1

Complemented Subspaces of Banach Spaces

1.1 Banach and Quasi-Banach Spaces

A quasinorm on a real or complex vector space X is a map $\|\cdot\|\colon X \longrightarrow \mathbb{R}^+$ that satisfies

- $\|x\| = 0 \implies x = 0$,
- $\|\lambda x\| = |\lambda|\|x\|$,
- $\|x + y\| \le \Delta(\|x\| + \|y\|)$,

for all $x, y \in X$, all scalars λ and some constant $\Delta \ge 1$, called the *modulus of concavity of the quasinorm*. If $\Delta = 1$, then $\|\cdot\|$ is a norm. A quasinorm induces a linear topology on the underlying space: the coarsest linear topology for which the unit ball $B_X = \{x \in X : \|x\| \le 1\}$ is a neighbourhood of the origin. A quasinormed space is a vector space equipped with a quasinorm; when the space is moreover complete, it is called a *quasi-Banach space*. If the quasinorm is a norm then it will be called a *normed space* and, if it is complete, a *Banach space*. An operator is a linear continuous map. Topologies induced by quasinorms are so uncomplicated that a linear map $u\colon X \longrightarrow Y$ acting between quasinormed spaces is continuous if and only if it is bounded, in the sense that

$$\|u\| = \sup_{\|x\| \le 1} \|u(x)\| < \infty. \tag{1.1}$$

The space of all operators from X to Y is denoted by $\mathfrak{L}(X, Y)$ and is a quasi-normed space endowed with the quasinorm (1.1). It is complete (or p-normed, see later) when Y is. The space $\mathfrak{L}(X, \mathbb{K})$ is the *dual* space of X, usually denoted by X^*, and is always a Banach space. We say that X has *separating dual* if, for every non-zero $x \in X$, there is $x^* \in X^*$ such that $\langle x^*, x\rangle \ne 0$. The space is said to have *trivial dual* if $X^* = 0$.

Why do we bother the reader with quasinorms instead of using just plain norms? Because one must face right from the start the awful truth: it is not

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possible to make a serious study of twisted sums of Banach spaces without considering quasinorms. That is simply the way things are. It will be explained at length through the book, but a motivating example can be given now: if everything one knows about a space Z is that it contains a Banach subspace Y such that Z/Y is a Banach space, it may simply happen that Z is not locally convex, but, even if it is, there is no way to specify a particular norm on Z, while it could be feasible to describe a quasinorm on it. Moreover, the necessity to distinguish quasinorms from the far more popular norms is because $\Delta > 1$ and has, in practice, side effects: the unit ball is no longer convex, and thus the Hahn–Banach theorem ceases to work, up to the point that a quasi-Banach space might have trivial dual or that a quasinorm is not necessarily continuous with respect to its own topology. In spite of these facts, there is no need to panic: the open mapping, closed graph and Banach–Steinhaus theorems work perfectly well on quasi-Banach spaces. Moreover, a simple remedy to correct the possible discontinuity of the quasinorm is to work with p-norms, $0 < p \leq 1$, which are quasinorms satisfying the additional inequality

- $\|x + y\|^p \leq \|x\|^p + \|y\|^p$.

Obviously p-norms are q-norms for $0 < q < p$. A p-norm satisfies the inequality $\big|\|x\|^p - \|y\|^p\big| \leq \|x - y\|^p$, and this makes it continuous. Even better than that, quasinorms can be judiciously replaced by p-norms:

1.1.1 Aoki–Rolewicz Theorem *Each quasinorm is equivalent to some p-norm.*

Indeed, if $\|\cdot\|$ has modulus of concavity Δ and $2^{1/p-1} = \Delta$ then

$$\|x\|_{(p)} = \inf\left\{\left(\sum_{1\leq i\leq n} \|x_i\|^p\right)^{1/p} : x = \sum_{1\leq i\leq n} x_i\right\} \tag{1.2}$$

defines a p-norm on X such that $\|\cdot\|_{(p)} \leq \|\cdot\| \leq 2\Delta\|\cdot\|_{(p)}$. This is somehow optimal since an elementary computation reveals that the modulus of concavity of a p-norm is at most $2^{1/p-1}$. There is moreover an effective way to detect when a quasinorm is equivalent to a p-norm for a pre-established p:

Lemma 1.1.2 *Let $0 < p \leq 1$. A quasinormed space X is isomorphic to a p-normed space if and only if there is a constant C such that, for finitely many $x_i \in X$, one has*

$$\Big\|\sum_i x_i\Big\| \leq C\left(\sum_i \|x_i\|^p\right)^{1/p}. \tag{1.3}$$

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Proof It is clear that X is isomorphic to a p-normed space precisely when there is a p-norm $|\cdot|$ on X such that $|\cdot| \le \|\cdot\| \le C|\cdot|$. If this holds, then given finitely many $x_1, \ldots, x_n \in X$, one has

$$\Big\| \sum_{1\le i\le n} x_i \Big\| \le C \Big| \sum_{1\le i\le n} x_i \Big| \le C \left(\sum_{1\le i\le n} |x_i|^p \right)^{1/p} \le C \left(\sum_{1\le i\le n} \|x_i\|^p \right)^{1/p}.$$

As for the converse, if (1.3) holds then the functional (1.2) is a p-norm satisfying $\|\cdot\|_{(p)} \le \|\cdot\| \le C\|\cdot\|_{(p)}$. □

A p-norm defines an invariant metric by the formula $d(x,y) = \|x-y\|^p$, and thus the Aoki–Rolewicz theorem implies that quasinormed spaces are metrisable. The absolute summability criterion for completeness of a p-norm is:

1.1.3 *Let X be a p-normed space. Then X is complete if and only if every sequence (x_n) such that $\sum_{n\ge1} \|x_n\|^p < \infty$ is summable in X.*

The Completion

The Aoki–Rolewicz theorem places quasinormed spaces among metric linear spaces; thus quasinormed spaces can be completed. Quasinorms, however, can be rather nasty functions for which there can be no natural extension to the completion. Thus, perhaps the simplest way to construct a completion $\kappa\colon X \longrightarrow \widehat{X}$ of a quasinormed space X is to first put an equivalent p-norm to then use the induced metric to construct a completion via equivalent classes of Cauchy sequences. The operations and the p-norm extend by uniform continuity. If for some reason we want to keep the original quasinorm on X, we can do so. Completions, like many other universal constructions in this book, come with a universal property: every operator u from X into a complete space Y factors through the inclusion $\kappa\colon X \longrightarrow \widehat{X}$ as $u = \widehat{u}\kappa$ with $\|\widehat{u}\| = \|u\|$ in the form

$$\begin{array}{ccc} X & \xrightarrow{\;\;\kappa\;\;} & \widehat{X} \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle \widehat{u}} & \\ & Y & \end{array} \tag{1.4}$$

The p-Banach Envelope

The proof of Lemma 1.1.2 suggests the following construction. Given a quasinormed space X and $p \in (0,1]$, the formula (1.2) defines a positively homogeneous and p-subadditive function such that $\|\cdot\|_{(p)} \le \|\cdot\|$. If $N_{(p)} = \{x \in X\colon \|x\|_{(p)} = 0\}$, then $\|\cdot\|_{(p)}$ becomes a genuine p-norm on $X/N_{(p)}$. The completion of $X/N_{(p)}$ with the natural extension of $\|\cdot\|_{(p)}$ is a p-Banach space called the *p-Banach envelope* of X and is denoted $X_{(p)}$. The universal property

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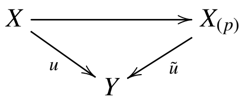
that corresponds to p-Banach envelopes is that every operator $u\colon X \longrightarrow Y$ from X into a p-Banach space Y factors as (the unlabeled arrow is the obvious map)



and $\|\tilde{u}\| = \|u\|$. The 1-Banach envelope is called the *Banach envelope*. It should be clear that the dual of a quasinormed space and that of its Banach envelope coincide so that $X_{(1)} = 0$ if and only if X has trivial dual. A good account of Banach envelopes is [289].

Some Fundamental Examples

Given a σ-finite measure space (S, μ), we write $L_0(\mu)$ for the space of real or complex measurable functions on S modulo almost everywhere equality. The space $L_0(\mu)$ comes with the topology of convergence in measure on sets of finite measure, for which a typical neighbourhood of zero is

$$\{f \in L_0(\mu) : \mu\{s \in A : |f(s)| > \varepsilon\} < \varepsilon\},$$

where $\mu(A) < \infty$ and $\varepsilon > 0$. This space, mostly used as an ambient space, is not locally bounded (unless S consists of a finite number of atoms) nor locally convex (unless μ is purely atomic); see 1.8.2.

1.1.4 By a (quasinormed) function space, we will mean a linear subspace $X \subset L_0(\mu)$ equipped with a quasinorm $\|\cdot\|$ such that

- if A has finite measure, then the characteristic function 1_A belongs to X,
- if $g \in X$ and $f \in L_0(\mu)$ is such that $|f| \leq g$, then $f \in X$ and $\|f\| \leq \|g\|$,
- the inclusion of X into $L_0(\mu)$ is continuous.

If X is complete, we call it a quasi-Banach function space.

The simplest such spaces are the Lebesgue spaces $L_p(\mu)$ of p-integrable functions quasinormed, when $0 < p < \infty$, with

$$\|f\|_p = \left(\int_S |f|^p d\mu\right)^{1/p}$$

and the space $L_\infty(\mu)$ of essentially bounded measurable functions endowed with the essential supremum norm. These are Banach spaces for $p \geq 1$ and quasi-Banach (actually p-Banach) spaces when $0 < p < 1$. When μ is a Lebesgue measure on $[0, 1]$, we omit it, and we just write L_p. The Hardy classes

H_p, closely related to the spaces L_p, are thoroughly studied in Duren's classic [163]. The space H_p is the space of analytic functions $f\colon \mathbb{D} \longrightarrow \mathbb{C}$ such that $\|f\|_p = \sup_{0<r<1} M(f,r) < \infty$, where

$$M(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p}.$$

If $f \in H_p$, then the limit $f^*(e^{it}) = \lim_{r\to 1} f(re^{it})$ exists for almost all $t \in (0, 2\pi]$ and $M(f,r)$ is increasing on $r \in (0,1)$. The mapping $f \longmapsto f^*$ defines an isometry between H_p and the closed subspace of $L_p(\mathbb{T})$ spanned by the exponentials e^{int} for $n = 0, 1, 2, \ldots$

The sequence spaces ℓ_p and the Schatten classes S_p have separating dual for all p, and the same is true for H_p since an obvious application of Cauchy's integral formula shows that for every $z \in \mathbb{D}$ and every $f \in H_p$, we have

$$|f(z)| \le 2^{1/p}(1-|z|)^{-p}\|f\|_p. \tag{1.5}$$

When a quasi-Banach space X has non-trivial dual, as is the case if X is a normed space, then $\mathfrak{L}(X,Y) \neq 0$ for all quasinormed spaces (or topological vector spaces) Y. This is no longer true when X is a quasi-Banach space:

1.1.5 *If* $0 < p < 1$, *then* $\mathfrak{L}(L_p, Y) = 0$ *for all* q*-normed spaces* Y *with* $q > p$.

The proof can be made easy: let Y be a q-normed space, $0 < p < q \le 1$, and let $u\colon L_p \longrightarrow Y$ be an operator. If f is a norm 1 element of L_p then, by the intermediate value theorem, there is $a \in (0,1)$ such that

$$\int_0^a |f|^p = \int_a^1 |f|^p = \frac{1}{2}.$$

Set $g = 1_{[0,a)}f$, $h = 1_{[a,1]}f$. Then $\|g\| = \|h\| = 2^{-1/p}$ and $f = g + h$, so

$$\|uf\| = \|ug + uh\| \le \left(\|ug\|^q + \|uh\|^q\right)^{1/q} \le \|u\|(2^{-q/p} + 2^{-q/p}) = 2^{1-q/p}\|u\|.$$

As f is arbitrary, this implies that $\|u\| \le 2^{1-q/p}\|u\|$, which is possible only if $\|u\| = 0$. The same argument works replacing L_p by any vector valued $L_p(X)$, no matter the quasi-Banach space X one considers.

1.2 Complemented Subspaces

A projection P on a quasinormed space X is an idempotent of $\mathfrak{L}(X)$, i.e., an operator $P\colon X \longrightarrow X$ such that $P^2 = P$. A closed subspace $Y \subset X$ is complemented if there exists a projection on X whose range is Y, usually called a projection onto Y. It is clear that a projection of X onto Y is just an extension of the identity $\mathbf{1}_Y$ to X.

Lemma 1.2.1 *A closed subspace Y of a quasi-Banach space X is complemented if and only if there exists a closed subspace V of X such that $Y \cap V = 0$ and $Y + V = X$.*

Proof If a projection P as above exists then $\ker P$ is closed, and letting $V = \ker P$, we have $X = Y \oplus V$. Conversely, if such a V exists, every $x \in X$ can be decomposed as $x = y + v$ with $y \in Y$ and $v \in V$ in a unique way, and we can define a mapping $P\colon X \longrightarrow X$ taking $y = P(x)$. This map is linear and idempotent and satisfies $P[X] = Y$. It is continuous since it has a closed graph. □

The subspace V in the lemma is sometimes called a *complement* of Y, which explains the terminology. It is clear that all complements of a given subspace are isomorphic as they are isomorphic to the quotient space X/Y. A quantified version of the notion of complemented subspace can be defined by means of the quasinorm of the projection: we say that Y is λ-complemented in X if there is a projection of X onto Y whose quasinorm is at most λ. The relative projection constant of Y in X is defined as $\lambda(Y, X) = \inf\{\|P\| : P \text{ is a projection of } X \text{ onto } Y\}$. While the finite-dimensional subspaces of a Banach space are all complemented, by the Hahn–Banach theorem, that is not always the case for quasi-Banach spaces, and indeed 1.1.5 shows that no finite-dimensional subspace of L_p can be complemented when $0 < p < 1$.

