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ON THE TWISTOR SPACE OF THE SIX-SPHERE

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The set of all complex lines of the right-handed Dirac spinor bundle of a standard six-sphere is the total space of the twistor fibration. The twistor space, endowed with its natural Kähler structure, is recognised to be a six-dimensional complex quadric. The relevant group is Spin (7), which acts transitively on the six-quadric, as a group of fiber-preserving isometries. We use a result due to Berard-Bérgery and Matsuzawa to show the existence of a non-Kähler, non symmetric, Hermitian-Einstein metric on the six-quadric, which is Spin (7)-invariant.

1. INTRODUCTION

The present paper was motivated by the following result, which was obtained independently by Berard-Bérgery and Matsuzawa (see [**1**, **9**]): *let $F \to B \to M$ be a Riemannian submersion with totally geodesic fibres. Assume that the metrics g_F g_B and g_M are Einstein, with Einstein constants E_F, E_B and E_M respectively, and $E_F > 0$. If g_B is not locally a Riemannian product of g_F and g_M, then the metric g_B^t obtained by scaling the metric on B in the direction of F by a factor $t > 0$ is Einstein if and only if $t = 1$ or $t = \frac{E_F}{E_M - E_F}$.*

Obviously the two metrics above are different if and only if $E_F \neq \frac{1}{2} E_M$.

Wang and Ziller in [**12**], pointed out that the only known examples which satisfy the assumptions of the theorem, with $E_F \neq \frac{1}{2} E_M$, are the Hopf fibrations:

$$\begin{aligned} S^3 &\to S^{4n+3} &&\to \mathsf{HP}^n, \\ S^2 &\to \mathsf{CP}^{2n+1} &&\to \mathsf{HP}^n, \\ S^7 &\to S^{15} &&\to S^8 \end{aligned}$$

The Riemannian metrics $g^1_{S^{4n+3}}$, $g^1_{\mathsf{CP}^{2n+1}}$ and $g^1_{S^{15}}$ are the standard symmetric Einstein metrics. And $g^t_{S^{4n+3}}$, $g^t_{\mathsf{CP}^{2n+1}}$, $g^t_{S^{15}}$, $t = \frac{E_F}{E_M - E_F}$, are the homogeneous, non symmetric Einstein metrics found by Jensen ([8]), Bourguignon-Karcher ([3]) and Ziller ([**14**]).

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In this paper we will give a new example where the theorem above applies. The standard six-sphere with constant sectional curvature 4 is viewed as a homogeneous space of the group Spin (7):

$$S^6 = \mathrm{Spin}(7)/SU(4).$$

The twistor space of S^6 is realised as the set of all complex lines of the right-handed Dirac spinor bundle. Hence the twistor fibration is given by:

$$\mathbb{CP}^3 \subset \frac{\mathrm{Spin}(7)}{U(3)} \longrightarrow S^6.$$

The normal homogeneous metric g on $\mathrm{Spin}(7)/U(3)$ is a bundle metric, and is Einstein with Einstein constant $E = 12$. It is well known that the natural almost complex structure J is integrable, and (J, g) is a Kähler structure. The fibres are totally geodesic complex submanifolds of constant sectional curvature and with Einstein constant $\check{E} = 8$.

If we let Spin (7) act on $\mathbb{R}^8$ via its faithful 8 dimensional representation, then it acts transitively on the Grassmannian of the oriented planes of $\mathbb{R}^8$ (see [**5**]), and $U(3)$ is the isotropy subgroup (see [**7**]). Therefore the twistor space of S^6 may be recognised to be a six-dimensional quadric of a complex projective space $\mathbb{CP}^7$ of constant holomorphic sectional curvature 4.

