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CONSTRUCTIONS OF Σ-GROUPS, RELATIVELY FREE Σ-GROUPS

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Abstract

A Σ-group is an abelian group on which is given a family of infinite sums having properties suggested by, but weaker than, those which hold for absolutely convergent series of real or complex numbers. Two closely related questions are considered. The first concerns the construction of a Σ-group from an arbitrary abelian group on which certain series are given to be summable, certain of these series being required to sum to zero. This leads to a Σ-theoretic construction of $\mathbf{R}$ from $\mathbf{Q}$ and in general of the completion of an arbitrary metrizable abelian group (with the associated unconditional sums) from that group. The second question is whether, in a given Σ-group, the values of the infinite sums may be determined solely from a knowledge of which series are summable. Such a Σ-group is said to be relatively free and it is shown that all metrizable abelian groups are relatively free.

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1. Introduction

A Σ-group is an abelian group on which is given a collection of infinite sums satisfying some natural conditions (see §3); the notion is due to Wylie [6]. A basic example of a Σ-group is the additive group $\mathbf{R}$ of real numbers together with all the absolutely convergent series on $\mathbf{R}$ and their sums. Two questions suggest themselves:

I. Can $\mathbf{R}$ as a Σ-group be constructed directly from the additive group $\mathbf{Q}$ of rational numbers together with appropriate additional data?

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II. Why is it that some (all?) infinite sums in $\mathbf{R}$ necessarily have the value they do have by virtue of algebraic manipulations alone? (For example if $s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ then $2s = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 1 + s$ and so $s = 1$.)

These and related questions were made precise and answered in [2, 4.5], within the context of Σ-groups in general. Our intention is to give an account of this work and of various further developments concerning the ideas involved.

Sections 2 and 3 are mainly devoted to introducing the concepts we shall use. In §3, the classes of complete Σ-groups and regular Σ-groups are defined and the properties of those Σ-groups given by unconditional summability in Hausdorff topological abelian groups are reviewed.

The work of the paper starts properly with Section 4, in which the "T-construction" is described. The idea here is that, given an abelian group A together with a suitable set S of series on A and subset K of S, one attempts to construct from these ingredients a Σ-group in which the series in S have sums, with value 0 if the series are in K. Besides producing $\mathbf{R}$ from $\mathbf{Q}$, and more generally the completion of any metrizable abelian group from that group, the T-construction leads to a proof of the fact that every regular Σ-group is embeddable in a complete regular Σ-group.

The second of the two questions posed above gives rise to the subject-matter of Section 5. We wish to consider those Σ-groups in which the value of each sum is determined algebraically, in some reasonable sense. The class of relatively free Σ-groups is defined and is shown to be closed under countable products and to contain all metrizable abelian groups; this latter fact provides an answer to the second question.

We are grateful to Isidore Fleischer for many helpful discussions on the subject matter of this paper, in particular for suggesting to us the present definition of a Σ-group (this was prior to our learning of Wylie's work) and for making available to us a preliminary version and a preprint of [3].

2. Series, s-monoids, sums, and s-kernels

To start with, our idea of a series on an abelian group A is simply that of a family $(a_i : i \in I)$ of elements of A. However, since we are only going to be considering the *unordered* sum (in some sense) of such a family, the indexing of the terms is irrelevant and we need only know the number of times a given element of A occurs as a term in the series; moreover, occurrences of 0 will not affect the sum and can be suppressed, or inserted, at will. We are thus led to define a *series* on A as an equivalence class of families of elements of A, two families being equivalent if they differ either in their indexing (via some bijection between the index sets involved) and/or terms=0. We could equivalently define

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a series on A as a function N from $A \setminus \{0\}$ to the class of cardinals, where $N(a)$ gives the number of times a occurs in the series. This approach was used in [2, 4.5] (for the case when every $N(a)$ is finite) and with it, series are the same as the elements of Rado's cardinal module Ω [5] except for the restriction to $A \setminus \{0\}$ as domain. Nevertheless in practise the picture of series as families is usually more suggestive and easier to work with, though then one does have to check that definitions phrased in terms of families apply unambiguously to the associated series (these checks are trivial and will not be mentioned explicitly).

The following terminology and notation will be used in connection with series α, β, γ, etc. on A: The *cardinality* of α is the number of (non-zero) terms of α; 0 denotes the unique series of cardinality 0; (a) denotes the singleton series with a as its only term; $\alpha + \beta$ denotes the series obtained by concatenating α and β; $\bigoplus_{i \in I} \alpha_i$ denotes the series obtained by concatenating the α_i, $i \in I$; $\kappa\alpha$ denotes the series obtained by concatenating κ copies of α, where κ is a cardinal (we only need this for $\kappa \leq \omega$); $-\alpha$ denotes the series obtained by replacing each term a of α with $-a$; $\alpha - \beta$ denotes $\alpha + (-\beta)$; if $\alpha = (a_i : i \in I)$ and F is a finite subset of I then $a(F)$ denotes $\Sigma_{i \in F} a_i$; β is a *subseries* of α if $\alpha = \beta + \gamma$ for some γ; β is a *contraction* of α, or α *contracts* to β, if $\alpha = (a_i : i \in I)$ and $\beta = (a(I_j) : j \in J)$ for some partition $(I_j : j \in J)$ of I into finite sets I_j; β is a *subcontraction* of α if β is a subseries of a contraction of α, equivalently, if β is a contraction of a subseries of α; $\mathrm{Ser}(A)$ denotes the class of all series on A; $\mathrm{Fin}(A)$ denotes the set of all finite series on A; if $f\colon A \to B$ is a function from A to an abelian group B such that $f(0) = 0$ and α is a series on A then $f(\alpha)$ denotes the series on B obtained by replacing each term a of α with $f(a)$.