Pełczyński's Decomposition Method and Minimality

Pełczyński's decomposition method appears in [376] to prove that all infinite-dimensional complemented subspaces of ℓ_p for $p \in [1, \infty)$ or c_0 are isomorphic to the whole space. We could not resist giving its proof.

Proposition 1.2.2 *Let X and Y be quasi-Banach spaces, each of them isomorphic to a complemented subspace of the other. Assume that either both X and Y are isomorphic to their squares or that Y is isomorphic to $\ell_p(Y)$ for some $0 < p < \infty$ or to $c_0(Y)$. Then, $X \simeq Y$.*

Proof The first case is as follows: let V be a complement of X in Y. Then

$$Y \simeq X \times V \simeq X \times X \times V \simeq Y \times X$$

and, for the same reason, $X \simeq X \times Y$. The second case is similar. We will give the proof for ℓ_p; the case of c_0 is analogous. First note that $Y \simeq \ell_p(Y) \simeq Y^2$.

Thus, as before, $X \simeq Y \times X$. Let V be a complement of X in Y such that $Y \simeq X \times V$. Then,

$$X \simeq Y \times X \simeq \ell_p(Y) \times X \simeq \ell_p(X \times V) \times X \simeq \ell_p(X) \times \ell_p(V) \simeq \ell_p(Y) \simeq Y. \quad \square$$

An example of Gowers [195] shows that the two hypotheses $X \simeq X^2$ and $Y \simeq Y^2$ cannot be simultaneously dropped. Now, a gliding hump argument shows that if X is one of the spaces c_0 or ℓ_p for $0 < p < \infty$, then every infinite-dimensional closed subspace of X contains a further subspace spanned by a basic sequence equivalent to a sequence of blocks of the canonical basis, and thus is isomorphic to the whole space. If, moreover, $1 \leq p < \infty$, that subspace can be taken as complemented. This shows that a complemented subspace of ℓ_p with $1 \leq p < \infty$ or c_0 is either finite-dimensional or isomorphic to the whole space. The result is also true for $p = \infty$, but for quite different reasons (see the discussion after 1.6.3). It is also true for $0 < p < 1$ because each *complemented* subspace (but not any subspace, see later) of ℓ_p contains a further complemented copy of ℓ_p; cf. [443; 444]. The non-separable versions (see the later proposition) also hold: the case $p = 1$ was obtained in [304] (see also [411, Corollary, p. 29] and [175]), the case $c_0(I)$ in [199], the case $0 < p < 1$ in [372] (notice that here the non-separable version holds, while the separable version does not: see the comment after Corollary 2.7.4) and the case $1 < p < \infty$ in [406].

Proposition 1.2.3 *Each complemented subspace of either $\ell_p(I)$, $0 < p < \infty$ or $c_0(I)$ is isomorphic to some $\ell_p(J)$ or $c_0(J)$. Each complemented subspace of ℓ_∞ is either finite-dimensional or isomorphic to ℓ_∞.*

There exist complemented subspaces of $\ell_\infty(I)$ that are not isomorphic to any $\ell_\infty(J)$ [413]; see also [22].

Definition 1.2.4 A quasi-Banach space X is said to be minimal (resp. complementably minimal) if each of its closed infinite-dimensional subspaces contains a copy of X (resp. a copy of X complemented in X). We say that X is prime if all its infinite-dimensional complemented subspaces are isomorphic to X and primary if $X = A \oplus B$ implies that either $A \simeq X$ or $B \simeq X$.

Proposition 1.2.5 *If X is a minimal Banach space then there is $C \geq 1$ such that every closed infinite-dimensional subspace of X contains a subspace that is C-isomorphic to X.*

Proof Throughout this proof, *subspace* means closed infinite-dimensional subspace. If no such C exists, fix $f \in \mathbb{R}^{\mathbb{N}}$ and obtain a decreasing sequence $Y_1 \supset Y_2 \supset \cdots$ such that X does not $f(n)$-embed into Y_n. Pick $(y_n)_{n \geq 1}$ a basic

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sequence in X with $y_n \in Y_n$, and let $E_n = \overline{[y_n, y_{n+1}, \dots]}$. There is a subspace $Z \subset E_1$ and a $C > 1$ such that Z is C-isomorphic to X and for which we may assume $\dim E_1/Z = \infty$. Set $H_n = Z \cap E_n$, and choose $H_n \subset F_n \subset E_n$ such that $k = \dim E_n/F_n = \dim Z/H_n \leq n$. Thus, for some k-dimensional $A \subset E_n$ and $B \subset Z$, we have that E_n is (at most) $(1 + \sqrt{n})$-isomorphic to $A \oplus_1 F_n$ and $\overline{[H_n + B]}$ is (at most) $(1 + \sqrt{n})$-isomorphic to $B \oplus_1 H_n$. Since A and B are (at most) n-isomorphic, it follows that $A \oplus_1 H_n$ is at most $n(1+\sqrt{n})^2$-isomorphic to Z, hence $Cn(1+\sqrt{n})^2$-isomorphic to X. And since $\overline{[H_n + B]} \subset E_n$, it turns out that X does actually $Cn(1+\sqrt{n})^2$-embed in E_n, in flagrant contradiction of the assumption that X does not $f(n)$-embed into Y_n for, say $f(n) = n^2(1+\sqrt{n})^2$. □

Thus, ℓ_p spaces, $0 < p < \infty$, and c_0 are prime. It is an open problem whether L_p spaces are prime for $0 < p < 1$: Kalton proved in [250] that they are primary and that there exists, up to isomorphisms, at most one complemented subspace of L_p other than L_p itself [253]; see also [256, Section 3]. The list of minimal or complementably minimal spaces is not long. All ℓ_p spaces, $0 < p < \infty$, and c_0 are minimal, and complementably minimal when $1 \leq p < \infty$ or c_0. The spaces ℓ_p for $p \in (0, 1)$ are not complementably minimal [397]. Schlumprecht's arbitrarily distortable space [426; 13] and its dual [87] are complementably minimal, as are its superreflexive variations [87] and their duals. Tsirelson's space T, or its dual, are not complementably minimal [88, pp. 54–59], but its p-convexified version T_p, $1 < p < \infty$, is complementably minimal [88]. It is obvious that every minimal space must be separable. A bit less clear is that minimal spaces must be subspaces of spaces with unconditional basis: by the Gowers dichotomy [196], a Banach space X contains either a subspace with unconditional basis or an H.I. subspace, in which case it cannot be isomorphic to any proper subspace. Since a space with unconditional basis contains either c_0, ℓ_1 or a reflexive subspace [334, Theorem 1.c.12 (a)], a minimal space must be either reflexive or a subspace of c_0 or ℓ_1. Thus, a complementably minimal space must be c_0, ℓ_1 or reflexive.

1.3 Uncomplemented Subspaces

It is a basic fact in functional analysis that each closed subspace of a Hilbert space has a complement, namely the *orthogonal* complement. A classical result by Lindenstrauss and Tzafriri [333] provides the converse: a Banach space all of whose subspaces are complemented is isomorphic to a Hilbert space. Thus, each non-Hilbert infinite-dimensional Banach space contains uncomplemented subspaces. The introduction of the nicely written paper

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[396] contains an historical account of the first discovered uncomplemented subspaces. The minimality of the spaces ℓ_p allows us to more or less easily locate some uncomplemented subspaces of ℓ_p: all subspaces not isomorphic to ℓ_p itself. For instance, if $Q\colon \ell_p \longrightarrow X$ is a quotient map onto an infinite-dimensional space X not isomorphic to ℓ_p then $\ker Q$ is uncomplemented in ℓ_p, since any complement of $\ker Q$ should be isomorphic to X and also complemented, which is impossible. Still, further insight into properties of ℓ_p spaces is needed to obtain such quotients. But it can be done. Worse yet, uncomplemented subspaces of ℓ_p exist and can even be isomorphic to ℓ_p:

Proposition 1.3.1 *For each $p \in (0, \infty)$ different from 2, the space ℓ_p contains an uncomplemented subspace isomorphic to ℓ_p.*

This result is local in nature, in the sense that it depends upon proving that for every $0 < p < \infty$, $p \neq 2$, there is a sequence of subspaces $E_n \subset \ell_p^n$ such that

- $\ell_p(\mathbb{N}, E_n)$ is isomorphic to ℓ_p,
- $\lambda(E_n, \ell_p^n) \longrightarrow \infty$ as $n \longrightarrow \infty$.

The uncomplemented copy of ℓ_p inside ℓ_p is thus $\ell_p(\mathbb{N}, E_n)$ inside $\ell_p(\mathbb{N}, \ell_p^n)$. Let us tell the story as p decreases. The case $2 < p < \infty$ is solved by Rosenthal in [417] finding subspaces E_n that are badly complemented but still uniformly isomorphic to the ℓ_p space of the corresponding dimension. Bennett, Dor, Goodman, Johnson and Newman [37] solved the case $1 < p < 2$ locating badly complemented subspaces of ℓ_p^n uniformly isomorphic to the corresponding Euclidean space. Then $\ell_p(\mathbb{N}, E_n) \simeq \ell_p(\mathbb{N}, \ell_2^k) \simeq \ell_p$, by Pełczyński's decomposition method (see later). In fact, Rosenthal had previously settled the case $1 < p < \frac{4}{3}$ in [409] using harmonic ideas of Rudin. The case $p = 1$, by far the most difficult and least understood of all, resisted until Bourgain's paper [48]; see also Section 2.2. Curiously enough, the result is almost trivial for $0 < p < 1$, since in this case one can take E_n of dimension 1! Indeed, let E_n be the line spanned by $s_n = \sum_{i=1}^n e_i$ in ℓ_p^n. Any projection of ℓ_p^n onto E_n has the form $P(x) = f(x)s_n$, where f is a linear functional on ℓ_p^n such that $f(s_n) = 1$. Clearly, $\|P\| = \|f\|\|s_n\|$, and the minimum is attained when $f(x) = n^{-1}\sum_{i=1}^n x(i)$, which gives $\|P\| = n^{1/p-1}$. A more sophisticated argument of Stiles [443, Theorem 2.3] produces a subspace X of ℓ_p isometric to ℓ_p without infinite-dimensional subspaces complemented in ℓ_p. The case of c_0 stands apart: every copy of c_0 inside c_0 is complemented by Sobczyk's theorem (Section 1.7), and quotients of c_0 contain c_0 complemented [334]:

Proposition 1.3.2 *Every quotient of c_0 is isomorphic to a subspace of c_0.*

If $(F_n)_n$ is a sequence of finite-dimensional Banach spaces increasingly badly complemented in ℓ_∞^n then $c_0(\mathbb{N}, F_n)$ cannot be isomorphic to c_0. Deciding the converse is an open question: if $c_0(\mathbb{N}, F_n) \simeq c_0$, must the F_n be uniformly isomorphic to $\ell_\infty^{\dim F_n}$?

The space c_0 is not complemented in its bidual ℓ_∞. Phillips proved it in [386] for c and Sobczyck in [439] for c_0 itself:

Proposition 1.3.3 *c_0 is not complemented in ℓ_∞.*

Proof A weak*-null sequence of extensions of the coordinate functionals must have weak null restrictions to c_0. The Schur property of ℓ_1 makes them norm null, and that is impossible. □

Indecomposable and H.I. Spaces

While the only complemented subspaces in a prime space are copies of the space itself, there is a much more extreme way of having very few complemented subspaces: not having complemented subspaces at all.

Definition 1.3.4 A Banach space X is said to be indecomposable if, whenever $X = A \oplus B$, either A or B is finite-dimensional. A Banach space is said to be *hereditarily indecomposable* (H.I.) if every subspace is indecomposable.

H.I. Banach spaces exist [197], can be uniformly convex [169] or $\mathscr{L}_\infty$-spaces [17] and solve Banach's unconditional basis problem: does every Banach space contain an unconditional basic sequence? No: H.I. spaces do not. Actually, the Gowers dichotomy theorem [196] states that every Banach space contains either an H.I. subspace or an unconditional basic sequence. Thus, H.I. spaces are deeply entwined with the structure of general Banach spaces and can no longer be regarded as an anecdotal pathology. An H.I. space cannot be isomorphic to any proper subspace: actually, every operator from an H.I. space to any of its proper subspaces must be strictly singular [197, §4 theorem and corollary]. Indecomposable spaces isomorphic to their hyperplanes (which also exist [198]) must be prime: indeed, their infinite-dimensional complemented subspaces are finite-codimensional, and a space isomorphic to its hyperplanes is also isomorphic to its finite-codimensional subspaces. Indecomposable spaces can be $C(K)$-spaces [297], large [299] and also arbitrarily large [301], while H.I. spaces must be subspaces of ℓ_∞ [396], hence they have at most the dimension of the continuum. Abandon Banach spaces and you will find stranger things: rigid spaces (whose only endomorphisms are the scalar multiples of the identity; see [404; 449] for surprising news), spaces without basic sequences (see Section 9.4.4) and so on.