Since the Einstein constant E of the base is 20, we see that the Riemannian submersion

$$\mathbb{CP}^3 \hookrightarrow Q_6 \rightarrow S^6$$

satisfies the assumptions of the theorem. Therefore: *scaling the metric g on Q_6 in the direction of $\mathbb{CP}^3$ by a factor $t = 2/3$ we get a* Spin (7)*-invariant, Hermitian–Einstein metric g' on the six-quadric.*

We compute the differential of the fundamental two form of (J, g'), and we show that (J, g') is not *Kähler*. Since every symmetric Riemannian metric on Q_6 is a Kähler metric *we deduce that g' is not symmetric.*

2. Preliminaries

We let spin (7) be the Lie subalgebra of su (8) whose elements are skew-Hermitian matrices of the form:

$$S(A, B) = \begin{vmatrix} A & \overline{B} \\ B & \overline{A} \end{vmatrix}$$

where: $A \in \mathrm{su}(4)$ and B is a 4×4 complex skew-symmetric matrix satisfying the following conditions:

$$\overline{B}^1_2 = B^3_4, \quad \overline{B}^1_3 = -B^2_4, \quad \overline{B}^1_4 = B^2_3. \tag{2.1}$$

The Lie subgroup Spin (7) $\to$ SU (8) corresponding to the Lie subalgebra spin (7) is isomorphic (see [4, 5]) to the universal covering group of SO (7).

The special unitary group SU (4) is regarded as a subgroup of Spin (7) by setting:

$$S \in \mathrm{SU}(4) \to \mathrm{S}(X, O) = \begin{Vmatrix} X & 0 \\ O & \overline{X} \end{Vmatrix}. \tag{2.2}$$

Hence the Lie subalgebra su (4) $\to$ spin (7) is given by all matrices of the form $\mathrm{S}(A, O)$, $A \in \mathrm{su}(4)$.

We let m be the vector subspace of spin (7), whose elements are of the form $S(O, B)$, and $B \in o(4, \mathbb{C})$ satisfies (2.1). In the following we will identify m with a six-dimensional Euclidean $\mathbb{R}^6$ as follows:

$$\begin{Vmatrix} X^1 \\ \vdots \\ \vdots \\ X^6 \end{Vmatrix} \to \begin{Vmatrix} 0 & -X^1+iX^2 & -X^3+iX^4 & -X^5+iX^6 \\ X^1-iX^2 & 0 & -X^5-iX^6 & X^3+iX^4 \\ X^3-iX^4 & X^5+iX^6 & 0 & -X^1-iX^2 \\ X^5-iX^6 & -X^3-iX^4 & X^1+iX^2 & 0 \end{Vmatrix}. \tag{2.3}$$

On m the inner product is obtained by using the Killing form. Then the adjoint representation of Spin (7) restricted to the subgroup SU (4) splits as a direct sum of two irreducible representations. The associated irreducible components of spin (7) are su(4) and m; furthermore the representation of SU (4) in m is just the 2 : 1 spin covering homomorphism of SU (4) $\simeq$ Spin (6) onto SO (6).

Now let us consider $U(3)$ as a subgroup of SU (4) given by:

$$Y \in U(3) \longmapsto \begin{Vmatrix} Y_1^1 & 0 & Y_2^1 & Y_3^1 \\ 0 & (\det Y)^{-1} & & 0 \\ Y_1^2 & 0 & Y_2^2 & Y_3^2 \\ Y_1^3 & 0 & Y_2^3 & Y_3^3 \end{Vmatrix}. \tag{2.4}$$

We thus have $\mathrm{su}(4) = u(3) \oplus n$, where n is identified with $\mathbb{C}^3$ by setting:

$$\begin{Vmatrix} Z^4 \\ Z^5 \\ Z^6 \end{Vmatrix} \longmapsto \begin{Vmatrix} 0 & -\overline{Z}^4 & 0 & 0 \\ Z^4 & 0 & Z^5 & Z^6 \\ 0 & -\overline{Z}^5 & 0 & 0 \\ 0 & -\overline{Z}^6 & 0 & 0 \end{Vmatrix}. \tag{2.5}$$

It is convenient now to identify m with $\mathbb{C}^3$ by setting:

$$\begin{Vmatrix} Z^1 \\ Z^2 \\ Z^3 \end{Vmatrix} \longmapsto \begin{Vmatrix} 0 & -\overline{Z}^1 & -Z^3 & Z^2 \\ \overline{Z}^1 & 0 & \overline{Z}^2 & \overline{Z}^3 \\ Z^3 & -\overline{Z}^2 & 0 & -Z^1 \\ -Z^2 & -\overline{Z}^3 & Z^1 & 0 \end{Vmatrix}. \tag{2.6}$$