The above notions enjoy quite a number of very easily verified properties which we shall use without explicit mention, remarking here only that, were it not a proper class, $\mathrm{Ser}(A)$ would be a commutative monoid under the above $+$, with 0 as identity element. In any case, we may speak of the submonoids of $\mathrm{Ser}(A)$, it being understood that these are sets.

An *s-monoid* on A is a submonoid S of $\mathrm{Ser}(A)$ which contains $\mathrm{Fin}(A)$ and is closed under minus and contraction. For example, $\mathrm{Fin}(A)$ is itself an s-monoid on A, being the smallest such. The following fact is simple but useful.

(2.1) *If S is an s-monoid on A and $\alpha + \beta$ and β are in S then α is in S.*

PROOF. $\alpha + \beta - \beta$ is in S and contracts to α.

An s-monoid S is said to be *subseries-closed* if all subseries of the series in S are themselves in S. If S is any s-monoid then the set $\hat{S}$ of all subseries of the series in S is easily seen to be the smallest subseries-closed s-monoid containing S.

(2.2) *Let S be an s-monoid on A.*

(1) *If α is in $\hat{S}$ and α contracts to some β in S then α is in S.*

(2) *α is in $\hat{S}$ if and only if $\alpha - \alpha$ is in S.*

PROOF. For (1), let $\alpha + \alpha'$ be in S; then $\alpha + \alpha' - \beta$ is in S and contracts to α'; hence α' is in S so that α is in S by (2.1). (2) follows from (1) since $\alpha - \alpha$ contracts to 0 and α is a subseries of $\alpha - \alpha$.

A *sum* on A is a monoid morphism $\Sigma\colon S \to B$, where S is an s-monoid on A and B is an abelian group, such that if α in S contracts to β then $\Sigma(\alpha) = \Sigma(\beta)$. For $\alpha = (a_i\colon i \in I)$ in S, $\Sigma(\alpha)$ is often written as $\Sigma_{i\in I} a_i$, or simply as $\Sigma_i a_i$.

A notion closely related to that of a sum is the following. If S is an s-monoid on A then an *s-kernel* in S is a submonoid K of S which is closed under minus and contraction and contains $\alpha - \alpha$ for all α in S. It is clear that if $\Sigma\colon S \to B$ is a sum on A then the *kernel* $\ker \Sigma$ of Σ, namely $\{\alpha \in S\colon \Sigma(\alpha) = 0\}$, is an s-kernel in S. Conversely, every s-kernel K in S is the kernel of a sum: as is easily checked, the condition $\alpha - \beta \in K$ defines a monoid congruence on S for which the corresponding quotient monoid S/K is an abelian group and the natural map $[-]$ from S to S/K is a sum on A with K as its kernel.

The set of s-kernels in S is closed under intersection and in particular there is a smallest s-kernel in S, which we denote by $k_0(S)$. $k_0(S)$ will be studied in detail in §5. The following facts are easily verified.

(2.3)(1) *If K is an s-kernel in S and α in S contracts to some β in K then α is in K.*

(2) *An s-kernel in S is also an s-kernel in $\hat{S}$.*

(3) $k_0(\hat{S}) = k_0(S)$.

We remark that the sum $[-]\colon S \to S/k_0(S)$ is a universal sum on A with domain S in the sense that if $f\colon S/k_0(S) \to B$ is a group morphism, where B is abelian, then $f[-]\colon S \to B$ is a sum and every sum $\Sigma\colon S \to B$ arises in this way from a unique such f.

An s-kernel K in an s-monoid S is said to be *$\bigoplus$-closed* in S if, whenever α_i is in K for all i and $\bigoplus_i \alpha_i$ is in S, then $\bigoplus_i \alpha_i$ is in K.

3. Σ-groups

A *Σ-group* is a triple (A, S, Σ) consisting of an abelian group A, an s-monoid S on A, and a sum $\Sigma\colon S \to A$ such that $\Sigma((a)) = a$ for all a in A. This notion of Σ-group is equivalent to Wylie's notion of a congregation [6] and is somewhat more general than that used in [2] and [4], which consider only those Σ-groups in the present sense which are complete and regular, as defined below.

A *morphism* from a Σ-group (A, S, Σ) to a Σ-group (B, T, Σ) is a group morphism $f\colon A \to B$ such that, for all α in S, $f(\alpha)$ is in T and $\Sigma(f(\alpha)) = f(\Sigma(\alpha))$.

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A morphism f from (A, S, Σ) to (B, T, Σ) is an *embedding* if α is in S for every series α on A such that $f(\alpha)$ is in T and $\Sigma(f(\alpha))$ is in $f(A)$; such an f is necessarily injective for if $f(a) = 0$ then $f(\omega(a))$ is in T with $\Sigma(f(\omega(a))) = 0$ so that $\omega(a)$ is in S and hence $a = 0$ since $\omega(a) - \omega(a)$ contracts to (a) as well as to 0. The following notion is essentially equivalent to that of an embedding: if (B, T, Σ) is a Σ-group and A is a subgroup of B then the *relativization* of (B, T, Σ) to A is the Σ-group (A, S, Σ) where S consists of the series on A which are in T and have their sums in A, and Σ is the restriction to S of the Σ on T. Note that a series on A is in $\hat{S}$ if and only if it is in $\hat{T}$ (use (2.2)(2)).