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1.4 Local Properties and Techniques

The so-called local theory studies the structure of (quasi-) Banach spaces by means of their finite-dimensional subspaces. Very often the asymptotic behaviour of quantitative information is the key. Of paramount importance is the:

1.4.1 Principle of Local Reflexivity *Let X be a Banach space, F a finite-dimensional subspace of X^{**}, G a finite-dimensional subspace of X^* and $\varepsilon > 0$. Then there is an operator $T: F \to X$ such that*

- $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$,
- $Tx = x$ *for every* $x \in F \cap X$,
- $\langle x^{**}, x^* \rangle = \langle x^*, Tx^{**} \rangle$ *for every* $x^* \in G$ *and every* $x^{**} \in F$.

The reader is referred to [227, § 9, p. 53] or [5, § 12.2] for proofs that have been refined over the years.

The $\mathscr{L}_p$-Spaces and Related Classes

Definition 1.4.2 Let $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. An infinite-dimensional Banach space X is said to be an $\mathscr{L}_{p,\lambda}$-space if every finite-dimensional subspace of X is contained in another finite-dimensional subspace of X whose Banach–Mazur distance to the corresponding ℓ_p^n is at most λ. A space X is said to be an $\mathscr{L}_p$-space if it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geq 1$.

$\mathscr{L}_p$-spaces can be considered the local version of $L_p(\mu)$-spaces and $\mathscr{L}_\infty$-spaces the local version of $C(K)$-spaces. An infinite-dimensional Banach space is an $\mathscr{L}_{\infty,1+}$-space if and only if its dual is isometric to $L_1(\mu)$ for some measure μ. The latter are usually called *Lindenstrauss spaces* and include, among other interesting classes – see Note 8.8.1 – all $C(K)$ spaces. If the reader tries to define $\mathscr{L}_p$-space for $0 < p < 1$ in the most standard form, there will be a traumatic moment when they discover that it is not even known whether L_p satisfies the definition. There is, however, a satisfactory notion of $\mathscr{L}_p$-space, due to Kalton, that works fine for $0 < p < 1$, but it has to wait until Chapter 5.

Type and Cotype

Our basic source for the study of type and cotype of Banach spaces is the masterpiece of Diestel, Jarchow and Tonge [153, Chapter 11]. In what follows we will work in the wider context of quasi-Banach spaces.

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Definition 1.4.3 A quasi-Banach space X is said to have type p for $0 < p \leq 2$ if there is a constant T such that for every finite sequence $(x_i)_{1\leq i\leq n}$ of points of X we have

$$\left(\int_0^1 \Big\| \sum_{1\leq i\leq n} r_i(t)x_i \Big\|^p dt\right)^{1/p} \leq T\left(\sum_{1\leq i\leq n} \|x_i\|^p\right)^{1/p}, \tag{1.6}$$

where $(r_i)_{i\geq 1}$ is the Rademacher sequence.

If $0 < p \leq 1$, this is just the randomised version of (1.3), which was used to characterise the p-normability of X. We are not really interested in type $p < 1$ because a quasi-Banach space has type $0 < p < 1$ if and only if it has an equivalent p-norm [252, Theorem 4.2]. Incidentally, the L_p quasinorm on the left-hand side of (1.6) can be replaced by any other L_q quasinorm with $q \in (0, \infty)$. This fact is due to Kahane when X is a Banach space [153, 11.1] and to Kalton in general [252, Theorem 2.1].

Proposition 1.4.4 *A quasi-Banach space having type $p > 1$ is locally convex.*

The proof is based on the behaviour of the following sequences, defined for any quasi-Banach space X:

$$a_n(X) = \sup_{\|x_i\|\leq 1} \Big\| \sum_{1\leq i\leq n} x_i \Big\|, \qquad b_n(X) = \sup_{\|x_i\|\leq 1} \inf_{\varepsilon_i=\pm 1} \Big\| \sum_{1\leq i\leq n} \varepsilon_i x_i \Big\|. \tag{1.7}$$

It is clear that both $(a_n)_n$ and $(b_n)_n$ are increasing and submultiplicative, that is, $a_{nm} \leq a_n a_m$ and $b_{nm} \leq b_n b_m$ for all $n, m \in \mathbb{N}$. The proof follows by simply assembling the three parts of the next result:

Lemma 1.4.5

- *If X has type $p > 1$, then $n^{-1}b_n \longrightarrow 0$.*
- *If $n^{-1}b_n \longrightarrow 0$, then $(n^{-1}a_n)_n$ is bounded.*
- *If $(n^{-1}a_n)_n$ is bounded, then X is locally convex.*

Proof The first part is obvious, since for $(x_i)_{1\leq i\leq n}$ in X and $p \in (0, \infty)$, one has

$$\inf_{\varepsilon_i \pm 1} \Big\| \sum_{1\leq i\leq n} \varepsilon_i x_i \Big\| \leq \left(\int_0^1 \Big\| \sum_{1\leq i\leq n} r_i(t)x_i \Big\|^p dt\right)^{1/p},$$

and so, if X has type p then $b_n(X) \leq Tn^{1/p}$. Let us check the third point. With no loss of generality, we may assume that X is r-normed for some $0 < r \leq 1$. Now, if $n^{-1}a_n \leq R$, then the ball of radius R contains the set of means

$$\left\{\frac{x_1 + \cdots + x_n}{n} : \|x_i\| \leq 1, n \in \mathbb{N}\right\}$$

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whose closure is exactly the closed convex hull of B_X. The same argument shows that X is locally convex if and only if $(n^{-1}a_n)_n$ has a bounded subsequence. The second point accounts for the lion's share of the proof. We keep assuming that X is r-normed. Pick $2n$ points $(x_i)_{1\leq i\leq 2n}$ from B_X and $\varepsilon_i = \pm 1$ such that $\|\sum_{1\leq i\leq 2n}\varepsilon_i x_i\| \leq b_{2n}$. Rearranging, we may assume that $\varepsilon_i = 1$ for $i \leq k \leq n$ and $\varepsilon_i = -1$ for $k < i \leq 2n$, that is, $\|x_1+\cdots+x_k-x_{k+1}-\cdots-x_{2n}\| \leq b_{2n}$. Writing $\sum_{1\leq i\leq 2n} x_i = 2\sum_{1\leq i\leq k} x_i - \sum_{1\leq i\leq 2n}\varepsilon_i x_i$ and using the r-subadditivity of the quasinorm, we obtain

$$\Big\|\sum_{i=1}^{2n} x_i\Big\|^r = 2^r\Big\|\sum_{i=1}^{k} x_i\Big\|^r + b_{2n}^r \leq 2^r a_n^r + b_{2n}^r \qquad\Longrightarrow\qquad a_{2n}^r \leq 2^r a_n^r + b_{2n}^r.$$

As (b_n) is increasing and submultiplicative and $n^{-1}b_n \longrightarrow 0$, there is some $s > 1$ such that $n^{-s}b_n$ is bounded, say, by C, and dividing by $(2n)^r$, we get

$$\left(\frac{a_{2n}}{2n}\right)^r \leq \left(\frac{a_n}{n}\right)^r + \left(\frac{C2^s n^s}{2n}\right)^r = \left(\frac{a_n}{n}\right)^r + C^r(2n)^{r(s-1)}.$$

If, for $n \geq 0$, we put $\alpha_n = 2^{-n}a_{2^n}$, then the preceding inequality becomes $\alpha_{n+1}^r - \alpha_n^r \leq C^r 2^{r(s-1)(n+2)}$, and since $\sum_n 2^{r(s-1)(n+2)} < \infty$, we obtain that (α_n^r) is bounded, which is enough. □

Banach spaces for which $n^{-1}b_n \longrightarrow 0$ are traditionally called *B-convex* and Banach spaces having type $p > 1$ are traditionally called spaces having non-trivial type. A very deep result [153, 13.10 Theorem plus 13.16 Theorem] states:

1.4.6 *Let X be an infinite-dimensional Banach space. The following are equivalent:*

(i) *X is B-convex,*
(ii) *X does not contain ℓ_1^n uniformly,*
(iii) *X has non-trivial type.*

We have established that quasi-Banach spaces satisfying (iii) are Banach spaces satisfying (i). A still deeper result by Pisier shows that for each infinite-dimensional B-convex Banach space, there is a constant C so that, for every n, there are operators $I\colon \ell_2^n \longrightarrow X$ and $P\colon X \longrightarrow \ell_2^n$ such that PI is the identity on ℓ_2^n and $\|I\|\|P\| \leq C$ [153, 19.3 Theorem]; in particular, X contains ℓ_2^n uniformly complemented. All this produces the following dichotomy [153, 13.3 and 19.3]:

1.4.7 *A Banach space either contains ℓ_1^n uniformly or contains ℓ_2^n uniformly complemented.*

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Definition 1.4.8 A (quasi-) Banach space X is said to have cotype q , where $2 \leq q < \infty$, if there is a constant C such that

$$\left(\sum_{1\leq i\leq n} \|x_i\|^q\right)^{1/q} \leq C\left(\int_0^1 \left\| \sum_{1\leq i\leq n} r_i(t)x_i\right\|^q dt\right)^{1/q} \tag{1.8}$$

for every $x_1, \ldots, x_n \in X$, where $(r_i)_{i\geq 1}$ is the Rademacher sequence.

The spaces L_p have type $\min(p, 2)$ and cotype $\max(2, p)$ for $0 < p < \infty$. It is relatively easy to prove that the dual of a type p space has cotype p^*, where p^* is given by $1 = 1/p + 1/p^*$ [153, 11.10 Proposition]. The converse is true for B-convex spaces [153, 13.17 Proposition] and false in general (consider the case of ℓ_1, which has cotype 2). Kwapien's theorem [153, 12.19 and 12.20] establishes:

1.4.9 Kwapien's theorem *A (quasi-) Banach space having type* 2 *and cotype* 2 *is isomorphic to a Hilbert space.*

The Maurey–Pisier *Great* Theorem (well, one of them) states:

1.4.10 Maurey–Pisier theorem *Every Banach space X contains almost isometric copies of* $\ell^n_{p(X)}$ *and* $\ell^n_{q(X)}$, *where* $p(X) = \sup\{p\colon X \text{ has type } p\}$ *and* $q(X) = \inf\{q\colon X \text{ has cotype } q\}$.

See [362, § 13] for a reasonably accessible proof. Spaces with $p(X) = 2 = q(X)$ have been called *near Hilbert* and will be encountered later. Kwapień's result is contained in Maurey's extension theorem [153, 12.22]: *every operator from a subspace of a type* 2 *space to a cotype* 2 *space can be extended to an operator on the whole space that still factorises through a Hilbert space.* The following definition should then come as no surprise:

1.4.11 Maurey extension property A Banach space X is said to have the Maurey extension property (MEP) if every Hilbert valued operator defined on a subspace of X can be extended to X.

Type 2 spaces have MEP, and thus Hilbert subspaces of type 2 spaces are complemented.

Ultraproducts

The Banach space ultraproduct construction originates in model theory and has been, and continues to be, the main channel of communication between logic and Banach space theory. Even emancipated from model theory, ultraproducts of Banach spaces have a surprisingly large number of applications, ranging from the local theory to the Lipschitz and uniform classification of Banach spaces. For a detailed study of this construction at the elementary level needed here, we refer the reader to Heinrich's survey paper [211] or Sims' notes [434]. A more complete exposition with the necessary model-theoretic background is [212]. Ultraproducts of quasi-Banach spaces have never had a comparable prestige; nevertheless, they are even more useful, for the same reasons as in the case of Banach spaces and because they provide an operative substitute for the bidual (which may well be trivial now). The ultraproduct construction is based on the notion of convergence along an ultrafilter, which we pause to explain.

Let I be a set and $\mathcal{F}$ a filter on I (a family of subsets that does not contain the empty set, is closed under finite intersections and such that if $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$). Given a topological space S, a function $f\colon I \longrightarrow S$, a filter $\mathcal{F}$ on I and $s \in S$, we write $\lim_{\mathcal{F}(i)} f(i) = s$ and say that f converges to s along $\mathcal{F}$ if $f^{-1}[V] \in \mathcal{F}$ for every neighbourhood V of s. The definition makes sense even if f is *only* defined on some $A \in \mathcal{F}$: just consider the limit along the family $\{B \in \mathcal{F}\colon B \subset A\}$, which is a filter on A. Keep this fact in mind since it will be used without further mention. An ultrafilter on I is a maximal filter with respect to inclusion. Not exactly trivial, but nonetheless graspable, is the very elegant characterisation of ultrafilters as those filters $\mathcal{F}$ such that for every partition of I into two (or finitely many) subsets, exactly one of them belongs to $\mathcal{F}$. The only ultrafilters that one will ever see explicitly are the principal, or fixed, ultrafilters: an ultrafilter $\mathcal{U}$ is fixed if it contains a finite set (hence a singleton) in which case there is $i \in I$ such that $\mathcal{U} = \{A \subset I\colon i \in A\}$. Otherwise, $\mathcal{U}$ is called free. An ultrafilter $\mathcal{U}$ is said to be *countably incomplete* if there is a decreasing sequence of elements of $\mathcal{U}$ with empty intersection. This happens if and only if there is a strictly positive function $f\colon I \longrightarrow (0, \infty)$ such that $f(i) \longrightarrow 0$ along $\mathcal{U}$. Every free ultrafilter on $\mathbb{N}$ is countably incomplete. The simplest way to produce a free ultrafilter is to start with a filter containing no finite subsets (for instance, the cofinite subsets of I when this is infinite) and use Zorn's lemma to refine it to an ultrafilter, or use the following variation that will appear over and over: when I carries a partial *directed* order, i.e. such that for every $i, j \in I$, there exists $k \in I$ such that $i, j \leq k$, the sets $\{j \in I\colon i \leq j\}$ generate the so-called order filter, and every ultrafilter containing it is free unless I has a maximal element. The key point is that every function with values on a compact (Hausdorff) space has a (unique) limit along any ultrafilter, and this is actually a topological version of the ultrafilter characterisation alluded to earlier.