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Then the adjoint representation of Spin(7) restricted to the subgroup U(3) decomposes as a direct sum of three irreducible representations, and $u(3) \oplus n \oplus m$ is the associated irreducible decomposition of spin(7). The representation of U(3) in m is given by

$$Y \in \mathrm{U}(3) \longmapsto \det(Y)Y \tag{2.7}$$

and the representation of $U(3)$ in n is:

$$Y \in \mathrm{U}(3) \longmapsto \det(Y)\overline{Y} \tag{2.8}$$

Now let ω denote the Maurer–Cartan form of Spin(7); we let θ be the $\mathbf{R}^6$-valued one-form obtained by using the decomposition spin(7) = su(4) $\oplus\, m$ and the identification (2.3). We let ψ be the su(4)-component of ω, and ϕ be the u(3)-component of ω. We denote by σ' and σ'' be the C^3-valued one-forms given by the m and n components of ω, modulo the identifications (2.6) and (2.5). Then the structure equations of Spin(7) can be written as follows:

$$\begin{aligned}
d\sigma^1 &= \left(\phi_2^2 + \phi_3^3\right) \wedge \sigma^1 + \bar{\phi}_1^2 \wedge \sigma^2 + \bar{\phi}_1^3 \wedge \sigma^3 + \bar{\sigma}^3 \wedge \sigma^5 - \bar{\sigma}^2 \wedge \sigma^6,\\
d\sigma^2 &= -\phi_1^2 \wedge \sigma^1 + \left(\phi_3^3 + \phi_1^1\right) \wedge \sigma^2 - \phi_3^2 \wedge \sigma^3 - \bar{\sigma}^3 \wedge \sigma^4 + \bar{\sigma}^1 \wedge \sigma^6,\\
d\sigma^3 &= -\phi_1^3 \wedge \sigma^1 + \bar{\phi}_3^2 \wedge \sigma^2 + \left(\phi_2^2 + \phi_1^1\right) \wedge \sigma^3 + \bar{\sigma}^2 \wedge \sigma^4 - \bar{\sigma}^1 \wedge \sigma^5,
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
d\sigma^4 &= \sigma^3 \wedge \sigma^2 - \sigma^2 \wedge \sigma^3 + \left(\phi_2^2 + \phi_3^3 + 2\phi_1^1\right) \wedge \sigma^4 + \phi_1^2 \wedge \sigma^5 + \phi_1^3 \wedge \sigma^6,\\
d\sigma^5 &= -\sigma^3 \wedge \sigma^1 + \sigma^1 \wedge \sigma^3 - \bar{\phi}_1^2 \wedge \sigma^4 + \left(\phi_1^1 + 2\phi_2^2 + \phi_3^3\right) \wedge \sigma^5 - \bar{\phi}_3^2 \wedge \sigma^5,\\
d\sigma^6 &= +\sigma^2 \wedge \sigma^1 - \sigma^1 \wedge \sigma^2 - \bar{\phi}_1^3 \wedge \sigma^4 + \phi_3^2 \wedge \sigma^5 + \left(\phi_1^1 + \phi_2^2 + 2\phi_3^3\right) \wedge \sigma^6.
\end{aligned} \tag{2.10}$$

Finally

$$\begin{aligned}
d\phi_1^1 + \phi_2^1 \wedge \phi_1^2 + \phi_3^1 \wedge \phi_1^3 &= \sigma^1 \wedge \bar{\sigma}^1 - \sigma^2 \wedge \bar{\sigma}^2 - \sigma^3 \wedge \bar{\sigma}^3 - \sigma^4 \wedge \bar{\sigma}^4,\\
d\phi_2^2 + \phi_1^2 \wedge \phi_2^1 + \phi_3^2 \wedge \phi_2^3 &= -\sigma^1 \wedge \bar{\sigma}^1 + \sigma^2 \wedge \bar{\sigma}^2 - \sigma^3 \wedge \bar{\sigma}^3 - \sigma^5 \wedge \bar{\sigma}^5,\\
d\phi_3^3 + \phi_1^3 \wedge \phi_3^1 + \phi_2^3 \wedge \phi_3^2 &= -\sigma^1 \wedge \bar{\sigma}^1 - \sigma^2 \wedge \bar{\sigma}^2 + \sigma^3 \wedge \bar{\sigma}^3 - \sigma^6 \wedge \bar{\sigma}^6,\\
d\phi_1^2 + \phi_1^2 \wedge \phi_1^1 + \phi_2^2 \wedge \phi_1^2 + \phi_3^2 \wedge \phi_1^3 &= -2\bar{\sigma}^1 \wedge \sigma^2 + \bar{\sigma}^5 \wedge \sigma^4,\\
d\phi_1^3 + \phi_1^3 \wedge \phi_1^1 + \phi_2^3 \wedge \phi_1^2 + \phi_3^3 \wedge \phi_1^3 &= -2\bar{\sigma}^1 \wedge \sigma^3 + \bar{\sigma}^6 \wedge \sigma^4,\\
d\phi_2^3 + \phi_1^3 \wedge \phi_2^1 + \phi_2^3 \wedge \phi_2^2 + \phi_3^3 \wedge \phi_2^3 &= -2\bar{\sigma}^2 \wedge \sigma^3 + \bar{\sigma}^6 \wedge \sigma^5,
\end{aligned} \tag{2.11}$$