The *product* $\prod_i (A_i, S_i, \Sigma)$ of a family of Σ-groups (A_i, S_i, Σ) is the Σ-group $(\prod_i A_i, S, \Sigma)$ where a series α on $\prod_i A_i$ is in S if and only if $\pi_i(\alpha)$ is in S_i and $\pi_i(\Sigma(\alpha)) = \Sigma(\pi_i(\alpha))$ for all i (π_i is the projection from $\prod_i A_i$ to A_i). In this case, a series α on $\prod_i A_i$ is in $\hat{S}$ if and only if $\pi_i(\alpha)$ is in $\hat{S}_i$ for all i (use (2.2)(2) again).

A Σ-group (A, S, Σ) is said to be: *complete* if S is subseries-closed; *regular* if, whenever α_i is in S for all i and $\bigoplus_i \alpha_i$ is in S, then $(\Sigma(\alpha_i) : i \in I)$ is in S and $\Sigma_i \Sigma(\alpha_i) = \Sigma(\bigoplus_i \alpha_i)$; and *discrete* if $S = \mathrm{Fin}(A)$. The *arity* of (A, S, Σ) is the supremum of the cardinalities of the series in S. If (B, T, Σ) is a Σ-group then a subgroup A of B is said to be Σ-*closed* (in B) if $\Sigma(\alpha)$ is in A for every series α on A which is in T. The following result is straightforward.

(3.1)(1) *A relativization of a complete Σ-group is complete if and only if the subgroup involved is Σ-closed. Relativizations of regular Σ-groups are regular.*

(2) *Products of complete Σ-groups are complete, and likewise for regular Σ-groups.*

If (A, S, Σ) is a regular Σ-group then $\ker \Sigma$ is clearly $\bigoplus$-closed in S, but we can say somewhat more.

(3.2) *In every regular Σ-group (A, S, Σ), $\ker \Sigma$ is $\bigoplus$-closed in $\hat{S}$.*

PROOF. Let α_i be in $\ker \Sigma$ for all i in I and let $\bigoplus_i \alpha_i$ be in $\hat{S}$, so that $\bigoplus_i \alpha_i + \bigoplus_j (b_j)$ is in S for some series $(b_j : j \in J) = \bigoplus_j (b_j)$. By regularity, $(\Sigma(\alpha_i) : i \in I) + (\Sigma((b_j)) : j \in J) = \bigoplus_j (b_j)$ is in S and has sum $\Sigma(\bigoplus_i \alpha_i + \bigoplus_j (b_j))$. Hence $\bigoplus_i \alpha_i$ is in S (by (2.1)) and $\Sigma(\bigoplus_i \alpha_i) = 0$.

The following result is proved by means of a similar argument.

(3.3) *If (A, S, Σ) is a regular Σ-group and if α_i is in S for all i in I and $\bigoplus_i \alpha_i$ is in $\hat{S}$ then $(\Sigma(\alpha_i) : i \in I)$ is in $\hat{S}$.*

To see that a Σ-group (A, S, Σ) with $\ker \Sigma$ $\bigoplus$-closed in $\hat{S}$ is not necessarily regular, consider the Σ-group with $A = \mathbf{R}$, $S =$ the set of all absolutely convergent series on $\mathbf{R}$ with all but finitely many terms in $\mathbf{Q}$, and $\Sigma =$ the usual Σ. Then (A, S, Σ) is complete and $\ker \Sigma$ is $\bigoplus$-closed in $S = \hat{S}$. Regularity fails, however, for if we split up the series $(2^{-n!}: n \in \omega)$ in S into infinitely many infinite subseries then each subseries has an irrational sum so that the series of these sums is not in S.

Let A be a Hausdorff topological abelian group. Then we may associate with A the Σ-group (A, S, Σ) in which $S =$ the set of all unconditionally summable series on A and $\Sigma =$ the unconditional sum (for a discussion of unconditional sums, see Bourbaki [1, Chapter III, §5]). The following two results review some of the properties of the Σ-groups which arise in this way.

(3.4) *If A is a Hausdorff topological abelian group then the associated Σ-group (A, S, Σ) is regular and if A is complete then (A, S, Σ) is complete. If B is a Hausdorff topological abelian group and A is a subgroup of B with the relative topology then the Σ-group associated with A is the relativization to A of the Σ-group associated with B. If A_i, i in I, is a family of Hausdorff topological abelian groups and A is their product (with the product topology) then the Σ-group associated with A is the product of the Σ-groups associated with the A_i.*

PROOF. The proof of Theorem 2 in [1, Chapter III, §5] gives regularity and the statement about relativization is obvious. The remaining two statements are Propositions 2 and 4 respectively in [1, Chapter III, §5].

(3.5) *Let A be a metrizable abelian group and let (A, S, Σ) be the associated Σ-group. Then*

(1) *(A, S, Σ) is of countable arity,*

(2) *(A, S, Σ) is complete if and only if A is complete,*

(3) *if $(s_n: n \in \omega)$ is a Cauchy sequence in A then there exist $n_0 < n_1 < n_2 < \cdots$ such that $(s_{n_{k+1}} - s_{n_k}: k \in \omega)$ is in $\hat{S}$, and if $s_n \to 0$ then the n_k can be chosen so that $(s_{n_k}: k \in \omega)$ is in $\hat{S}$.*

PROOF. (1) is the Corollary to Proposition 1 in [1, Chapter III, §5], one uses the first part of (3) for the "only if" in (2), and (3) itself is easy once one takes an invariant metric for A (such exists by Proposition 2 in [1, Chapter IX, §3]) and chooses the n_k so that the relevant sequences tend to 0 sufficiently rapidly.

