon > 0$ if $\alpha \geqq (\gamma+1)^{-1}$ and with $\varepsilon < \gamma\alpha(2(1-\alpha)-\alpha\gamma)^{-1}$ if $\alpha < (1+\gamma)^{-1}$.

Proof of Theorem 2. The general argument is similar to that of Theorem 1. Here instead of (13), we have the following identity:

$$\frac{a'_+}{a} x f(x) + \frac{d}{dt}\{V + \tfrac{1}{2}\varphi'' x^2 - \varphi' x x'\} = \tfrac{1}{2}\varphi''' x^2 + \frac{a'_-}{a} x f(x),$$

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from which we have the following inequality:

$$(16)\qquad \gamma\frac{a'}{a}V+\frac{d}{dt}\{(1+\gamma)V+\tfrac{1}{2}\varphi''x^2-\varphi'xx'\} = \tfrac{1}{2}\varphi'''x^2+\frac{a'}{a}\,(xf(x)-2\gamma F(x)).$$

Integrating (16) from t_0 to t_k, we obtain

$$(17)\qquad \gamma(1-\varepsilon)L\log a(t_k) \leqq |c_0|+B^2\int_{t_0}^{t_k}|\varphi'''|\,dt,$$

where c_0 is some appropriate integration constant. Using (5) and (6), one easily derives a contradiction from (17).

REMARK 1. Theorem 1 is a nonlinear extension of some stability conditions recently obtained for the linear equation:

$$(18)\qquad x''+a(t)x = 0.$$

However, even in the special case of equation (18), Theorem 1 is an improvement over its predecessors where it is assumed that $a'(t) \geqq 0$ instead of (3), (cf. Meir, Willett and Wong [3] and an independent result for the case $\frac{1}{2} \leqq \alpha < 1$ by Chang [1].) The assumption (3), or its stronger substitute that $a'(t) \geqq 0$, is essential here and in [3] as compared to the result of Lazer [2] where no such assumption is made.

REMARK 2. Assumptions (A_3) and (A_4) are easily realized if $f(x)$ is non-decreasing in x. As typical examples, one may take $f(x) = x^\lambda$, where λ is the quotient of two odd integers and $\lambda > 0$, or take

$$f(x) = \begin{cases} x & |x| \leqq 1,\\ \dfrac{x}{|x|^\mu} & |x| > 1,\end{cases}$$

with $1 \leqq \mu < 2$.

REMARK 3. It is easily verified that the elementary functions $a(t) = t^\sigma$, $\sigma > 0$, e^t, and $\log t$ satisfy both (4) and (6). An example is given in [3] which satisfies (6) but not (4).

REMARK 4. Results on asymptotic properties of solutions of (1) may be transferred to the following slightly more general equation:

$$(19)\qquad (p(t)x')'+q(t)f(x) = 0, \qquad p(t) > 0,$$

by standard Louiville transformations. The transformation necessary depends on the convergence and divergence of the integral

$$\int^{\infty} \frac{dt}{p(t)}.$$

In case

$$\int^{\infty} \frac{dt}{p(t)} = \infty,$$

we let

$$s = \int^{t} \frac{d\tau}{p(\tau)}$$

and $y(s) = x(t)$ and transform (19) into:

$$\frac{d^2y}{ds^2} + p(t)q(t)f(y) = 0,$$

which is of the form of equation (1). On the other hand, if

$$\int^{\infty} \frac{dt}{p(t)} < \infty,$$

we let

$$s = \left(\int_t^{\infty} \frac{d\tau}{p(\tau)}\right)^{-1} \quad \text{and} \quad y(s) = x(t)\left(\int_t^{\infty} \frac{d\tau}{p(\tau)}\right)$$

and transform (19) into

$$\frac{d^2y}{ds^2} + \frac{p(t)q(t)}{s^4} f(y) = 0,$$

which is again of the form of equation (1). To preserve asymptotic properties under Louiville transformations, it is essential here that s tends to infinity as t does.

REMARK 5. Finally, we note that the present hypothesis does not imply that equation (1) is globally asymptotically stable, i.e. all solutions and their derivatives tend to zero as t tends to infinity. In fact, the interesting fact is that every non-trivial solution $x(t)$ of (1) satisfies

(20) $$\limsup_{t\to\infty} |x'(t)| > 0.$$

To see this, define an energy-like function $W(t, x)$ as follows

(21) $$W(t, x) = x'^2 + 2a(t)\int_0^x f(u)du.$$

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Using (1), we obtain

$$\begin{aligned} W'(t, x) &= 2a'(t)\int_0^x f(u)du \\ &\geqq -a'_-(t)2\int_0^x f(u)du \\ &\geqq -\frac{a'_-(t)}{a(t)}W(t, x), \end{aligned}$$

from which it follows that

$$W(t, x) \geqq W(t_0, x_0)\exp\left(-\int_{t_0}^t \frac{a'_-(\tau)}{a(\tau)}d\tau\right). \tag{22}$$

Since for every non-trivial solution we must have $W(t_0, x_0) > 0$, (22) yields $W(t, x) \geqq \zeta^2 > 0$ for all t. Let $\{t_k\}$ be the sequence of zeros of $x(t)$ such that $t_k \to \infty$. We have from (21) that $|x'(t_k)| \geqq \zeta > 0$ for all k, and in particular (20) holds.

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