where $\sigma' =^T \left(\sigma^1\sigma^2\sigma^3\right)$, $\sigma'' =^T \left(\sigma^4\sigma^5\sigma^6\right)$ and $\phi = (\phi_b^a)_{a,b=1,2,3}$

3. The spinor structure of S^6

Now let us consider the homogeneous space Spin (7)/ SU (4) and the SU (4)-principal fibre bundle:

$$p\colon \mathrm{SU}(4) \hookrightarrow \mathrm{Spin}(7) \to \mathrm{Spin}(7)/\mathrm{SU}(4). \tag{3.1}$$

Then the $\mathbf{R}^6$-valued one form $\theta =^T (\theta^1 \dots \theta^6)$ is a tensorial one-form in Spin (7), which transforms according to the representation $\mathrm{SU}(4) = \mathrm{Spin}(6) \xrightarrow{\lambda} \mathrm{SO}(6)$. Hence the quadratic form $ds^2 = \sum_i (\theta^i)^2$ and the exterior form $\theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^6$ are projectable on $\mathrm{Spin}(7)/SU(4)$, and they define a Riemannian metric g and a volume form.

The su (4)-valued one form Ψ defines a connection form in the bundle (3.1), and the o(6)-valued one-form $\lambda(\psi)$ satisfies the following identity:

$$d\theta = -\lambda(\psi) \wedge \theta. \tag{3.2}$$

By using the structure equations (2.10) and (2.11) one has

$$d\lambda(\psi)^i_j + \lambda(\psi)^i_k \wedge \lambda(\psi)^k_j = 4\theta^i \wedge \theta^j. \tag{3.3}$$

It follows that Spin (7)/ SU (4), when endowed with the invariant metric g is a space of constant curvature $+4$. We may go further and see that the principal fibre bundle (3.1) is merely the spin double cover of the oriented orthonormal frame bundle of Spin (7)/ SU (4) under the metric.

For this reason, we will, from now on, speak interchangably of Spin (7)/ SU (4) and S^6, even though we have given no explicit isometry between them.

We now consider the right-handed Dirac spinor bundle of S^6. This is the rank-4 Hermitian complex vector bundle associated with the principal fibre bundle (3.1):

$$\sum = \mathrm{Spin}(7) X_{\mathrm{SU}(4)} \mathrm{C}^4 \to S^6. \tag{3.4}$$

Since the structure group is SU (4), then the Dirac spinor bundle is naturally equipped together with a complex orientation. Furthermore, the connection ψ induces a Hermitian covariant derivative acting on the cross sections of $\sum$, and it preserves the complex orientation.

4. The twistor fibration of S^6

Let (M, g, Vol) be an even-dimensional oriented Riemannian manifold. Then the set $\mathcal{T}(M)$ of all complex-orthogonal structures on the tangent spaces of M^n whose orientation is compatible with the fixed volume form, endowed with the natural topology, is the total space of the twistor bundle $\tau\colon \mathcal{T}(M) \to M$.

Eells and Salamon in [6] show that if (M, g, Vol) is a four-dimensional spin manifold, then the twistor bundle may be viewed as the set of all complex lines of the right-handed Dirac spinor bundle of any spinor structure. In [11] and [13] it is proven that the same construction holds for six-dimensional spin manifolds.