It is not true in general that if A is a Hausdorff topological abelian group such that the associated Σ-group is complete then A is complete. The following proposition will make it easy to give a counterexample.

(3.6) *Let A be a Hausdorff topological abelian group in which each countable intersection of neighbourhoods of 0 is again a neighbourhood of 0. Then the associated Σ-group is discrete.*

PROOF. Suppose that some infinite series $(a_i: i \in I)$ of non-zero terms is unconditionally summable in A, where without loss of generality we may take I to be countably infinite, and for each i in I let N_i be a neighbourhood of 0 not containing a_i. Then $\bigcap_i N_i$ is a neighbourhood of 0 containing none of the a_i contrary to Proposition 1 of [1, Chapter III, §5].

Since discrete Σ-groups are trivially complete, to obtain our counterexample we need only find an incomplete topological group satisfying the hypothesis of (3.6), and the following is such: let A be the subgroup of $\mathbb{Z}^{\omega_1}$ consisting of the elements of finite support (countable support would do as well), order A lexicographically, and give it the order topology.

4. Constructions of Σ-groups

A triple (A, S, K) with A an abelian group, S an s-monoid on A, and K an s-kernel in S will be called an ASK. Given an ASK (A, S, K), we want to construct from it a Σ-group in which the series in S become summable, to 0 if they are in K. As in §2, let S/K be the abelian group $\{[\alpha] \colon \alpha \in S\}$, where $[\alpha] = \{\beta \in S \colon \alpha - \beta \in K\}$, and let T be the s-monoid on S/K consisting of all series of the form $([\alpha_i] \colon i \in I)$ where the α_i and $\bigoplus_i \alpha_i$ are in S; we wish to define a sum $\dot{\Sigma} \colon T \to S/K$ on S/K by putting $\dot{\Sigma}_i[\alpha_i] = [\bigoplus_i \alpha_i]$.

(4.1) *Let (A, S, K) be an ASK and suppose that K is $\bigoplus$-closed in S. Then $\dot{\Sigma} \colon T \to S/K$ is well defined and the resulting $(S/K, T, \dot{\Sigma})$ is a Σ-group.*

PROOF. It is easily verified that $\dot{\Sigma}$ is well-defined, that it is a monoid morphism, and that $\dot{\Sigma}(([\alpha])) = [\alpha]$ for all $[\alpha]$ in T. Let $([\alpha_i] \colon i \in I)$ be in T, where the α_i are as above, and let $(I_j \colon j \in J)$ be a partition of I into finite sets, so that $([\alpha_i] \colon i \in I)$ contracts to $(\dot{\Sigma}_{i \in I_j}[\alpha_i] \colon j \in J)$. Then $\dot{\Sigma}_j \dot{\Sigma}_{i \in I_j}[\alpha_i] = [\bigoplus_j \bigoplus_{i \in I_j} \alpha_i] = [\bigoplus_i \alpha_i] = \dot{\Sigma}_i[\alpha_i]$, and thus $(S/K, T, \dot{\Sigma})$ is a Σ-group.

If (A, S, K) is an ASK in which K is $\bigoplus$-closed in S, we denote the Σ-group $(S/K, T, \dot{\Sigma})$ by $T(A, S, K)$. Note that if in addition S is subseries-closed then so is T, so that $T(A, S, K)$ is a complete Σ-group. In order for $T(A, S, K)$ to be regular, a slight strengthening of the condition that K is $\bigoplus$-closed in S appears to be necessary.

(4.2) *If (A, S, K) is an ASK with K $\bigoplus$-closed in $\hat{S}$ then the Σ-group $T(A, S, K)$ is regular.*

PROOF. Let (A, S, K) be as stated. We first obtain a subsidiary result, namely that if $\alpha_i - \beta_i$ is in K for all i in I, $\bigoplus_i \alpha_i$ is in S, and $\bigoplus_i \beta_i$ is in $\hat{S}$, then $\bigoplus_i \beta_i$ is in S. The hypothesis here implies that $\bigoplus_i(\alpha_i - \beta_i) = \bigoplus_i \alpha_i - \bigoplus_i \beta_i$ is in S, hence also in $K \subseteq S$, so that $\bigoplus_i \beta_i$ is in S by (2.1). Now write $T(A, S, K) = (S/K, T, \dot{\Sigma})$ and let ξ_j, j in J, and $\bigoplus_j \xi_j$ be in T; we want to show that $(\dot{\Sigma}(\xi_j) \colon j \in J)$ is in T with sum$=\dot{\Sigma}(\bigoplus_j \xi_j)$. We may write

$\xi_j = ([\alpha_i]: i \in I_j)$ and $\bigoplus_j \xi_j = ([\beta_i]: i \in I)$ where I is the disjoint union of the I_j, $\alpha_i - \beta_i$ is in K for all i in I, $\bigoplus_{i\in I_j} \alpha_i$ is in S for all j in J, and $\bigoplus_{i\in I} \beta_i$ is in S. Then $\bigoplus_{i\in I_j} \beta_i$ is in $\hat{S}$ and hence also in S by what we have already proved. It follows that $\dot{\Sigma}(\xi_j) = [\bigoplus_{i\in I_j} \beta_i]$ and since $\bigoplus_j \bigoplus_{i\in I_j} \beta_i = \bigoplus_i \beta_i$ is in S, $(\dot{\Sigma}(\xi_j): j \in J)$ is in T with sum$=[\bigoplus_i \beta_i] = \dot{\Sigma}(\bigoplus_j \xi_j)$.

