Ergod. Th. & Dynam. Sys., (2021), **41**, 2879–2880
doi:10.1017/etds.2020.86

CORRIGENDUM

Strongly ergodic equivalence relations: spectral gap and type III invariants – CORRIGENDUM

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doi:10.1017/etds.2017.108, Published by Cambridge University Press,
27 November 2017.

1. *Introduction*

There is a gap in the proof of [**HMV17**, Theorem 4.1] which was noticed by Stefaan Vaes. The gap is in the very last line of the proof: we apply [**HMV17**, Proposition 4.3] to conclude that $\mathcal{R} \times_\Omega G$ is strongly ergodic, but in order to do this, we have first to check that $\mathcal{R} \times_\Omega G$ is ergodic. Let us explain how the proof can be corrected.

2. *Correction of the proof of [**HMV17**, Theorem 4.1]*

First, we generalize slightly [**HMV17**, Proposition 4.3] in the following form.

LEMMA 2.1. *Let $\mathcal{R}$ be an equivalence relation on a standard measure space X. Let $\Gamma \curvearrowright X$ be a non-singular action of an amenable countable discrete group by automorphisms of $\mathcal{R}$. If the equivalence relation generated by $\mathcal{R}$ and $\mathcal{R}(\Gamma \curvearrowright X)$ is strongly ergodic, then there exists some $\mathcal{R}$-invariant subset $\emptyset \neq Y \subset X$ such that $\mathcal{R}_Y$ is strongly ergodic.*

Proof. We reduce to the case where $\mathcal{R}$ is ergodic, which was already proved in the original version. Since the equivalence relation generated by $\mathcal{R}$ and $\mathcal{R}(\Gamma \curvearrowright X)$ is strongly ergodic, the action of Γ on $\mathrm{L}^\infty(X)^{\mathcal{R}}$ must be strongly ergodic. But since Γ is amenable, this is only possible if the action of Γ on $\mathrm{L}^\infty(X)^{\mathcal{R}}$ is transitive, i.e. $\mathrm{L}^\infty(X)^{\mathcal{R}}$ is Γ-equivariantly isomorphic to $\ell^\infty(\Gamma/H)$ for some subgroup $H < \Gamma$. Take an H-invariant and $\mathcal{R}$-invariant subset $\emptyset \neq Y \subset X$ corresponding to an H-invariant atom of $\mathrm{L}^\infty(X)^{\mathcal{R}}$. Then $\mathcal{R}_Y$ is ergodic and the equivalence relation generated by $\mathcal{R}_Y$ and $\mathcal{R}(H \curvearrowright Y)$ must be strongly ergodic. By the ergodic case which is already known, we conclude that $\mathcal{R}_Y$ is strongly ergodic. □

We also need to use the notion of Mackey range of a cocycle. Let $\mathcal{R}$ be an equivalence relation on a standard measure space X, G a locally compact second countable abelian group and $\Omega \in \mathrm{Z}^1(\mathcal{R}, G)$ a 1-cocycle. Then the translation action of G on $X \times G$ preserves the skew-product equivalence relation $\mathcal{R} \times_\Omega G$. In particular, it induces an action $\eta_\Omega : G \curvearrowright \mathrm{L}^\infty(X \times G)^{\mathcal{R} \times_\Omega G}$. This action η_Ω is called the *Mackey range* of Ω. It is ergodic if and only if $\mathcal{R}$ is ergodic.

We sketch the proof of the following well-known observation which relates the kernel of $[\widehat{\Omega}]$ to the kernel of the Mackey range η_Ω of the cocycle Ω. Recall that a G-action is *transitive* if it is conjugate to $G \curvearrowright G/H$ for some closed subgroup $H < G$.

PROPOSITION 2.2. *Let $\mathcal{R}$ be an ergodic equivalence relation on a standard measure space X. Let $\Omega \in Z^1(\mathcal{R}, G)$ be a measurable 1-cocycle with values in a second countable locally compact abelian group G. Let $\eta_\Omega : G \curvearrowright L^\infty(X \times G)^{\mathcal{R} \times_\Omega G}$ be its Mackey range. Then we have*

$$\ker[\widehat{\Omega}] \subset (\ker \eta_\Omega)^\perp = \{p \in \widehat{G} \mid \langle p, g \rangle = 1 \text{ for all } g \in \ker \eta_\Omega\}$$

and the equality $\ker[\widehat{\Omega}] = (\ker \eta_\Omega)^\perp$ *holds if η_Ω is transitive.*

Proof. Take $p \in \ker[\widehat{\Omega}]$. Then $\widehat{\Omega}(p) = \partial u$ for some $u \in L^0(X, \mathbf{T})$. By definition of the skew-product construction, we get the relation $\partial(1 \otimes p) = \partial(u \otimes 1)$, where we view $1 \otimes p$ as an element of $L^0(X \times G, \mathbf{T})$. Therefore, we have $\partial(1 \otimes p) = \partial(u \otimes 1)$. This means that $u^* \otimes p$ is $\mathcal{R} \times_\Omega G$-invariant. Since $\eta_\Omega(g)(u^* \otimes p) = \overline{\langle p, g \rangle}(u^* \otimes p)$ for all $g \in G$, we conclude that $\langle p, g \rangle = 1$ for all $g \in \ker \eta_\Omega$.

Now suppose that η_Ω is transitive. Then η_Ω is conjugate to $G \curvearrowright G/H$ where $H = \ker \eta_\Omega$. Take $p \in H^\perp$. Since we can identify $L^\infty(X \times G)^{\mathcal{R} \times_\Omega G}$ with $L^\infty(G/H)$ in a G-equivariant way, we can find some $\mathcal{R} \times_\Omega G$-invariant function f on $X \times G$ such that $g \cdot f = \overline{\langle p, g \rangle} f$ for all $g \in G$. This forces f to be of the form $f = u^* \otimes p$ where we view p as a function on G. Then, as above, we get $\widehat{\Omega}(p) = \partial u$ and hence $p \in \ker[\widehat{\Omega}]$. □

We can finally correct the end of the proof of [HMV17, Theorem 4.1].

Proof. (...) Moreover, $\mathcal{Q}$ is generated by $\mathcal{R} \times_\Omega G$ and the orbit equivalence relation $\mathcal{R}(H \curvearrowright X \times G)$ of the translation action $H \curvearrowright X \times G$ on the second coordinate. By Lemma 2.1, we conclude that $(\mathcal{R} \times_\Omega G)_Y$ is strongly ergodic for some $\mathcal{R} \times_\Omega G$-invariant subset $\emptyset \neq Y \subset X \times G$. In particular, this implies that $L^\infty(X \times G)^{\mathcal{R} \times_\Omega G}$ is atomic, and hence that η_Ω is transitive. Since $[\widehat{\Omega}]$ is injective, Proposition 2.2 implies that $\mathcal{R} \times_\Omega G$ is ergodic. We conclude that $Y = X$ and we are done. □

Acknowledgements. We are grateful to Stefaan Vaes for pointing out the gap in the proof of [HMV17, Theorem 4.1]. CH is supported by Institut Universitaire de France and FY2019 JSPS Invitational Fellowship for Research in Japan (long term).

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