To avoid unnecessary complications, we will work with families of p-Banach (instead of arbitrary quasi-Banach) spaces: this yields the continuity of the quasinorms and, more importantly, a uniform bound for the moduli of concavity. So, let $(X_i)_{i\in I}$ be a family of p-Banach spaces indexed by I, and let $\mathcal{U}$ be an ultrafilter on I. The space of bounded families $\ell_\infty(I, X_i)$ is a p-Banach space, and $c_0^{\mathcal{U}}(I, X_i) = \{(x_i) \in \ell_\infty(X_i) : \lim_{\mathcal{U}(i)} \|x_i\| = 0\}$ is a closed subspace of $\ell_\infty(I, X_i)$. The ultraproduct of the spaces $(X_i)_{i\in I}$ following $\mathcal{U}$ is defined as the quotient $[X_i]_{\mathcal{U}} = \ell_\infty(I, X_i)/c_0^{\mathcal{U}}(I, X_i)$. We denote by $[(x_i)]$ the element of $[X_i]_{\mathcal{U}}$ which has the family (x_i) as a representative. Using the continuity of p-norms, it is not difficult to show that $\|[(x_i)]\| = \lim_{\mathcal{U}(i)} \|x_i\|$. It is clear that two bounded families $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ define the same element of $[X_i]_{\mathcal{U}}$ if the set $\{i \in I\colon x_i = y_i\}$ belongs to $\mathcal{U}$. As we remarked before (claiming falsely that the fact will be used without further mention), to define an element of $[X_i]_{\mathcal{U}}$, one just needs a bounded family $(x_i)_{i\in A}$ defined *only* on some subset $A \in \mathcal{U}$. When $X_i = X$ for all $i \in I$, we denote the ultraproduct by $X_{\mathcal{U}}$ and call it the ultrapower of X following $\mathcal{U}$. The diagonal mapping $X \longrightarrow X_{\mathcal{U}}$ sending each $x \in X$ to $[(x)]$ is an isometric embedding, and so each p-Banach space is isometric to a subspace of its ultrapowers. If $T_i\colon X_i \longrightarrow Y_i$ is a uniformly bounded family of operators, where X_i and Y_i are all p-Banach spaces, the ultraproduct operator $[T_i]_{\mathcal{U}}\colon [X_i]_{\mathcal{U}} \longrightarrow [Y_i]_{\mathcal{U}}$ is given by $[T_i]_{\mathcal{U}}[(x_i)] = [T_i(x_i)]$. Quite clearly, $\|[T_i]_{\mathcal{U}}\| = \lim_{\mathcal{U}(i)} \|T_i\|$.

Definition 1.4.12 An ultrasummand is a quasi-Banach space complemented in all its ultrapowers through the diagonal embedding.

The following result confirms the intuition that the unit ball of an ultrasummand enjoys a kind of 'compactness' (after all, a projection of $X_{\mathcal{U}}$ onto X must select one point of X from each bounded family of points).

Proposition 1.4.13 *Let X be a Banach space. The following are equivalent!*

(i) *X is an ultrasummand,*
(ii) *X is complemented in its bidual,*
(iii) *X is a complemented subspace of some dual space.*

Proof We prove the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (ii). The first one is trivial. To prove (iii) $\Longrightarrow$ (i), we assume that Y is a Banach space whose dual contains X and that $P\colon Y^* \longrightarrow X$ is a bounded projection. Now, if $\mathcal{U}$ is an ultrafilter on I and $X_{\mathcal{U}}$ is the corresponding ultrapower, we can define a projection $L\colon X_{\mathcal{U}} \longrightarrow X$ as $L[(x_i)] = \lim_{\mathcal{U}(i)} x_i$, where the limit is taken in the weak* topology of Y^*. The proof of (iii) $\Longrightarrow$ (i) relies on the principle of local reflexivity, which is responsible for embedding X^{**} as a very well-placed subspace of a suitable ultrapower of X: indeed, consider the order in $\mathscr{F}(X^{**}) \times \mathscr{F}(X^*) \times (0, \infty)$ given by $(F, G, \varepsilon) \le (F', G', \varepsilon')$ if $F \subset F', G \subset G'$

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and $\varepsilon' \leq \varepsilon$. Let $\mathfrak{U}$ be a free ultrafilter refining the order filter on $\mathscr{F}(X^{**}) \times \mathscr{F}(X^*) \times (0, \infty)$. Given $F \in \mathscr{F}(X^{**}), G \in \mathscr{F}(X^*)$ and $\varepsilon > 0$, we consider the operator $T_{(F,G,\varepsilon)}: F \longrightarrow X$ provided by the principle of local reflexivity. We define a map $\triangle: X^{**} \longrightarrow X_{\mathfrak{U}}$ by letting $\triangle(x^{**}) = [(x_{(F,G,\varepsilon)})]$, where $x_{(F,G,\varepsilon)} = T_{(F,G,\varepsilon)}(x^{**})$ if $x^{**} \in F$, and 0 otherwise. Clearly, $\triangle$ is a linear isometry of X^{**} into $X_{\mathfrak{U}}$. Note that $x_{(F,G,\varepsilon)}$ and $T_{(F,G,\varepsilon)}(x^{**})$ agree 'eventually', and so the linearity of $\triangle$ is not a problem due to our choice of $\mathfrak{U}$. The isometric copy thus obtained is moreover 1-complemented via the operator $\nabla: X_{\mathfrak{U}} \longrightarrow X^{**}$ sending $[(x_{(F,G,\varepsilon)})]$ to the weak* limit of $(x_{(F,G,\varepsilon)})$ along $\mathfrak{U}$. Clearly, ∇ is a well-defined, contractive operator and $\nabla\triangle = \mathbf{1}_{X^{**}}$, since the family $x_{(F,G,\varepsilon)}$ converges to x^{**} in the weak* topology along $\mathfrak{U}$ because each $x^* \in X^*$ eventually falls in G. □

Thus, reflexive and $L_1(\mu)$-spaces are ultrasummands. Since c_0 is not complemented in ℓ_∞, it cannot be an ultrasummand, and the same happens to any space containing c_0 complemented. To present typical non-locally convex ultrasummands, note that the only property of a dual Banach space which is needed to carry out the proof of the implication (iii) $\implies$ (i) in the preceding result is that the weak* topology is a linear topology weaker than the norm topology and makes the unit ball compact. Quasi-Banach spaces admitting a weaker-than-the-quasinorm topology, making the unit ball compact, are termed *pseudoduals*. One has

1.4.14 *The spaces ℓ_p, H_p and S_p are pseudoduals for all $0 < p < \infty$ and, therefore, ultrasummands.*

We only sketch the proof. The case ℓ_p is clear since the topology of pointwise convergence makes its unit ball compact. As for the Hardy classes H_p, the estimate (1.5) is exactly what we need to invoke Montel's theorem on normal families to conclude that the ball of H_p is compact under the topology of pointwise convergence on points of the open disc. The case of the Schatten classes S_p is because the quasinorm of S_p is lower semicontinuous with respect to the weak operator topology of $\mathfrak{L}(H)$ [157, Corollary 2.3]. Nevertheless, L_p emphatically refuses to be an ultrasummand for $0 < p < 1$ (see Note 1.8.3 for details).

1.5 The Dunford–Pettis, Grothendieck, Pełczyński and Rosenthal Properties

Although the borders between global and local properties are somewhat permeable, we now discuss some 'global' properties important in the study

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of Banach spaces mainly because of their connections with the structure of $\mathscr{C}$-spaces. A terse exposition can be found in [22, Appendix A1].

Definition 1.5.1 A Banach space X is said to have

- the Dunford–Pettis property (DPP) if every weakly compact operator defined on X sends weakly convergent sequences to convergent sequences,
- Pełczyński's property (V) if every operator defined on X is either weakly compact or an isomorphism on a copy of c_0,
- Rosenthal's property (V) – the same as before, replacing c_0 by ℓ_∞,
- Grothendieck's property if every operator from X to a separable Banach space is weakly compact.

Dunford and Pettis themselves established that $L_1(\mu)$ and $C(K)$ spaces have DPP. The extension to $\mathscr{L}_\infty$-spaces and the name are due to Grothendieck; see [151]. It is clear that a complemented subspace of a space with the DPP has DPP and that an infinite-dimensional space with DPP cannot be reflexive since, otherwise, weakly convergent sequences must be convergent, which makes the unit ball compact. General background about the Dunford–Pettis property can be found in [151] and [102, Chapter 6]. Pełczyński proved in [376, Theorem 5] that $\mathscr{C}$-spaces have property (V). This property clearly passes to quotients and, since Johnson and Zippin showed in [234] that every separable Lindenstrauss space is a quotient of $C[0,1]$ (plus the obvious fact that each separable subspace of a Lindenstrauss space is contained in a separable Lindenstrauss subspace), all Lindenstrauss spaces have property (V). However, not all $\mathscr{L}_\infty$-spaces have property (V) (see Section 10.5 for accessible counterexamples). The combination of property (V) and DPP shows that *every infinite-dimensional complemented subspace of a Lindenstrauss space contains* c_0. Reflexive spaces enjoy Grothendieck's property for obvious reasons, but so do injective spaces [152, VII, Theorem 15]. The information we need is that $\mathscr{L}_\infty$-spaces with Grothendieck's property do not contain separable complemented subspaces [22, Proposition 2.8]. Ultrasummands of type $\mathscr{L}_\infty$ or, equivalently, injective Banach spaces [22, Proposition 1.5] even have Rosenthal's property (V) [22, Proposition 2.8].

1.6 $C(K)$-Spaces and Their Complemented Subspaces

We focus on the following idea: the homeomorphism type of a compact space is independent of the particular realisation of it one encounters. The web formed of the seven compacta appearing in this book that we describe now has been woven with that silk:

1. ω^N is the only countable compact whose Nth derived set is one point.
2. ω^ω is the only countable compact whose ωth derived set is one point.
3. The Cantor set Δ is the only compact that is totally disconnected perfect and metrisable, regardless of whether it appears as $\{0,1\}^{\mathbb{N}}$, as $\{-1,0,1\}^{\mathbb{N}}$ or wearing other clothes.
4. The unit interval $[0,1]$.
5. $\beta\mathbb{N}$ is the Stone–Čech compactification of the discrete space $\mathbb{N}$, i.e. the only compact space containing a dense copy of $\mathbb{N}$ such that every bounded function on $\mathbb{N}$ extends to a continuous function on $\beta\mathbb{N}$. Thus, $\ell_\infty = C(\beta\mathbb{N})$. $\beta\mathbb{N}$ can be obtained as the Stone space of $\mathcal{P}(\mathbb{N})$, aka the space of ultrafilters on $\mathbb{N}$.
6. $\mathbb{N}^*$ is $\beta\mathbb{N}\setminus\mathbb{N}$. Two continuous functions on $\beta\mathbb{N}$ coincide on $\mathbb{N}^*$ if and only if their difference converges to 0 on $\mathbb{N}$, and thus $C(\mathbb{N}^*)$ can be identified with ℓ_∞/c_0. Under CH, $\mathbb{N}^*$ is the only totally disconnected Hausdorff F-space without isolated points of weight $\mathfrak{c}$ and such that every non-empty G_δ subset has a non-empty interior by Parovičenko's *other* theorem, see [453, Chapter 3, p. 80–83]. Parovičenko's *first* theorem [45] asserts that $\mathbb{N}^*$ maps continuously onto each compact space of weight $\aleph_1$ or less.
7. The unit ball B_X^* of the dual of a Banach space X is endowed with the weak* topology.

The passage from the compact K to the corresponding $C(K)$-space tears the cobweb apart:

1.6.1 Classification of separable $\mathscr{C}$-spaces *Let K be a metrisable compact space. If K is uncountable then $C(K)$ is isomorphic to $C[0,1]$. If K is countable then $C(K)$ is isometric to $C(\alpha)$ for some countable ordinal α.*

The first assertion is Milutin's theorem [364], which in particular means that $C[0,1] \simeq C\Delta) \simeq C(B_X^*)$ for all separable X. No proof for Milutin's theorem is perhaps clearer than the one presented in [5, §4.4]. The second assertion is a famous theorem of Mazurkiewicz and Sierpiński [360, Théorème 1], see also [430, Theorem 8.6.10]. Moreover, so far as separability is involved, $C(\Delta)$ is the guy to deal with:

Lemma 1.6.2 *Let K be any compact metric space. Then there exist positive contractive operators $S : C(K) \longrightarrow C(\Delta)$ and $R: C(\Delta) \longrightarrow C(K)$ such that $R1_\Delta = 1_K, S1_K = 1_\Delta$ and $RS = \mathbf{1}_{C(K)}$.*

The problem of identifying the complemented subspaces of $\mathscr{C}$-spaces was hermetically open until Plebanek and Salguero [393] betrayed our trust by finding a complemented subspace of a non-separable $C(K)$ space that is not isomorphic to a $C(K)$-space while this book was in print.