Now let (A, S, Σ) be any regular Σ-group. Then $(A, \hat{S}, \ker \Sigma)$ is an ASK in which, by (3.2), $\ker \Sigma$ is $\bigoplus$-closed in $\hat{S}$ and so $T(A, \hat{S}, \ker \Sigma)$ may be formed.

(4.4) *For any regular Σ-group (A, S, Σ), the map $\phi\colon A \to \hat{S}/\ker \Sigma$ defined by $\phi(a) = [(a)]$ is an embedding of (A, S, Σ) into the complete regular Σ-group $T(A, \hat{S}, \ker \Sigma)$.*

PROOF. We have $T(A, \hat{S}, \ker \Sigma) = (\hat{S}/\ker \Sigma, T, \dot{\Sigma})$ as above. To see that ϕ is a Σ-group morphism, let $\alpha = (a_i: i \in I)$ be in S. Then $\phi(\alpha) = ([(a_i)]: i \in I)$ is in T since each (a_i) and $\bigoplus_i (a_i) = \alpha$ are in S; moreover $\dot{\Sigma}(\phi(\alpha)) = \dot{\Sigma}_i[(a_i)] = [\bigoplus_i (a_i)] = [\alpha] = \phi(\Sigma(\alpha))$ (the last equality holds since $\alpha - (\Sigma(\alpha))$ is in $\ker \Sigma$). To see that ϕ is an embedding, let $\alpha = (a_i: i \in I)$ be a series on A such that $\phi(\alpha)$ is in T with $\dot{\Sigma}(\phi(\alpha)) = \phi(a)$ for some a in A. Then there exist series α_i in $\hat{S}$ with $\bigoplus_i \alpha_i$ also in $\hat{S}$ such that each $\alpha_i - (a_i)$, and also $\bigoplus_i \alpha_i - (a)$, are in $\ker \Sigma$, from which it follows that the α_i and $\bigoplus_i \alpha_i$ are actually in S. Since $\Sigma(\alpha_i) = a_i$ for all i, the regularity of (A, S, Σ) shows that $\alpha = (a_i: i \in I)$ is in S.

We remark that the map ϕ in this result gives an isomorphism of (A, S, Σ) onto $T(A, S, \ker \Sigma)$.

If (A, S, Σ) is a regular Σ-group then $T(A, \hat{S}, \ker \Sigma)$ is its *regular completion*, denoted by $RC(A, S, \Sigma)$. The regular completion of a regular Σ-group was originally constructed by Fleischer [3]; it may be described intrinsically as follows.

(4.5) *Let f be an embedding of a regular Σ-group (A, S, Σ) into a complete regular Σ-group (B, T, Σ) and suppose that for all $(b_i: i \in I)$ in T there exist series α_i on A such that $\bigoplus_i f(\alpha_i)$ is in T and $\Sigma(f(\alpha_i)) = b_i$ for all i. Then (B, T, Σ) is isomorphic to $RC(A, S, \Sigma)$.*

PROOF. Write $RC(A, S, \Sigma)$ as $(\hat{S}/\ker \Sigma, T', \dot{\Sigma})$ and define $g\colon \hat{S}/\ker \Sigma \to B$ by $g([\alpha]) = \Sigma(f(\alpha))$ (since α is in $\hat{S}$, $f(\alpha)$ is in $\hat{T} = T$). To see that g is a Σ-group morphism, let $([\alpha_i]: i \in I)$ be in T', where the α_i are as in the definition of the T-construction. Then the $f(\alpha_i)$ and $\bigoplus_i f(\alpha_i) = f(\bigoplus_i \alpha_i)$ are in T so that, by the regularity of (B, T, Σ), $(\Sigma(f(\alpha_i)): i \in I)$ is in T with sum $= \Sigma(\bigoplus_i f(\alpha_i)) = \Sigma(f(\bigoplus_i \alpha_i))$, that is, $(g([\alpha_i]): i \in I)$ is in T with sum $= g([\bigoplus_i \alpha_i]) = g(\dot{\Sigma}_i[\alpha_i])$ as required. g is injective for if $\Sigma(f(\alpha)) = 0$ then the fact that f is an embedding shows that α is in $\ker \Sigma$. Also g is surjective: if b is in B then (b) is in T and hence there exists a series α on A such that $f(\alpha)$ is in T and $\Sigma(f(\alpha)) = b$; since

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$\Sigma(f(\alpha - \alpha)) = 0$, $\alpha - \alpha$ is in S and so α is in $\hat{S}$, whence $g([\alpha])$ is defined and equals $\Sigma(f(\alpha)) = b$. The fact that every sum in (B, T, Σ) is the image under g of some sum in $RC(A, S, \Sigma)$ is shown similarly.

A particular case of this result is the following.

(4.6) *Let A be a metrizable abelian group and let B be its completion. Then the Σ-group associated with B is isomorphic to the regular completion of the Σ-group associated with A.*

PROOF. By a slight extension of the argument suggested for (3.5)(3), it can be seen that every sum in B is representable as an iterated sum of elements of A, as the hypothesis of (4.5) requires.

The example given at the end of §3 shows that (4.6) does not hold for all Hausdorff topological abelian groups.

5. Relatively free Σ-groups

A Σ-group (A, S, Σ) will be said to be *relatively free* if $\ker \Sigma = k_0(S)$ (this is stronger than the similar property introduced in [2, 4.5.14]). In order to study relatively free Σ-groups, an explicit description of the series in $k_0(S)$ is desirable and condition (v) in (5.6) below gives a useful criterion. The following notions and lemmas are needed.