For a $2n$-dimensional sphere S^{2n} the twistor bundle $\mathcal{T}(S^{2n})$ is given by

$$\tau \colon \frac{\mathrm{SO}(2n)}{\mathrm{U}(n)} \hookrightarrow \frac{\mathrm{SO}(2n+1)}{\mathrm{U}(n)} \to S^{2n}. \tag{4.1}$$

Hence $\mathcal{T}(S^{2n})$ is the smooth algebraic variety of all n-dimensional totally isotropic linear subspaces of $\mathbb{C}^{2n+1}$.

For $n = 6$, using the spinorial approach we get $\mathcal{T}(S^6) = \mathbb{P}(\sum)$, and hence the twistor fibration is given by

$$\tau \colon \mathbb{C}\mathbb{P}^3 \hookrightarrow \frac{\mathrm{Spin}(7)}{\mathrm{U}(3)} \to S^6. \tag{4.2}$$

It is well-known that $\mathrm{Spin}(7)$ acts transitively on the Grassmannian of the oriented planes of $\mathbb{R}^6$, via the faithful 8-dimensional representation of $\mathrm{Spin}(7)$ in $\mathrm{SO}(8)$ (see [5]). The isotropy subgroup is exactly $\mathrm{U}(3)$ (see [7]), and hence we may recognise $\mathcal{T}(S^6)$ to be the six-dimensional complex quadric (see [13], where the same result is obtained with different methods).

We now study the geometry of Q_6 determined by the following principal fibre bundle:

$$\tau \colon \mathrm{U}(3) \hookrightarrow \mathrm{Spin}(7) \to Q_6. \tag{4.3}$$

First we notice that σ' and σ'' are tensorial one forms with respect to (4.3), and they transform according to the representations (2.7) and (2.8). Hence we define an almost Hermitian $\mathrm{Spin}(7)$-invariant structure on Q_6 by setting

$$\begin{aligned} \pi^*\left(\Lambda^{(10)}Q_6\right) &= \mathrm{Span}\,(\sigma^1 \dots \sigma^6), \\ \pi^*(g') &= \sum_{n=1}^{12} (\eta^a)^2, \\ \pi^*(\Phi) &= -i\sum_{n=1}^{6} \sigma^i \wedge \bar{\sigma}^i, \end{aligned} \tag{4.4}$$

where $\sigma^1 = \eta^1 + i\eta^7$, $\sigma^2 = \eta^2 + i\eta^8$, $\sigma^3 = \eta^3 + i\eta^5, \dots, \sigma^6 = \eta^6 + i\eta^{12}$, and Φ denotes the fundamental two form.

The structure equations (2.9) and (2.10) imply that the almost-Hermitian structure is a complex Kähler structure.

If we let $\left(\kappa_a^b\right)_{a,b=1\dots 6}$ the restriction of the curvature form of g' on Spin(7), then $-2i \sum_{a=1}^{6} \kappa_a^a$ is a projectable two-form. Its projection ρ is the Ricci form, and an easy computations shows that:

$$-2i \sum_a \kappa_a^a = -12i \sum_a \sigma^a \wedge \bar{\sigma}^a = 12\pi^*(\Phi). \tag{4.5}$$

Therefore g' is an Einstein metric with Einstein constant $E' = 12$. Since Q_6 is simply connected, then it admits a unique (up to homothety) invariant Kähler–Einstein structure (see [**10**]). Therefore we have that (Q_6, g', J) is the six-dimensional complex quadric of a complex projective space $\mathbb{CP}^7$ of constant holomorphic sectional curvature 4.

The bundle $\Lambda^{(10)}(Q_6)$ splits as a direct sum $\mathcal{V} \oplus \mathcal{H}$, where $\mathcal{H}$ is the rank-3 complex bundle of semi-basic (for $\tau\colon Q_6 \to S^6$) complex linear (1, 0)-forms, and $\mathcal{V} = \mathcal{H}^{\perp}$. We notice that

$$\begin{aligned} \pi^*(\mathcal{H}) &= \operatorname{Span}\left(\sigma^1 \sigma^2 \sigma^3\right), \\ \pi^*(\mathcal{V}) &= \operatorname{Span}\left(\sigma^4 \sigma^5 \sigma^6\right). \end{aligned} \tag{4.6}$$

Equations (2.10) imply that the subbundle $\mathcal{V} \to \Lambda^{(10)}(Q_6)$ is a holomorphic, not integrable distribution.