1.6.3 *This is (most of) what is currently known:*

(a) *Every complemented subspace of ℓ_∞ is isomorphic to ℓ_∞.*

(b) *A subspace of $c_0(I)$ is complemented if and only if it is isomorphic to $c_0(J)$ for some J, and if and only if it is an $\mathscr{L}_\infty$-space.*

(c) *Every complemented subspace of $C(\omega^\omega)$ is isomorphic to either c_0 or $C(\omega^\omega)$.*

(d) *A complemented subspace of $C[0,1]$ with non-separable dual must be isomorphic to $C[0,1]$.*

Assertion (a) results from a combination of the DPP and Rosenthal's property (V) of ℓ_∞: together they yield that complemented subspaces of ℓ_∞ contain ℓ_∞, necessarily complemented, and therefore, by Pełczyński's decomposition method, they must be isomorphic to ℓ_∞. Incidentally, this is the case $p = \infty$ in Proposition 1.2.3. The first part of (b) for countable I is consequence of the DPP and Pełczyński's property of c_0 plus Pełczyński's decomposition method ('only if'). The 'if' part is a particular case of Sobczyk's theorem (the main topic of Section 1.7). The general non-separable version is due to Suárez Granero [199]. The second part of (b) is due to Godefroy, Kalton and Lancien [190, Remark 5.4], who proved that an $\mathscr{L}_\infty$-subspace of $c_0(I)$ is isomorphic to some $c_0(J)$. Part (c) is due to Benyamini [40]. The proof of (d) is built on a beautiful lemma of Rosenthal [415]: *an operator $T\colon C[0,1] \longrightarrow X$ for which $T^*[X^*]$ is non-separable fixes a copy of* $C[0,1]$. Therefore, let X be a complemented subspace of $C[0,1]$, and let $P\colon C[0,1] \longrightarrow X$ be a projection. If X has a non-separable dual then $P^*[X^*]$ is non-separable, and thus P fixes a copy of $C[0,1]$. Therefore, X contains a copy of $C[0,1]$. Now we isolate a useful result of Pełczyński [380, Theorem 1]

1.6.4 *A subspace of a separable $C(K)$ that contains a copy of $C(K)$ must also contain a complemented copy of $C(K)$.*

The proof of (d) can be completed as follows: if X is complemented in, and contains a copy of, $C[0,1]$, apply Pełczyński's decomposition method using c_0-vector sums. □

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1.7 Sobczyk's Theorem and Its Derivatives

It is a straightforward consequence of the Hahn–Banach theorem that when Y is a subspace of a Banach space X, every operator $u\colon Y \longrightarrow \ell_\infty$ can be extended, with the same norm to X. In particular, but in the end equivalently, ℓ_∞ is 1-complemented in any Banach space containing it. The following fundamental result of Sobczyk establishes that c_0 has a similar *homological* property for separable spaces, although doubling the norm of the extension. What follows is not Sobczyk's original proof but the beautiful proof of Veech [451], one of the masterpieces in THE BOOK, which Diestel will be annotating now.

1.7.1 Sobczyk's theorem *If X is a separable Banach space and Y is a subspace of X, then every operator $u\colon Y \longrightarrow c_0$ has a 2-extension to X.*

Proof Assume $\|u\| = 1$ and write $u(y) = (\langle y_n^*, y\rangle)_{n\geq 1}$, where $(y_n^*)_{n\geq 1}$ is a weak*-null sequence in the unit ball of Y^*. The strategy is to find a weak*-null sequence of extensions of these functionals. For each n, let $x_n^* \in X^*$ be a Hahn–Banach extension of y_n^*. Let D be the set of weak*-accumulation points of the sequence $(x_n^*)_n$, and recall that the dual ball of a separable space is weak*-metrisable by some metric d. The ridiculously simple observation 'a sequence such that every subsequence contains a further subsequence converging to zero is itself convergent to zero' yields $\lim_n d(x_n^*, D) = 0$. Choose $f_n \in D$ such that $d(x_n^*, f_n) \leq d(x_n^*, D) + \frac{1}{n}$. The sequence $(x_n^* - f_n)_n$ is weak*-null and extends (y_n^*) since an accumulation point of (x_n^*) must vanish on Y. Thus, the mapping $U\colon X \longrightarrow c_0$ given by $U(x) = (\langle x_n^* - f_n, x\rangle)_{n\geq 1}$ is an extension of u and, quite clearly, $\|U\| \leq 2$. □

The bound 2 cannot be improved because the norm of any projection of c onto c_0 is at least 2, as can be seen just considering the 'obvious projection' with kernel $[1_{\mathbb{N}}]$. The notion looming over Sobczyk's theorem is isolated in the next definition.

Definition 1.7.2 A Banach space E is separably injective if, for every subspace Y of a separable space X, every operator $u\colon Y \longrightarrow E$ has an extension $U\colon X \longrightarrow E$. If the extension can be achieved with $\|U\| \leq \lambda\|u\|$ then E is said to be λ-separably injective.

It is easy, though not entirely trivial, to see that each separably injective space must be λ-separably injective for some $\lambda \geq 1$ [22, Proposition 1.6]. The theory of separably injective spaces is surprisingly rich with examples and applications, as can be deduced from the mere existence of [22]. The theory of separably injective spaces concerns non-separable Banach spaces, in view of the outstanding

1.7.3 Zippin's theorem *c_0 is the only separable separably injective Banach space, up to isomorphism.*

Despite its pristine formulation, the proof requires a clever combination of Banach space machinery and delicate computations. Following Wittgenstein's mandate, we remain silent on the issue and refer to Zippin [466].

En Route to Non-separable Versions of Sobczyk's Theorem

The only well-known aspect of Sobczyk's result is the scalar separable case. To handle large non-separable spaces, we introduce a further batch of properties.

Definition 1.7.4 A Banach space is weakly compactly generated (WCG) if it contains a weakly compact subset with dense linear span.

Reflexive spaces and separable spaces are the two basic types of WCG spaces. The space $c_0(I)$ is WCG for all I since the inclusion $\ell_2(I) \longrightarrow c_0(I)$ has dense range, and so is $L_1(\mu)$ for finite μ since the inclusion $L_2(\mu) \longrightarrow L_1(\mu)$ has dense range. Let us present an accurate formulation for the idea that WCG spaces have many complemented subspaces.

Definition 1.7.5 A Banach space X has the separable complementation property (SCP) if every separable subspace of X is contained in a complemented separable subspace of X.

All WCG spaces have SCP [148, Chapter IV, Lemma 2.4], but there are many more, as can be seen in [396].

Definition 1.7.6 A projectional resolution of the identity (PRI) on a Banach space X is a system of projections $(P_\alpha)_{\omega\leq\alpha\leq\mu}$, where $\mu = \dim(X)$ such that

- $\|P_\alpha\| = 1$,
- $P_\alpha P_\beta = P_{\min(\alpha,\beta)}$,
- $\dim P_\alpha[X] \leq |\alpha|$,
- $\overline{\bigcup_{\beta<\alpha} P_\beta[X]} = P_\alpha[X]$,
- $P_\mu[X] = \mathbf{1}_X$.

It is not simple to prove, but every WCG space admits a PRI [148, VI, Theorem 2.5]; however, it is simple that a Banach space with dimension $\aleph_1$ and a PRI has the SCP. Being obvious that Sobczyk's theorem works in SCP spaces, it works in WCG spaces with dimension $\aleph_1$. A straight projectional-resolution-free proof can be given:

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Proposition 1.7.7 *c_0 is complemented in every WCG superspace (or, more generally, in a Banach space with the property that the closure of a countable set is metrisable in the weak topology).*

Proof Since weakly compact sets in a separable space are metrisable, separable sets in a WCG space are metrisable in the weak topology too. Veech's proof applies. □

We now consider special types of $\mathscr{C}$-spaces with PRI.

Definition 1.7.8 A compact space K is said to be

- Eberlein if it is homeomorphic to a weakly compact set of a Banach space,
- Corson if it is homeomorphic to a compact subset of some $\Sigma(I)$, the subspace of all countably supported elements of $[0, 1]^I$,
- Valdivia if there exist an I and an embedding of $\varphi\colon K \longrightarrow [0, 1]^I$ such that $\varphi[K] \cap \Sigma(I)$ is dense in $\varphi[K]$.

It would be very hard, if not impossible, to show the implications Eberlein $\Longrightarrow$ Corson $\Longrightarrow$ Valdivia more clearly than [148, VI. Theorem 7.2], and the same is true for [148, VI. Lemma 7.4]: K is an Eberlein compact if and only if $C(K)$ is WCG. Focusing now on the largest class of Valvidia compacta, we have the following decomposition [148, VI. Lemma 7.5]:

Lemma 1.7.9 *Let $K \subset [0, 1]^I$ be a Valdivia compact for which we assume that $K \cap \Sigma(I)$ is dense in K. Assume that $K \cap \Sigma(I)$ contains a dense subset of cardinality μ. Then there exists an increasing family $(I_\alpha)_{\omega \le \alpha \le \mu}$ of subsets of I such that, for every $\omega \le \alpha < \mu$,*

(a) $|I_\alpha| \le \alpha$,
(b) $I_\alpha = \bigcup_{\beta<\alpha} I_{\beta+1}$,
(c) $I_\mu = I$.

Moreover, if we define $R_L\colon [0, 1]^I \to [0, 1]^I$ by $R_L(x) = 1_L x$ when $L \subset I$, then $K_\alpha = R_{I_\alpha}[K] \subset K$ is a Valdivia compact.

Taking the embeddings $R^\circ_{I_\alpha}\colon C(K_\alpha) \longrightarrow C(K), R^\circ_{I_\alpha}(f) = f \circ R_{I_\alpha}$, associated to the family $(I_\alpha)_\alpha$ leads to [148, VI. Theorem 7.6 and Remark 7.7, p. 256]:

Proposition 1.7.10 *If K is a Valdivia compact, then $C(K)$ has a PRI $(P_\alpha)_\alpha$ such that $P_\alpha[C(K)]$ is isometric to $C(K_\alpha)$, with K_α again a Valdivia compact.*

And thus we are ready to discuss:

Sobczyk's Theorem for $c_0(I)$

Sobczyk's theorem steadily percolates out from the separability reservoir, first to copies of c_0 in WCG spaces. Then comes the smashing surprise that Sobczyk's original proof 'remains valid for $c_0(I)$', with the meaning that copies of $c_0(I)$ inside spaces X such that $X/c_0(I)$ is separable must be complemented. See [440] but also [79] for a lively discussion and full details. This leads to the question, first formulated by Yost, of whether copies of $c_0(I)$ must be complemented in WCG spaces (or beyond). Our first example of set theoretic considerations bursting into a seemingly remote domain appears while trying to answer this question: Yost's problem has a positive solution if $|I| < \aleph_\omega$, while there exists an Eberlein compact K such that $C(K)$ admits an uncomplemented copy of some $c_0(I)$ with $|I| = \aleph_\omega$ (the latter result shows that the former is optimal). We present now the positive part of the answer and postpone the negative part to Proposition 2.2.15.

Definition 1.7.11 A Banach space X is said to be K-Sobczyk if every κ-isomorphic copy of $c_0(I)$ inside X is $K\kappa$-complemented.

For instance, separable Banach spaces are 2-Sobczyk. We now begin our practice of using 'long' decompositions based on ordinals. To this end, let us explain the meaning of $x = \sum_{\alpha<\mu} x(\alpha)$: form the function $\widehat{x}\colon [0,\mu) \longrightarrow X$ inductively defined with $\widehat{x}(0) = 0$, then

$$\begin{cases} \widehat{x}(\beta) = \widehat{x}(\alpha) + x(\alpha+1), & \text{if } \beta = \alpha+1 \text{ is a successor ordinal,} \\ \widehat{x}(\beta) = x(\beta) + \lim_{\alpha\to\beta} \widehat{x}(\alpha), & \text{if } \beta \text{ is a limit ordinal,} \end{cases}$$

and finally asking for $x = \lim_{\beta\to\mu} \widehat{x}(\beta)$. Let X be a Banach space and μ an ordinal. We say that a family $(X_\alpha)_{\alpha<\mu}$ of subspaces, indexed by the ordinals below μ, is a decomposition of X if every point $x \in X$ can be written in a unique way as $x = \sum_{\alpha<\mu} x(\alpha)$, where $x(\alpha) \in X_\alpha$. A cardinal μ is said to be *regular* when the union of fewer than μ sets of cardinality smaller than μ has cardinality smaller than μ.

Lemma 1.7.12 *Let μ be an uncountable regular cardinal, and let X be a Banach space which admits a decomposition $(X_\alpha)_{\alpha<\mu}$. Assume that for some constants $K, M < \infty$ and each $\alpha < \mu$, we have*

(a) $\dim(X_\alpha) < \mu$,
(b) *is a projection* $P_\alpha\colon X \longrightarrow \overline{[X_\beta : \beta < \alpha]}$ *such that* $\|P_\alpha\| \le M$,
(c) $\overline{[X_\beta : \beta < \alpha]}$ *is K-Sobczyk.*

Then X is $2MK$-Sobczyk.

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Proof For $x \in X$, denote $\operatorname{supp}(x) = \{\alpha < \mu \colon x(\alpha) \neq 0\}$ and observe that $|\operatorname{supp}(x)| \leq \aleph_0$. Since $\operatorname{cf}(\mu) = \mu > \aleph_0$, there exists $\beta < \mu$ such that $x(\alpha) = 0$ for all $\beta < \alpha < \mu$. Let Y be a κ-isomorphic copy of $c_0(I)$ in X. For each $i \in I$, let $y_i \in X$ be the isomorphic image of $e_i \in c_0(I)$. Let us dispose of the case where $|I| < \mu$. Since each y_i has countable support, there is some $\nu < \mu$ such that $\overline{[X_\beta \colon \beta < \nu]}$ contains every y_i, hence the whole of Y; the result follows using the M-projection P_ν provided by (b) followed by any K-projection $\overline{[X_\beta \colon \beta < \nu]} \longrightarrow Y$, whose existence is guaranteed by (c). For the remainder of the proof, we assume $|I| = \mu$. As the only relevant property of I is its cardinality, we treat it also as a cardinal (namely μ), and we identify $I = \{i_\alpha \colon \alpha < \mu\}$. We maintain the different names, I and μ, mostly for notational (and psychological) reasons.