By a *series of partial sums* (SPS) of a series α on an abelian group A, we mean a series σ on A obtained as follows: let $\alpha = (a_i : i \in I)$, let $(I_j : j \in J)$ be a partition of I into countably infinite sets, and for each j in J let $(F_{j,k} : k \in \omega)$ be a strictly increasing sequence of finite subsets of I_j with I_j as their union; then put $\sigma = (a(F_{j,k}) : j \in J, k \in \omega)$. In such a situation, we say that $(F_{j,k} : j \in J, k \in \omega)$ is an *ascension to*, and that σ is *based on*, the partition $(I_j : j \in J)$.

(5.1) *If σ is an SPS of α then $\sigma - \sigma$ and α have a common contraction.*

PROOF. Let σ be as above; then $(a(F_{j,0}) : j \in J) + (a(F_{j,k+1}) - a(F_{j,k}) : j \in J, k \in \omega)$ is a common contraction of $\sigma - \sigma$ and α.

(5.2) *Let an SPS σ of $\alpha = (a_i : i \in I)$ be based on the partition $(I_j : j \in J)$ of I and let $(I'_r : r \in R)$ be a coarser partition of I into countably infinite sets. Then σ contracts to an SPS of α based on $(I'_r : r \in R)$.*

PROOF. Let σ be an above and for each r in R let $j_r(0), j_r(1), \dots$ be an enumeration of the j's in J for which $I_j \subseteq I'_r$. Then the contraction

$$\begin{aligned}(a(F_{j_r(0),0}), a(F_{j_r(0),1}) + a(F_{j_r(1),0}), \\ a(F_{j_r(0),2}) + a(F_{j_r(1),1}) + a(F_{j_r(2),0}), \dots : r \in R)\end{aligned}$$

of σ is as required.

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(5.3) *Let an SPS σ of $\alpha = (a_i : i \in I)$ be based on the partition $(I_j : j \in J)$ of I and let $(G_{j,k} : j \in J, k \in \omega)$ be any ascension to $(I_j : j \in J)$. Then there exists an ascension $(G'_{j,k} : j \in J, k \in \omega)$ to $(I_j : j \in J)$ contained in $(G_{j,k} : j \in J, k \in \omega)$ such that the corresponding SPS τ of α is a subcontraction of $\sigma + \alpha$.*

PROOF. Let σ be as above again and let $(F'_{j,k} : j, k)$ and $(G'_{j,k} : j, k)$ be ascensions to $(I_j : j)$ contained in $(F_{j,k} : j, k)$ and $(G_{j,k} : j, k)$ respectively such that $F'_{j,0} \subset G'_{j,0} \subset F'_{j,1} \subset G'_{j,1} \subset \cdots$ for all j in J. Then $(G'_{j,k} : j, k)$ is as stated since $\tau = (a(G'_{j,k}) : j, k) = (a(F'_{j,k}) + a(G'_{j,k} \setminus F'_{j,k}) : j, k)$ is a contraction of $(a(F'_{j,k}) : j, k) + (a(G'_{j,k} \setminus F'_{j,k}) : j, k)$, the first series here being a subseries of σ and the second being a subcontraction of α.

(5.4) *If σ is an SPS of α and α contracts to β then some subcontraction of $\sigma + \alpha$ is an SPS of β.*

PROOF. Let α be indexed by I. By (5.2) we may without loss of generality take σ to be based on a partition $(I_j : j)$ of I which is coarser than the partition of I into finite sets giving rise to the contraction of α to β. Let $(G_{j,k} : j, k)$ be an ascension to $(I_j : j)$ constructed by taking successive finite unions within each I_j of these finite sets and let τ be as in (5.3); then τ is an SPS of β as well as of α.

(5.5) *If β contracts to α and to 0 then some subcontraction of β is an SPS of α.*

PROOF. Since β contracts to 0 if and only if 0 is an SPS of β, this is a particular case of (5.4).

(5.6) *Let S be an s-monoid on an abelian group A and let α be in S. Then the following conditions are equivalent:*

(i) *α is in $k_0(S)$,*

(ii) *$\Sigma(\alpha) = 0$ for every sum Σ on A with domain S,*

(iii) *α is a contraction of $\beta - \beta$ for some β in S,*

(iv) *some β in S contracts to α and to 0,*

(v) *some SPS of α is in $\hat{S}$.*

PROOF. (i) and (ii) are equivalent by what was said in §2 about sums and s-kernels. The set of α in S which satisfy (iii) is easily verified to be an s-kernel in S, and thus (i) implies (iii). (iii) clearly implies (iv), and (iv) implies (v) by (5.5). Finally, (v) implies (i): if σ is an SPS of α in $\hat{S}$ then $\sigma - \sigma$ is in S by (2.2)(2) and is thus in $k_0(S)$, from which it follows by (5.1) and (2.3)(1) that α is in $k_0(S)$.

The equivalence of (i) and (iv) here leads to the observation that for α, β in S, $\alpha - \beta$ is in $k_0(S)$ if and only if there exists γ in S which contracts to both

α and β. (iii) may be pictured as saying that α can be written as a sum which "telescopes" to 0.