The fibres $\mathbb{CP}^3 \to Q_6$ are the maximal connected integral submanifolds of the exterior differential system

$$\alpha = 0 \qquad \forall \alpha \in \mathcal{H}.$$

If we let $j = \mathbb{CP}^3 \to Q_6$ denote the inclusion, the $j^*(\mathrm{Spin}(7)) \to \mathbb{CP}^3$ is a U(3)-principal fibre bundle, and $\sum_{A=4}^{6} \sigma^A \wedge \bar{\sigma}^A$ is a projectable bilinear form representing the induced metric $j^*(g')$. Furthermore, $\sum_{A=4}^{6} \sigma^A \wedge \bar{\sigma}^A$ gives rise to a well defined complex-structure, and $j^*(g')$ is a Kähler metric.

Notice that $-2i \sum_{A=4}^{6} \kappa_A^A$ is a projectible two-form, and its projection is the Ricci form of the fibre.

Since

$$2i \sum_{A=1}^{6} \kappa_A^A = -8i \left(\sum_A \sigma^A \wedge \sigma^A \right)$$

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we deduce that $j^*(g')$ is an Einstein metric with Einstein constant $\check{E} = 8$. Finally we notice that (2.10) implies that the fibres are totally geodesic submanifolds. Since the Einstein constant of g is $E = 20$, and obviously the projection τ is a Riemannian submersion, then we may apply the theorem of Berard Bergery and Matzuzawa and we get the following.

THEOREM. *If we scale the standard metric on Q_6 in the direction of the fibres by a factor $t = \frac{2}{3}$, then we have a Hermitian,* Spin (7)*-invariant Einstein metric g'' on Q_6.*

The principal fibre bundle

$$\pi\colon \mathrm{U}(3) \to \mathrm{Spin}\,(7) \to Q_6$$

is a reduction of the isotropy bundle $\mathrm{SO}(2) \times \mathrm{SO}(6) \to \mathrm{SO}(8) \to Q_6$, whose elements are the frames of $\mathrm{SO}(8)$ adapted to the twistor projection $\pi\colon Q_6 \to S^6$.

Therefore, (4.3) is also a reduction of the unitary frame bundle of the Hermitian manifold (Q_6, g'', J). The restriction on Spin (7) of the canonical one-form of g'' is the $\mathbf{C}^6$-valued one form given by:

$$\left(\sigma^1, \sigma^2, \sigma^3, \sqrt{\frac{2}{3}}\sigma^4, \sqrt{\frac{2}{3}}\sigma^5, \sqrt{\frac{2}{3}}\sigma^6\right). \tag{4.7}$$

Then $-i\big(\sigma^1 \wedge \bar\sigma^1 + \sigma^2 \wedge \bar\sigma^2 + \sigma^3 \wedge \bar\sigma^3 + \frac{2}{3}\sigma^4 \wedge \bar\sigma^4 + \frac{2}{3}\sigma^5 \wedge \bar\sigma^5 + \frac{2}{3}\sigma^6 \wedge \bar\sigma^6\big)$ is a projectible two-form, representing the fundamental two-form Φ' of (g', J). Using (2.10) we obtain:

(4.8)
$$\begin{aligned}\pi^*(d\Phi') = \frac{2}{3}\,&(\sigma^2 \wedge \sigma^3 \wedge \bar\sigma^4 - \sigma^1 \wedge \sigma^3 \wedge \bar\sigma^5 \\ &+\sigma^1 \wedge \sigma^2 \wedge \bar\sigma^6 - \bar\sigma^2 \wedge \bar\sigma^3 \wedge \sigma^4 + \bar\sigma^1 \wedge \bar\sigma^3 \wedge \sigma^5 - \bar\sigma^1 \wedge \bar\sigma^2 \wedge \sigma^5).\end{aligned}$$

Therefore $d\phi' \neq 0$, and hence g'' is not a Kähler metric. A fortiori g'' cannot be symmetric or isometric with the standard metric of Q_6.

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