Claim 1 For every $\alpha < \mu$, we have $|\{i \in I \colon y_i(\alpha) \neq 0\}| < \mu$.

Proof of Claim 1 Let $J = \{i \in I \colon y_i(\alpha) \neq 0\}$. Let ν be the smallest cardinal of a subset spanning a weak*-dense subspace of X_α^*. Since $\nu \leq \dim(X_\alpha) < \mu$ (because of (a)), the vectors $\{y_j\}_{j \in J}$ can be separated from 0 using no more than ν functionals of X_α^*. So, the unit basis $\{e_j\}_{j \in J}$ of $c_0(J)$ can be separated from 0 using ν functionals of $\ell_1(J)$, which cannot be if $|J| = \mu > \nu$. □

Since μ is regular, for each $\alpha < \mu$, one has

$$|\{i \in I \colon \operatorname{supp}(y_i) \cap [0, \alpha) \neq \emptyset\}| < \mu. \tag{1.9}$$

This allows us to introduce a correspondence between points of I, certain subsets of I and limit ordinals below μ that 'stabilises supports' as follows. Given $\beta, \gamma \leq \mu$, we write $I[\beta, \gamma) = \{i \in I \colon \operatorname{supp} y_i \subset [\beta, \gamma)\}$.

Claim 2 Given $\beta < \mu$ and $j \in I[\beta, \mu)$, there exist a limit ordinal $\rho = \rho(\beta, j)$ with $\beta < \rho < \mu$ and a subset $J(\beta, j) \subset I[\beta, \mu)$ containing j, with $|J(\beta, j)| < \mu$, such that

- for every $i \in J(\beta, j)$, we have $\operatorname{supp} y_i \subset [\beta, \rho)$,
- if $i \in I[\beta, \mu) \backslash J(\beta, j)$, then $\operatorname{supp} y_i \cap [\beta, \rho) = \emptyset$.

Proof of Claim 2 Take any limit ordinal $\alpha_1 \in [\beta, \mu)$ such that $[\beta, \alpha_1)$ contains $\operatorname{supp} y_j$. By (1.9), the set $\{i \in I \colon \operatorname{supp}(y_i) \cap [\beta, \alpha_1) \neq \emptyset\}$ has cardinality smaller than μ. As μ is regular, we can find another limit ordinal $\alpha_2 \in (\alpha_1, \mu)$ such that

$$i \in I[\beta, \mu) \text{ and } \operatorname{supp} y_i \cap [\beta, \alpha_1) \neq \emptyset \implies \operatorname{supp} y_i \subset [\beta, \alpha_2).$$

Iterating the argument, we obtain a strictly increasing sequence of limit ordinals $(\alpha_k)_{k \geq 1}$ with $\alpha_k < \mu$ for all k in such a way that

$$i \in I[\beta, \mu) \text{ and } \operatorname{supp} y_i \cap [\beta, \alpha_k) \neq \emptyset \implies \operatorname{supp} y_i \subset [\beta, \alpha_{k+1}).$$

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We set the limit ordinal $\rho = \sup_{k\geq 1} \alpha_k$. Clearly, $\rho < \mu$, and the set

$$J = \Big\{ i \in I[\beta,\mu) \colon \ \operatorname{supp} y_i \cap [\beta,\rho) \neq \emptyset \Big\} = \bigcup_{k\geq 1} \Big\{ i \in I[\beta,\mu) \colon \ \operatorname{supp} y_i \cap [0,\alpha_k) \neq \emptyset \Big\}$$

has the required properties. □

The main piece of the proof is the following:

Claim 3 There exists an increasing family of ordinals $(\rho_\alpha)_{\alpha<\mu}$ such that

- $\rho_\alpha < \mu$ is a limit ordinal for all $\alpha > 0$ and $\rho_0 = 0$,
- the sets $I_\alpha = I[\rho_\alpha, \rho_{\alpha+1})$ form a partition of I, with $|I_\alpha| < \mu$ for all $\alpha < \mu$.

Proof of Claim 3 Note that μ, being an infinite cardinal, cannot be a successor ordinal, so $\alpha + 1 < \mu$ for $\alpha < \mu$. The proof is by transfinite induction on α. For the initial step we set $\beta = 0$ and $j = i_0$ in Claim 2. Then $I_0 = J(0, i_0)$, $\rho_0 = 0$ and $\rho_1 = \rho(0, i_0)$. Note that $i_0 \in I_0$ and that ρ_1 is a limit ordinal. Let us perform the inductive step: to this end, assume that for some $\eta < \mu$, we have already obtained an increasing family of limit ordinals $(\rho_{\alpha+1})_{\alpha<\eta}$ such that

($\dagger_\eta$) $|I_\alpha| < \mu$ for all $\alpha < \eta$ and $i_\beta \in \bigcup_{\alpha\leq\beta} I_\alpha$ for all $\beta < \eta$,
($\ddagger_\eta$) if $i \notin \bigcup_{\alpha<\eta} I_\alpha$, then $\operatorname{supp} y_i \bigcap \bigcup_{\alpha<\eta} [\rho_\alpha, \rho_{\alpha+1}) = \emptyset$,

and let us focus on i_η. If i_η is already in $\bigcup_{\alpha<\eta} I_\alpha$, there is nothing to do but wait: set $\rho_\eta = \rho_{\eta+1} = \sup_{\alpha<\eta} \rho_{\alpha+1}$. Needless to say, in this case, we have $I_\eta = \emptyset$, but we are at peace with that. Otherwise, if i_η is not yet in $\bigcup_{\alpha<\eta} I_\alpha$, we distinguish two cases, as is by now customary:

• If $\eta = \gamma + 1$ is a successor, use Claim 2 with $j = i_\eta$ and $\beta = \rho_\eta = \rho_{\gamma+1}$ and set $\rho_{\eta+1} = \rho(\rho_\eta, \eta)$. Note that I_η corresponds to the output set $J(\rho_\eta, \eta)$.

• If η is a limit ordinal, set $\beta = \sup_{\alpha<\eta} \rho_\alpha$ and $j = i_\eta$ in Claim 2 and $\rho_\eta = \beta$, $\rho_{\eta+1} = \rho(\beta, j)$. This yields $I_\eta = J(\beta, j)$, and everything works fine.

This finishes the induction process. Iterating the construction until $\eta = \mu$ proves the claim. □

Finally, we use Claim 3 to conclude the proof. It is clear that $\overline{[y_i \colon i \in I_\alpha]}$ is a subspace of $\overline{[X_\beta \colon \rho_\alpha \leq \beta < \rho_{\alpha+1}]}$ κ-isomorphic to $c_0(I_\alpha)$, so hypothesis (c) provides a projection $R_\alpha \colon \overline{[X_\beta \colon \rho_\alpha \leq \beta < \rho_{\alpha+1}]} \longrightarrow \overline{[y_i \colon i \in I_\alpha]}$ with norm at most $K\kappa$. Since

$$(P_{\rho_{\alpha+1}} - P_{\rho_\alpha})[X] = \overline{[X_\beta \colon \rho_\alpha \leq \beta < \rho_{\alpha+1}]}$$

and $\lim_{\alpha\to\mu} \|(P_{\rho_{\alpha+1}} - P_{\rho_\alpha})x\| = 0$ for each $x \in X$, it is possible to define a projection $Q \colon X \longrightarrow Y$ taking

$$Q(x) = \sum_{\alpha<\mu} R_\alpha (P_{\rho_{\alpha+1}} - P_{\rho_\alpha})(x).$$

Quite clearly, $\|Q\| \leq 2MK\kappa$. □

We are thus ready to provide a lavish solution to Yost's problem.

Proposition 1.7.13 *Let $m < \omega$. If K is a Valdivia compact of weight at most $\aleph_m$ then $C(K)$ is 2^{m+1}-Sobczyk.*

Proof Note that the dimension of $C(K)$ equals the weight of K. The proof proceeds by induction on m. Before we begin, recall that Proposition 1.7.10 asserts that $C(K)$ spaces with K a Valdivia compact are overt examples of spaces with a PRI decomposition *and* that the spaces in the decomposition can be chosen $C(S)$-spaces with S Valdivia compact again. Now trust us, just sit at the peak of the induction roller coaster and let yourself go down with it: when $m = 0$, separable ($C(K)$ or not) spaces are 2-Sobczyk; when $m = 1$, Lemma 1.7.12 shows that Banach spaces with a PRI and dimension $\aleph_1$ ($C(K)$ or not) are 4-Sobczyk. This includes $C(K)$ spaces with K a Valdivia compact. And from that point on, recall that we only need to consider $C(K)$-spaces with K Valdivia and apply Lemma 1.7.12. □

A class of compacta $\mathcal{K}$ produces two classes of Banach spaces: one by the simple method of isolating those Banach spaces X such that $B_X^* \in \mathcal{K}$, the other by generation – an element of $\mathcal{K}$ spans a dense subspace of X. Sometimes the match is perfect, as in the case of Eberlein compacta, but not always; that explains the formulation of the next result.

Proposition 1.7.14 *Let X be a Banach space such that B_X^* is a Valdivia compact. If $m < \omega$, then every M-isomorphic copy of $c_0(\aleph_m)$ in X is $2^{m+1}M$-complemented.*

Proof We show that when K is a Valdivia compact, any copy of $c_0(\aleph_m)$ inside $C(K)$ is contained in a complemented subspace $C(K_m)$ of $C(K)$ such that K_m is a Valdivia compact with $\dim C(K_m) = \aleph_m$ in order to then apply Proposition 1.7.13 to get a 2^{m+1} projection. To do that, just let yourself go down the cardinal slide: let I be a set of cardinality $\aleph_m$ and $K \subset [0, 1]^I$ a Valdivia compact. Let ω_m be the first ordinal with cardinal $|\omega_m| = \aleph_m$, let $\mu = \dim C(K)$, and pick a subset $H = \{h_i : i < \omega_m\}$ of $K \cap \Sigma(I)$ that norms $c_0(I)$. Use Lemma 1.7.9 to obtain the increasing family $\{I_\alpha : \omega \leq \alpha \leq \mu\}$ and then Proposition 1.7.10 to get a PRI $(P_\alpha)_{\omega\leq\alpha\leq\mu}$ on $C(K)$ for which $P_\alpha[C(K)] = C(K_\alpha)$ with each $K_\alpha \subset K$ a Valdivia compact. By construction, P_{ω_m} is an isometry on $c_0(I)$, and $\dim C(K_m) = \aleph_m$. □

In due course, Proposition 2.2.15 will show that the preceding result is optimal. On the other hand, despite that ordinal spaces $[0, \alpha]$ are not Valdivia compacta for $\alpha \geq \omega_2$ [147, II. Proposition 2], the decomposition method of Lemma 1.7.12 still works, providing:

Proposition 1.7.15 *The space $C[0, \omega_2]$ is 2^3-Sobczyk.*

Proof The space $C[0, \omega_2]$ admits a PRI $(P_\alpha)_{\omega \leq \alpha < \omega_m}$ given by

$$P_\alpha(f)(\beta) = \begin{cases} f(\beta) & \text{if } \beta \leq \alpha \\ f(\alpha) & \text{if } \beta \geq \alpha \end{cases}$$

(details can be seen in [148, p. 259]). Taking into account that the range of P_α is isometric to $C[0, \alpha]$ for $\alpha < \omega_2$, and that $[0, \alpha]$ is a Valdivia compact, we are ready for induction: Proposition 1.7.13 yields that it is 4-Sobczyk, and the induction Lemma 1.7.12 applies to obtain that $C[0, \omega_2]$ is 8-Sobczyk. □

It could be interesting to determine relations between I and α so that copies of $c_0(I)$ are complemented in $C[0, \alpha]$. A remarkable example in [302, Theorem 2.7] isolates a compact scattered space K of height 3 and Lindelöf (every open cover contains a countable subcover) in its weak topology that contains an uncomplemented copy of $c_0(\aleph_1)$.