(5.7) *Relativizations of relatively free Σ-groups are relatively free.*

PROOF. Let (A, S, Σ) be a relativization of a relatively free Σ-group (B, T, Σ) and let α in S have $\Sigma(\alpha) = 0$. Then α is in $k_0(T)$ and hence some SPS σ of α is in $\hat{T}$. But σ is a series on A and so σ is in $\hat{S}$. Thus α is in $k_0(S)$.

We next show that countable products of relatively free Σ-groups are relatively free and for this we need a further lemma on SPS.

(5.8) *Let α be a series on a countable product $A = \prod_{n\in\omega} A_n$ of abelian groups and for each n let σ_n be an SPS of $\pi_n(\alpha)$. Then there exists an SPS σ of α such that for each n, $\pi_n(\sigma)$ is a subcontraction of $\sigma_n + (n+1)\pi_n(\alpha)$.*

PROOF. Let α be indexed by I. Since the number of A_n is countable and likewise for each set in any partition on which an SPS is based, (5.2) entitles us to suppose without loss of generality that there is a single partition $(I_j : j \in J)$ of I on which every σ_n is based. Let $(G_{j,k} : j, k)$ be the ascension (to $(I_j : j)$) which produces the SPS σ_0 of $\pi_0(\alpha)$. By (5.3), $(G_{j,k} : j, k)$ contains an ascension $(G'_{j,k} : j, k)$ such that the corresponding SPS, τ_1 say, of $\pi_1(\alpha)$ is a subcontraction of $\sigma_1 + \pi_1(\alpha)$. Then by (5.3) again, $(G'_{j,k} : j, k)$ contains an ascension $(G''_{j,k} : j, k)$ giving rise to an SPS τ_2 of $\pi_2(\alpha)$ which is a subcontraction of $\sigma_2 + \pi_2(\alpha)$. Continuing in this way, we obtain a descending sequence of ascensions $(G^{(n)}_{j,k} : j, k)$ to $(I_j : j)$ such that for each n the corresponding SPS τ_n of $\pi_n(\alpha)$ is a subcontraction of $\sigma_n + \pi_n(\alpha)$. Then $(G^{(k)}_{j,k} : j, k)$ will be an ascension to $(I_j : j)$ such that for each n the corresponding SPS of $\pi_n(\alpha)$ differs in at most its terms $\pi_n(a(G^{(k)}_{j,k}))$, j in J, $k < n$, from a subseries of τ_n. The result follows on taking σ to be the SPS of α corresponding to $(G^{(k)}_{j,k} : j, k)$.

(5.9) *Countable products of relatively free Σ-groups are relatively free.*

PROOF. Let (A, S, Σ) be the product $\prod_n (A_n, S_n, \Sigma)$ of countably many relatively free Σ-groups and let α in S have $\Sigma(\alpha) = 0$. Then for each n, $\Sigma(\pi_n(\alpha)) = 0$ and by (5.6) some SPS σ_n of $\pi_n(\alpha)$ is in $\hat{S}_n$. Let σ be as in (5.8); then each $\pi_n(\sigma)$ is in $\hat{S}_n$ and hence σ is in $\hat{S}$. Thus α is in $k_0(S)$.

This result has a corollary which we shall need later on in discussing an example.

(5.10) *Let (A_i, S_i, Σ), i in I, be a family of relatively free Σ-groups of countable arity, let A be the subgroup of $\prod_i A_i$ consisting of the elements of countable support, and let (A, S, Σ) be the relativization to A of $\prod_i (A_i, S_i, \Sigma)$. Then (A, S, Σ) is relatively free.*

PROOF. Let α in S have $\Sigma(\alpha) = 0$ and let $(I_j : j \in J)$ be a partition of I into countable sets such that the support of each term of α is contained in some I_j (the existence of such a partition depends upon both of the above countability assumptions). Then $\alpha = \bigoplus_j \alpha_j$ where α_j consists of the terms of α whose supports are contained in I_j. Clearly $\Sigma(\alpha_j) = 0$ for each j and since we may regard α_j as a series on $\prod_{i \in I_j} A_i$, (5.9) shows that there exists an SPS σ_j of α_j in $\hat{S}$. Then $\sigma = \bigoplus_j \sigma_j$ will be an SPS of α in $\hat{S}$, and so α is in $k_0(S)$.

We now obtain various results on countable series and on Σ-groups of countable arity. First we deal with the very special case of a singleton series.

(5.11) *A singleton series (a) has only one SPS, namely $\omega(a)$. For any s-monoid S on A, (a) is in $k_0(S)$ if and only if $\omega(a)$ is in S.*

PROOF. The first statement is evident from the definition of an SPS. The second follows from (5.6) and the fact that $\omega(a) - \omega(a)$ contracts to $\omega(a)$, showing that if $\omega(a)$ is in $\hat{S}$ then it is in S.

Define a *classical SPS* of a countable series $\alpha = (a_n : n \in \omega)$ to be an SPS of α of the form $(s_{n_k} : k \in \omega)$ where $n_0 < n_1 < n_2 < \cdots$ and $s_n = a_0 + \cdots + a_n$. (This notion definitely involves the ordering $a_0, a_1, a_2, \ldots$ of the terms of α, which is not the case for SPS in general.)

(5.12) *Let $\alpha = (a_n : n \in \omega)$ be a countable series on A.*

(1) *If σ is an SPS of α then some subcontraction of $\sigma + \alpha$ is a classical SPS of α.*

(2) *For any s-monoid S on A, α is in $k_0(S)$ if and only if α is in S and some classical SPS of α is in $\hat{S}$.*

PROOF. In (1), (5.2) shows that σ contracts to an SPS σ' of α based on the partition of ω into one part and the conclusion of (1) is obtained by applying (5.3) to σ' and the ascension $(\{n : n \leq k\} : k \in \omega)$. (2) follows from (5.6) and (1).