1.8 Notes and Remarks

1.8.1 Topological Stuff

The reader can skip this section now and eventually return when some non-Hausdorff space pops up. Because non-Hausdorff topologies will pop up. Indeed, there are places in this book where non-Hausdorff linear topologies are unavoidable. Let X be a linear space. A mapping $\varrho\colon X \longrightarrow \mathbb{R}_+$ is said to be a semi-quasinorm if it is positively homogeneous (that is, $\varrho(\lambda x) = |\lambda|\varrho(x)$ for every $x \in X$ and every scalar λ) and there is a constant Δ such that $\varrho(x + y) \leq \Delta(\varrho(x) + \varrho(y))$ for all $x, y \in X$. If, moreover, one has $\varrho(x + y)^p \leq \varrho(x)^p + \varrho(y)^p$ for all $x, y \in X$ then we say that ϱ is a semi-p-norm, or just a seminorm when $p = 1$. Yes, right: this is nothing different from a quasinorm, just omitting the requirement that if an element has 'size' zero, it has to be zero. Let us agree that a semi-quasinormed space is a linear space X endowed with a semi-quasinorm ϱ. In a semi-quasinormed space, we can form the linear topology for which the sets $\{x \in X\colon \varrho(x) \leq \varepsilon\}$ are a fundamental system of neighbourhoods of zero as in the quasinormed case. Needless to say, such a topology is Hausdorff precisely when ϱ is a quasinorm. There is also a uniform structure whose (basic) neighbourhoods of the diagonal are the sets $\{(x, y) \in X \times X : \varrho(y - x) \leq \varepsilon\}$. It turns out that X is complete if and only if

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every Cauchy sequence converges. This can be taken as the definition, if one prefers. Of course, by a Cauchy sequence, we mean a sequence (x_n) such that for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $\varrho(x_n - x_m) < \varepsilon$ for all $n, m \geq k$. And (x_n) converges to x if $\varrho(x - x_n) \to 0$ as $n \to \infty$. Note that if (x_n) converges to x and $\varrho(x' - x) = 0$, then (x_n) converges to x' too. By the very definition, $\ker\varrho = \{x \in X : \varrho(x) = 0\}$ is a closed subspace of X and $\varrho(x)$ essentially depends only on the class of x in $X/\ker\varrho$, because when $y \in \ker\varrho$, $\Delta^{-1}\varrho(x) \leq \varrho(x + y) = \Delta\varrho(x)$. When ϱ is a semi-p-norm, we actually have $\varrho(x + y) = \varrho(x)$. In any case, ϱ induces a quasinorm $\varrho[x] = \inf_{\varrho(y)=0} \varrho(x + y)$ on the quotient space $X/\ker\varrho$. Since any linear projection $X \longrightarrow \ker\varrho$ is continuous, $\ker\varrho$ is complemented in X, and X is linearly isomorphic to $\ker\varrho \times X/\ker\varrho$ endowed with the product topology, corresponding to the functional $(y, [x]) \longmapsto \max(\varrho(y), \varrho[x]) = \varrho[x]$. No open mapping theorem exists for non-Hausdorff spaces: consider the formal identity $X \longrightarrow Y$, where X is your favourite Banach space and Y is the same space with the trivial seminorm. The two basic examples of semi-quasinormed spaces to keep in mind are:

• The quotient of a quasinormed space X by a possibly non-closed subspace Y endowed with the quotient semi-quasinorm

$$\|x + Y\| = \inf_{y \in Y} \|x + y\|.$$

The class of x in X/Y is zero if and only if $x \in Y$. However, we have $\|x+Y\| = 0$ if and only if x belongs to the closure of Y in X. It is clear that if X is complete (a quasi-Banach space), then so is X/Y, no matter if it is Hausdorff or not.

• The space $\mathsf{Q}(X, Y)$ of homogeneous maps $\Phi\colon X \longrightarrow Y$ acting between two quasinormed spaces X, Y such that

$$Q(\Phi) = \sup_{x,y \neq 0} \frac{\|\Phi(x + y) - \Phi(x) - \Phi(y)\|}{\|x\| + \|y\|} < \infty.$$

It is clear that $\Phi \in \mathsf{Q}(X, Y) \longmapsto Q(\Phi) \in \mathbb{R}_+$ is a semi-quasinorm whose modulus of concavity does not exceed that of Y and that $Q(\Phi) = 0$ if and only if Φ is linear (maybe unbounded). These maps will have their moments of glory in Chapter 3 and after that throughout the book.

A subset B of a topological vector space X is said to be bounded if it is absorbed by all neighbourhoods of zero; i.e. for every neighbourhood V of zero, there is $\lambda > 0$ such that $B \subset \lambda V$. A topological vector space is said to be locally bounded if it has a bounded neighbourhood of the origin. Semi-quasinormed spaces are locally bounded because the unit ball $B_X = \{x \in X : \varrho(x) \leq 1\}$ is a bounded neighbourhood of zero. Conversely, if B is a bounded

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symmetric neighbourhood of zero in X, the functional $\|x\| = \inf\{t > 0\colon x \in tB\}$ is a semi-quasinorm giving the topology of X.

1.8.2 Orlicz, Young, Fenchel and L_0 Too

The terminology in this section may differ from that used in other, more respectable texts. An Orlicz function is a continuous, increasing function $\varphi\colon [0,\infty) \longrightarrow [0,\infty)$ vanishing only at zero *and* satisfying the 'Δ_2-condition': there is a constant C such that $\varphi(2t) \le C\varphi(t)$ for all $t \ge 0$. We do not require Orlicz functions to be convex or that $\varphi(t) \to \infty$ as $t \to \infty$, conditions which are basically equivalent to the fact that the associated Orlicz space is a Banach space (see, for instance, Lindenstrauss-Tzafriri [334, pp. 137]). If (S,μ) is a measure space then the associated Orlicz space $L_\varphi(\mu)$ is the space of those measurable functions $f\colon S \longrightarrow \mathbb{K}$ such that

$$|f|_\varphi = \int_S \varphi(|f(s)|)d\mu(s) < \infty.$$

Although $|\cdot|_\varphi$ need not be homogeneous or subadditive, it defines a linear topology on $L_\varphi(\mu)$ for which the sets $\{f\colon |f|_\varphi < \varepsilon\}$ are a neighbourhood base at zero. The condition $\lim_{\lambda\to 0}\sup_t \frac{\varphi(\lambda t)}{\varphi(t)} = 0$ guarantees that $L_\varphi(\mu)$ is a locally bounded space, in which case, the functional $\|f\|_\varphi = \inf\{r > 0\colon |r^{-1}f|_\varphi \le 1\}$ is a quasinorm, called the *Luxemburg quasinorm*. When μ is the counting measure on $\mathbb{N}$, we obtain the so-called Orlicz sequence spaces.

Let V be a finite-dimensional linear space, possibly of dimension 1. A Young function $\Phi\colon V \longrightarrow \mathbb{R}^+$ is an even convex function such that $\Phi(tv) \to \infty$ for each $v \in V$ as $|t| \to \infty$. We do not require that Φ vanish only at zero. A family $\Phi_k\colon V_k \longrightarrow \mathbb{R}^+$ of Young functions defines a *modular* sequence space

$$h((\Phi_k)_k) = \left\{ v \in \prod_{k\ge 1} V_k\colon \sum_{k=1}^{\infty} \Phi_k(tv_k) < \infty \text{ for all } t > 0 \right\},$$

equipped with the Luxemburg norm

$$\|v\|_{(\Phi_k)_k} = \inf\left\{ t > 0\colon \sum_{k=1}^{\infty} \Phi_k(v_k/t) \le 1 \right\}.$$

If $V_k = \mathbb{K}^n$ for some n and all k and all the Φ_k agree with some Young function $\Phi\colon \mathbb{K}^n \longrightarrow \mathbb{R}_+$, then the modular sequence space $h(\Phi)$ is called a Fenchel-Orlicz space; if, moreover, $n = 1$, then $h(\Phi)$ agrees with the small Orlicz space as defined in [334, bottom of p. 137]. There is a considerable overlap between Orlicz sequence spaces and modular spaces in the locally convex zone. These subtleties will be necessary only in Chapter 8: note that, according to our fussy definitions, c_0 is a modular sequence space, but not an Orlicz space.

The space L_0 of all measurable functions on the unit interval deserves a special mention. The topology of convergence in measure is metrisable: set

$$|h|_0 = \int_0^1 \frac{|h|}{1+|h|}dt$$

so that the formula $d(f,g) = |f-g|_0$ defines a complete (invariant) metric on L_0. Thus, L_0 is an Orlicz function space in the wide sense adopted earlier. However, L_0 is not locally bounded. More yet:

Proposition *Each operator from L_0 to a quasinormed space is zero.*

Proof The key is that $|f|_0 \le \lambda(\text{supp}(f))$ regardless of the values assumed by f. Let $u\colon L_0 \longrightarrow Y$ be an operator, where Y is a p-normed space. Take $\delta > 0$ such that $\|u(f)\| \le 1$ for $|f|_0 \le \delta$. Divide $[0,1]$ into n subintervals $I_1, \dots, I_n$ of measure less than δ. Pick any $f \in L_0$. Then $f = \sum_{i=1}^n f_i$, where $f_i = 1_{I_i} f$. For each scalar c, we have

$$\|u(cf)\|^p \le \sum_{1\le i\le n} \|u(cf_i)\|^p \le n,$$

and since n is fixed and c arbitrary, we see that $\|u(f)\| = 0$. □

1.8.3 Ultrapowers of L_p When $0 < p < 1$

We emphatically concluded Section 1.4 with the assertion that L_p is not an ultrasummand if $0 < p < 1$. The following result shows that L_p is uncomplemented even in its 'countable' ultrapowers. The proof, a re-elaboration of a quip of Kalton [255, Proof of Lemma 8.1], will be eased by recalling that, given a family $(S_i)_{i\in I}$ of sets and an ultrafilter $\mathcal{U}$ on I, the *set theoretic ultraproduct* $\langle S_i\rangle_{\mathcal{U}}$ is the set $\prod_i S_i$ factored by the equivalence relation $(s_i) = (t_i) \iff \{i \in I : s_i = t_i\} \in \mathcal{U}$. The class of (s_i) in $\langle S_i\rangle_{\mathcal{U}}$ will be denoted $\langle(s_i)\rangle$.

Proposition *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then $\mathfrak{L}((L_p)_{\mathcal{U}}, Y) = 0$ for every separable quasi-Banach space Y.*

Proof Let us treat the Lebesgue measure on $[0,1]$ as a probability and, accordingly, the elements of L_p as random variables. Our first observation is that if $f \in L_p$ is simple then there is a sequence of 'Rademacher-like' functions (r_n) mutually independent and independent with f such that $\lambda\{t\colon r_n(t) = \pm 1\} = \frac{1}{2}$ for all $n \in \mathbb{N}$, where λ is Lebesgue measure on the unit interval: just write $f = \sum_k a_k 1_{A_k}$ with A_k a partition of $[0,1]$ and work on each A_k separately. Take

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finitely many non-zero scalars c_n. Applying Khintchine's inequality [153, p. 10] on each A_k, we get

$$\Big\| \sum_n c_n r_n f \Big\|_p \le \|f\|_p \|(c_n)\|_2. \tag{1.10}$$

Now, let $\mathcal{U}$ be a free ultrafilter on a countable set I and form the ultrapower $(L_p)_{\mathcal{U}}$. For notational reasons, it's better to keep using I instead of $\mathbb{N}$ for the index set of the ultrafilter. Take a normalised $f \in (L_p)_{\mathcal{U}}$ and a representative (f_i) with f_i simple and $\|f_i\|_p = 1$ for all $i \in I$. For each i we select a 'Rademacher' sequence $(r_i^n)_{n\ge 1}$ of mutually independent functions which are, moreover, independent with f_i and $\lambda\{t\colon r_i^n(t) = \pm 1\} = \frac{1}{2}$ for all n and i. Let $n\colon I \longrightarrow \mathbb{N}$ be any function. Consider the class of $(r_i^{n(i)})$ in the ultrapower $(L_\infty)_{\mathcal{U}}$ and write $[(r_i^{n(i)})]f = [(r_i^{n(i)} f_i)]$. The class of $(r_i^{n(i)})$ in $(L_\infty)_{\mathcal{U}}$ depends only on the class of $(n(i))$ in the set theoreric ultrapower $\langle\mathbb{N}\rangle_{\mathcal{U}}$. Thus, if $\alpha = \langle(n(i)\rangle$, then $r_\alpha = [(r_i^{n(i)})]$ is correctly defined as an element of $(L_\infty)_{\mathcal{U}}$, and so is the product $r_\alpha f = [(r_i^{n(i)} f_i)]$ for every $f \in (L_p)_{\mathcal{U}}$.

Claim 1 If $T\colon (L_p)_{\mathcal{U}} \longrightarrow Y$ is an operator and f and $\{r_\alpha\colon \alpha \in \langle\mathbb{N}\rangle_{\mathcal{U}}\}$ are as before, then there is $y \in Y$ such that for every $\varepsilon > 0$, the set

$$\{\alpha \in \langle\mathbb{N}\rangle_{\mathcal{U}} : \|y - T(r_\alpha f)\| \le \varepsilon\}$$

has the cardinality of the continuum.

Proof of Claim 1 Assume Y is q-normed for the remainder of the proof and write it as the union of countably many balls of radius 1. Since $\langle\mathbb{N}\rangle_{\mathcal{U}}$ has the cardinality of the continuum (use an almost disjoint (Definition 2.2.9) family of size $\mathfrak{c}$), some of its members must contain $T(r_\alpha f)$ for 'continuum many' αs. Take that ball and write it as the union of countably many balls of radius $\frac{1}{2}$, and so on. Continuing in this way, we get a sequence of closed balls $(B_n)_{n\ge 1}$ in Y such that $B_{n+1} \subset B_n$ for every $n \in \mathbb{N}$; B_n has radius 2^{-n}, and for every n, the cardinality of the set of those $\alpha \in \langle\mathbb{N}\rangle_{\mathcal{U}}$ for which $T(r_\alpha f) \in B_n$ is the continuum. The intersection of the balls yields the point we were looking for. □

Claim 2 The point in Claim 1 has to be zero.