(5.13) *Let (A, S, Σ) be a Σ-group of countable arity. Then the following conditions are equivalent:*

(i) *(A, S, Σ) is relatively free,*

(ii) *for all α in $\ker \Sigma$, some classical SPS of α is in $\hat{S}$,*

(iii) *for all $(a_n : n \in \omega)$ in S, there exist $n_0 < n_1 < n_2 < \ldots$ such that $(\Sigma_{n > n_k} a_n : k \in \omega)$ is in $\hat{S}$.*

PROOF. (i) and (ii) are equivalent by (5.12)(2). Suppose that (ii) holds and that $\alpha = (a_n : n \in \omega)$ is in S with $\Sigma(\alpha) = s$ say. Then $(-s) + \alpha$ is in $\ker \Sigma$ and by (ii) there exist $n_0 < n_1 < n_2 < \cdots$ such that $(-s + s_{n_k} : k \in \omega)$ is in $\hat{S}$, which is (iii). (iii) implies (ii) similarly.

(5.14) *The Σ-group associated with an arbitrary metrizable abelian group is relatively free.*

PROOF. We have countable arity from (3.5)(1), and (5.13)(ii) holds by virtue of the second statement in (3.5)(3).

Let A be a metrizable abelian group, let B be its completion, and let (A, S, Σ) and (B, T, Σ) be the associated Σ-groups. Then, combining the preceding result with (4.6), we see that (B, T, Σ) is isomorphic to $T(A, \hat{S}, k_0(S))$. This shows in particular that we may construct $\mathbf{R}$ with its usual Σ-structure by forming $T(\mathbf{Q}, S', k_0(S'))$ where S' consists of all series $(a_n : n \in \omega)$ on $\mathbf{Q}$ for which the partial sums of $(|a_n| : n \in \omega)$ are bounded.

(5.15) *Let (A, S, Σ) be a regular Σ-group of countable arity such that for every series $(a_n : n \in \omega)$ in S there exists natural numbers m_n, n in ω, such that $m_n \to \infty$ and $\bigoplus_n m_n(a_n)$ is in $\hat{S}$. Then (A, S, Σ) is relatively free.*

PROOF. We may assume without loss of generality that m_n takes the values $0, 1, 2, \ldots$ Successively, With possible repetitions. Let n_k denote the largest n for which $m_n = k$; then $\bigoplus_n m_n(a_n) = \bigoplus_k (a_n : n > n_k)$. Since this series is given to be in $\hat{S}$ and each $(a_n : n > n_k)$ is in S, $(\Sigma_{n > n_k} a_n : k \in \omega)$ is in $\hat{S}$ by (3.3), and the conclusion follows by (5.13).

A Σ-group (A, S, Σ) is defined to be *adic* if, for all $(a_i : i \in I)$ in S and all families $(m_i : i \in I)$ of natural numbers, $\bigoplus_i m_i(a_i)$ is also in S. The following result is immediate from (5.15).

(5.16) *Every regular adic Σ-group of countable arity is relatively free.*

In conclusion, we describe an example which will show, amongst other things, that in the preceding result the hypothesis of countable arity cannot be omitted. Let (B, T, Σ) be the Σ-group $\mathbf{Z}^{\omega_1}$ with the product Σ-structure, each copy of $\mathbf{Z}$ having the discrete Σ-structure, and let (A, S, Σ) be the relativization of (B, T, Σ) to the subgroup A of B consisting of the elements of countable support. (B, T, Σ) and (A, S, Σ) both have the same arity, namely ω_1, and the facts stated in (3.4) show that they are the Σ-groups associated respectively with the two topological groups: B with the product topology (the factors $\mathbf{Z}$ being discrete) and A with the relative topology, hence that both are regular, with (B, T, Σ) being complete—in fact it is clearly adic. (A, S, Σ) is not complete and indeed we easily see from (4.5) that its regular completion is isomorphic to (B, T, Σ). (5.10) shows that (A, S, Σ) is relatively free, but (B, T, Σ) is not relatively free. To see this let e and e_ν, ν in ω_1, be the elements of B defined by $\pi_\mu(e) = 1$ and $\pi_\mu(e_\nu) = \delta_{\mu\nu}$ for all μ, and let $\alpha = (e) - (e_\nu : \nu \in \omega_1)$; then α is in $\ker \Sigma$ but it is not in $k_0(T)$. For let σ be any SPS of α. Then the term e of α is coupled, via the partition on which σ is based, with only countably many of the $-e_\nu$ in α and thus if we take $-e_{\nu_0}$ to be distinct from each of these $-e_\nu$, $\pi_{\nu_0}(\sigma)$ will have infinitely many non-zero terms. It follows that σ cannot be in T and so α is not in $k_0(T)$. (It can be shown that $k_0(T)$ consists of all series in T of the form $\bigoplus_i \alpha_i$ with each α_i countable and in $\ker \Sigma$. Hence $k_0(T)$ is $\bigoplus$-closed in T and so (B, T, Σ) fails to be relatively free even in the weaker sense of [2, 4.5.14].)

This example shows that an uncountable product of relatively free Σ-groups is not necessarily relatively free (in fact the argument shows that no product with uncountably many non-trivial factors is relatively free) and likewise for the Σ-group associated with an arbitrary Hausdorff topological abelian group, for a regular adic Σ-group of uncountable arity, and for the regular completion of a relatively free regular Σ-group.

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