Proof of Claim 2 Let $y \in Y$ a point for which the conclusion of Claim 1 is true. Take a sequence $(\alpha_n)_n$ of different indices in $\langle\mathbb{N}\rangle_{\mathcal{U}}$ such that $\|y - T(r_{\alpha_n} f)\|_p^q \le 2^{-n}$. Then, for every integer n, we have

$$\Big\| ny - \sum_{k=1}^n T(r_{\alpha_k} f)\Big\|_p^q \le \sum_{k=1}^n \frac{1}{2^k} \le 1 \quad \Longrightarrow \quad \overbrace{\Big\| \sum_{k=1}^n T(r_{\alpha_k} f)\Big\|_p^q \ge n^q \|y\|^q - 1}^{(\dagger)}.$$

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But if we apply (1.10) 'coordinatewise' to $\sum_{k=1}^{n} r_{\alpha_k} f$ with $c_k = 1$ for $1 \le k \le n$, we obtain

$$\Big\| \sum_{k=1}^{n} r_{\alpha_k} f \Big\|_p \le n^{1/2} \qquad \Longrightarrow \qquad \Big\| \sum_{k=1}^{n} T(r_{\alpha_k} f) \Big\|_p^q \le \|T\|^q n^{q/2},$$

which is compatible with (†) only if $\|y\| = 0$. This proves Claim 2. □

To conclude the proof, pick any normalised $f \in (L_p)_{\mathfrak{U}}$ and construct the corresponding family (r_α) as before. For each $\varepsilon > 0$, there is some α such that $\|T(r_\alpha f)\| \le \varepsilon$. Note that, by independence, $\|(1 + r_\alpha) f\|^p = 2^{p-1}\|f\|^p$, that is, $\|(1 + r_\alpha) f\| = 2^{1-1/p}\|f\|$. Recalling that Y is q-normed, we have

$$\|Tf\|^q \le \|T(1 + r_\alpha) f\|^q + \|T(r_\alpha f)\|^q \le \varepsilon^q + \|T\|^q 2^{q-q/p}\|f\|^q.$$

But ε is arbitrary, and so $\|T\| \le 2^{1-1/p}\|T\|$; that is, $T = 0$. □

This result trivially implies that every compact operator on L_p with values in a quasi-Banach space is zero if $0 < p < 1$, which is a result of Pallaschke. We will not explore this line of research here since anyone interested in operators on non-locally convex spaces should begin with Chapters 7 and 8 of [283] or with [166]. The conclusion we prefer to draw instead is that the proof of the proposition depends only on that fact that the cardinality of the set theoretic ultrapower $\langle \mathbb{N} \rangle_{\mathfrak{U}}$ is greater than the dimension of the target space Y; it actually works when $\dim(Y) < \mathfrak{c}$. On the other hand, $\langle \mathbb{N} \rangle_{\mathfrak{U}}$ can be as large as we want:

Lemma *For every cardinal $\aleph$, there is an ultrafilter $\mathfrak{U}$ on some index set such that $|\langle \mathbb{N} \rangle_{\mathfrak{U}}| \ge \aleph$.*

Proof Let A be any set and $I = \text{fin}(A)$ the set of all non-empty finite subsets of A ordered by inclusion. Let $\mathfrak{U}$ be any ultrafilter refining the order filter of I. Let $\langle F \rangle_{\mathfrak{U}}$ denote the set theoretic ultraproduct of the family $\{F : F \in \text{fin}(A)\}$ following $\mathfrak{U}$, that is, the 'elements' of $\langle F \rangle_{\mathfrak{U}}$ are (classes of) families (a_F) of the product space $\prod_{\text{fin}(A)} F$, where a_F belongs to F for every F and $\langle (a_F) \rangle = \langle (b_F) \rangle$ if the set of those F for which $a_F = b_F$ belongs to $\mathfrak{U}$. There is an obvious embedding of A into $\langle F \rangle_{\mathfrak{U}}$. Get the idea? Thus, $|A| \le |\langle F \rangle_{\mathfrak{U}}| \le |\langle \mathbb{N} \rangle_{\mathfrak{U}}|$. □

Corollary *If Y is an ultrasummand, then $\mathfrak{L}(L_p, Y) = 0$ for $0 < p < 1$.*

Proof Let $T : L_p \longrightarrow Y$ be an operator, and let $\mathfrak{U}$ be an ultrafilter on some index set I for which the cardinality of $\langle \mathbb{N} \rangle_{\mathfrak{U}}$ is strictly greater than $\dim(Y)$. Let $P : Y_{\mathfrak{U}} \longrightarrow Y$ be a projection through the diagonal embedding. The hypotheses imply that the composition $PT_{\mathfrak{U}} : (L_p)_{\mathfrak{U}} \longrightarrow Y$ is zero, and so is T. □

1.8.4 Sobczyk's Theorem Strikes Back

According to Veech's proof, Sobczyk's theorem is the statement that every weak*-null sequence on a subspace of a separable Banach space can be extended to a weak*-null sequence on the whole space. The norm of the elements in the sequence of extended functionals, however, doubles. This fact makes it apparently impossible to produce a proof à la Hahn–Banach obtaining a suitable extension to one more dimension and then iterating the argument. Such a proof is almost possible, nonetheless, and it was obtained in passing by Kalton [273, Section 5]. Following, pretty badly, Behrends [35], we call an ordered space as *groundless* when any decreasing sequence of elements has a lower bound. The groundless set we need is the space $\mathcal{P}^*_\infty(\mathbb{N})$ of all infinite subsets of the integers modulo finite sets endowed with the order $[A] \leq [B]$ if $B\backslash A$ is finite. It is not obvious that it is obvious that $\mathcal{P}^*_\infty(\mathbb{N})$ is groundless: if $(A_n)_{n\geq 1}$ is a sequence of infinite subsets of $\mathbb{N}$ such that $[A_{n+1}] \leq [A_n]$ for all n, then $A = \{k_n : n \in \mathbb{N}\}$, where $k_n \in \bigcap_{i\leq n} A_i$ is infinite and $[A] \leq [A_n]$ for all n. A mapping $f : P \longrightarrow Q$, acting between ordered sets, is order preserving if $y \leq x$ implies $f(y) \leq f(x)$. We say that a point $x \in P$ is stationary if $f(y) = f(x)$ for all $y \leq x$.

Behrends' lemma *Let $f : P \longrightarrow Q$ be an order-preserving map, where P is groundless. Then f has a stationary point in the following cases:*

- *Q is a subset of $\mathbb{R}^{\mathbb{N}}$.*
- *$Q = \mathcal{K}(M)$ is the set of all compact subspaces of a metric space M ordered by inclusion.*

Proof We first show the result for $Q = \mathbb{R}$: since P is groundless, there is $p_0 \in P$ such that f must be bounded on $\{y \in P : y \leq p_0\}$. If $m = \inf\{f(y) : y \leq p_0\} = \inf f(y_n)$ and $y \leq y_n$ for all n, then $f(a) = f(b)$ for all $a, b \leq y$. The result for $\mathbb{R}^{\mathbb{N}}$ follows by diagonalisation. The second case also follows by taking into account that $\mathcal{K}(M)$ is 'countably determined' in the sense that there is an order-preserving, injective mapping $g : \mathcal{K}(M) \longrightarrow [-1, 1]^{\mathbb{N}}$. Indeed, let $(h_n)_{n\in\mathbb{N}}$ be a dense sequence in the unit ball of the real-valued $C(M)$ and set $g(K) = (\sup_{s\in K} h_n(s))_{n\in\mathbb{N}}$. □

Let X be a Banach space, and let K_n be a sequence of weak*-compact convex subsets of the unit ball of X^*. If A is an infinite subset of $\mathbb{N}$, we set

$$K_A = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{i\in A, i\geq n} K_i}. \tag{1.11}$$

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This definition is formally identical to the definition of an upper limit and, clearly, K_A depends only on $[A]$. One thus has an order-preserving function

$$K_\bullet : \mathcal{P}^*_\infty(\mathbb{N}) \longrightarrow \mathcal{K}(B^*_X).$$

Lemma *Assume that K_A is weak*-metrisable.*

(a) *If $x^* \in K_A$ then x^* is the weak*-limit of a sequence $(x^*_i)_{i\in A}$ with $x^*_i \in K_i$ for all $i \in A$ if and only if, for every infinite subset $B \subset A$, one has $x^* \in K_B$.*
(b) *There is an infinite subset $B \subset A$ with the property that $K_M = K_B$ for every $M \subset B$. For this B, the set K_B is convex.*

Proof The 'only if' implication in (a) is clear. For the converse, given $x^* \in K_A$, we can arrange a decreasing sequence of weak* open sets $(V_k)_{k\geq 0}$ such that $x^* = \bigcap_{k\geq 1}(V_k \cap K_A)$, with $V_0 = X^*$. The hypothesis implies that for each k the set $\{i \in A : K_i \cap V_k \neq \emptyset\}$ is cofinite in A. Pick an increasing sequence $(j(k))_k$ of indices in A such that $K_i \cap V_k \neq \emptyset$ for $i \geq j(k)$ and define a sequence $(x_i)_{i\in A}$ taking $x^*_i \in K_i \cap V_k$ if $j(k) \leq i < j(k+1)$. Clearly, $x^*_i \longrightarrow x^*$ weak* as i increases in A. The first part of (b) is an obvious application of Behrend's lemma to the map $K_\bullet : \mathcal{P}^*_\infty(A) \longrightarrow \mathcal{K}(K_A)$ defined as $B \longrightarrow K_B$. The convexity of K_B follows from part (a). □

It is plain that if the ambient compact $K_{\mathbb{N}}$ is metrisable then each infinite set contains a stationary subset, from which a kind of à-la-Hahn–Banach-better-than-the-original Sobczyk-like theorem follows:

Corollary *Let Y be a closed subspace of a real Banach space X such that X/Y is separable, and let $\tau : Y \longrightarrow c_0$ be an operator. If, for every $x \in X$, there is a λ-extension $T_x : Y + [x] \longrightarrow c_0$, then there is a λ-extension $T : X \longrightarrow c_0$.*

Proof Assume $\|\tau\| = 1$, and write $\tau = (\tau_n)$ as a sequence of functionals. Let K_n be the set of all extensions of τ_n with norm at most λ, and consider the family of compacta K_A defined by (1.11). The separability of X/Y makes the bounded subsets of $Y^\perp = \{x^* \in X^* : x^*|_Y = 0\}$ weak*-metrisable, as well as K_A for every $\mathcal{P}^*_\infty(\mathbb{N})$. Assume that no weak*-null extension of (τ_n) with norm at most λ exists. Applying the first part of the previous lemma with $A = \mathbb{N}$, we get an infinite subset $B \subset \mathbb{N}$ such that $0 \notin K_B$. Without loss of generality, we can assume this B is the stationary set appearing in (b), and so K_B is convex. We can thus separate K_B from 0 by an element $x \in X$; say $\langle x^*, x\rangle \geq \varepsilon$ for some $\varepsilon > 0$ and all $x^* \in K_B$. The hypothesis yields a λ-extension $T_x : Y + [x] \longrightarrow c_0$. If $(f_n)_{n\geq 1}$ is the corresponding sequence of functionals, we have $f_n \in K_n$ for all n since $\|f_n\| \leq \lambda$ and $\langle f_n, x\rangle \to 0$ as $n \to \infty$. If f is a weak*-accumulation point of the subsequence $(f_n)_{n\in B}$, then $f \in K_B$ and $\langle f, x\rangle = 0$, which contradicts the choice of x. □

The previous approach only really makes sense for $\lambda < 2$, because for $\lambda = 2$, we already have Sobczyk's theorem. For $\lambda = 2$, it would provide a proof for Sobczyk's theorem if it were true that c_0-valued operators extend to one more dimension, doubling the norm. But proving that is actually as hard as Sobczyk's theorem! A nice loophole would exist were it obvious, clear or at least true that hyperplanes of Banach spaces are 2-complemented, but this is false; see comments and examples before Definition 2.1.6. Two more remarkable results about c_0 complementation deserve mention [394, Theorem 3]: *for every $n \in \mathbb{N}$, there exists a $6(n+1)$-Sobczyk space that is not n-Sobczyk*, from which it follows [394, Corollary 4] that there exists a Banach space X admitting a countable chain of subspaces $(Y_n)_{n\geq 0}$ such that $X = \overline{\bigcup_n Y_n}$, $Y_0 = c_0$, Y_n is complemented in Y_{n+1} but c_0 is not complemented in X – even if every copy of c_0 in X contains a subspace isomorphic to c_0 and complemented in X [182].

Sources

General references on quasi-Banach spaces are [283; 269; 408]. The result 1.1.5 is from Day [144], but the proof we present is in the spirit of [303, §15.9.9]. Proposition 1.2.5 is from [84]. The proof of Proposition 1.3.3 is a clever insight in Phillips' proof taken from an exercise in Bourbaki [47, 55, Exercice 16]. Two long-standing problems have been whether complemented subspaces of spaces with unconditional basis have unconditional basis and whether every Banach space contains an unconditional basic sequence. The first one is still open [83, Problem 1.8], while the second was solved by Gowers and Maurey [197], leading, with the aid of W. B. Johnson, to the discovery of H.I. spaces, thoroughly studied by Argyros and his group. Precisely, that H.I. spaces are subspaces of ℓ_∞ is due to Argyros, although a proof can be traced back to Plichko-Yost [396], and the proof presented in the text has been taken from [2]. The principle of local reflexivity is the wondrous creation of Lindenstrauss and Rosenthal [331]. Besides WCG spaces, the list of known spaces with SCP includes weakly sequentially complete Banach lattices [187], Banach spaces with the commuting bounded approximation property [83], duals of Asplund spaces [224, p. 38]; see also [207, Theorem 3.42], spaces of continuous functions on any ordinal [244, Theorem 1.6] and Plichko spaces [243]. Just in case the definition has momentarily slipped our minds, recall that a Banach space X is called Plichko if there is a dense subset $A \subset X$ and norming subset $B \subset X^*$ such that for every $x^* \in B$, the set $\{x \in A : \langle x^*, x\rangle \neq 0\}$ is countable. There also exist $\mathscr{C}$-spaces with SCP that are not Plichko [309]. Proposition 1.7.7 is from Yost [460, corollary], who invented the Veech topological spaces: those in which every separable subset is metrisable. The paper [396] contains a lot of additional information about the SCP property. Proposition 1.7.13 has been taken from the Argyros versus Spain paper [16], where much more results and examples can be found. The results and ideas in 1.8.4 are from Kalton [273], although we followed a different path through Behrends' lemma, which has been obviously taken from [35, Section 2], even if Behrends generously attributes the idea to Hagler and Johnson (see [152, p. 231]).