9

Entropy generation and decoherence of quantum fields

In Chapter 4 we studied particle creation in an external field, building from the basic concepts and techniques of quantum field theory in a dynamical background field or spacetime to the point where we can recognize that particle creation is in general a non-Markovian process. We derived a quantum Vlasov equation for the rate of particle creation in a changing electric field, and discussed cosmological particle creation from a changing background spacetime. In these processes we pointed out an intrinsic relation between the number and phase of a system in a particular quantum state. We presented a squeezed-state description of particle creation and discussed the conditions under which particle number may increase and others when it may decrease. These discussions bring out some basic issues in the statistical mechanics of quantum fields. In this chapter we will discuss two of these, entropy generation from particle creation and decoherence of quantum fields in the transition from quantum to classical. We will show that dissipation and fluctuations (or noise) in quantum field systems are the primary causes responsible in each of these processes.

In this chapter we shall adopt natural units $\hbar = c = k_{\mathrm{B}} = 1$.

9.1 Entropy generation from particle creation

In discussing the problem of entropy generation from cosmological particle creation [Park69, Zel70, Hu82] we are confronted by the following apparent paradox: on the one hand textbook formulae suggest that entropy (S) is proportional to the number (N) of particles produced (e.g. $S \propto N$ for photons). On the other hand, from quantum field theory, particle pairs created in the vacuum will remain in a pure state and one should not expect any entropy generation. Inquiry into this paradox led to serious subsequent investigations into the statistical properties of particles and fields [Hu84, HuKan87, HuPav86, Kan88a, Kan88b]. These early inquiries in the 1980s of the theoretical meaning of entropy of quantum fields were conducive to gaining a better understanding of the statistical mechanical properties of quantum fields and useful for practical calculations such as for a relativistic plasma of particles and fields in heavy ion experiments, or in finding the entropy content of primordial gravitons in the early universe.

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9.1.1 Choice of representations and initial conditions

Many different schemes were proposed in the 1990s for entropy generation from particle production. Brandenberger, Mukhanov and Prokopec [BrMuPr92, BrMuPr93] suggested a coarse graining of the field by integrating out the rotation angles in the probability functional, while Gasperini and Giovannini [GasGio93, GasGioVen93] considered a squeezed vacuum in terms of new variables which give the maximum and minimum fluctuations, and suggested a coarse graining by neglecting information about the subfluctuant variable (defined in Section 4.2). Keski-Vakkuri [Kes94] studied entropy generation from particle creation with many particle mixed initial states. Matacz [Mat94] considered a squeezed vacuum of a harmonic oscillator system with time-dependent frequency, and, motivated by the special role of coherent states, modeled the effect of the environment by decohering the squeezed vacuum in the coherent state representation. Kruczenski, Oxman and Zaldarriaga [KrOxZa94] used a procedure of setting the off-diagonal elements in the density matrix to zero before calculating the entropy. Despite the variety of coarse-graining measures used, in the large squeezing limit (late times) these approaches all give an entropy of $S = 2r$ per mode, where r is the squeezing parameter. This result which gives the number of particles created at late times agrees with that obtained earlier by Hu and Pavon [HuPav86].

Noteworthy in this group of work is that the *representation of the state* of the quantum field and the *coarse graining in the field* are stipulated, not derived. What is implicitly assumed or glossed over in these approaches is the important process of decoherence – the suppression of the off-diagonal components of a reduced density matrix in a certain basis. It is a necessary condition for realizing the quantum-to-classical transition, see [Zur81, Zur82, Zur91, JooZeh85, CalLeg85, UnrZur89, HuPaZh92, Zur93]. The deeper issues are to show explicitly how the entropy of particle creation depends on the choice of specific initial state and/or particular ways of coarse graining, and to understand how natural or how plausible these choices of the initial state representation or the coarse-graining measure are in different realistic physical conditions [Hu94a].[1] To answer these questions, one needs to work with a more basic theoretical framework incorporating statistical mechanics and quantum fields. We shall treat the decoherence and entropy/uncertainty issues with the quantum open system concept [Davies76, LinWes90, Wei93] and the influence functional formalism introduced in Chapters 3 and 5. Our discussion of the different ways of defining the entropy of quantum fields is adapted from [CaHuRa00], while our open systems

[1] This includes conditions when, for example, the quantum field is at a finite temperature or is out of equilibrium, interacting with other fields, or that its vacuum state is dictated by some natural choice, for example, in the earlier quantum cosmology regime such as the Hartle–Hawking boundary condition leading to the Bunch–Davies vacuum in de Sitter spacetime.

treatment of entropy generation follows that of Koks *et al.* [Kok96, KoMaHu97]. Notable later work on related subjects includes that of Kiefer *et al.* [KiPoSt00] and Campo and Parentani [CamPar04].

9.1.2 Coarse graining the environment in an open system

In the quantum Brownian motion paradigm the role of the Brownian particle can be played by a detector, a designated mode of a quantum field, such as the homogeneous inflaton field, or the scale factor of the background spacetime (as in minisuperspace quantum cosmology), while the bath could be a set of coupled oscillators, a quantum field, or just the high-frequency sector of the field, as in stochastic inflation. The statistical properties of the system are depicted by the reduced density matrix (rdm) formed by integrating out the details of the bath. One can use the rdm or the associated Wigner function to calculate the statistical average of physical observables of the system, such as the uncertainty or the entropy functions. The von Neumann entropy of an open system is given by

$$S_{\rm CG} = -{\rm Tr}[\rho_R(t) \ln \rho_R(t)], \tag{9.1}$$

The entropy function constructed from the reduced density matrix (or the reduced Wigner function) of a particular state measures the information loss of the system in that state to the environment (or, in the phraseology of [ZuHaPa03], the "stability" characterized by the loss of predictive power relative to the classical description). One can study the entropy increase for a specific state, or compare the entropy at each time for a variety of states characterized by the squeeze parameter. Interaction with the environment changes the system's dynamics from unitary to dissipative, the energy loss being measured by the viscosity function, which governs the relaxation of the system into equilibrium with the environment. The entropy function for such open systems can also be used [AndHal93, Hal93, AnaHal95, HuZha93b, HuZha95, ZuHaPa03] as a measure of how close different quantum states can lead to a classical dynamics. For example, the coherent state being the state of minimal uncertainty has the smallest entropy function [ZuHaPa03] and a squeezed state in general has a greater uncertainty function [HuZha93b, HuZha95]. One can thus use the uncertainty to measure how classical or "nonclassical" a quantum state is.

With regard to the issue of entropy of quantum fields raised at the beginning, we can ask, what is the difference of this more rigorous definition based on open-system dynamics and those obtained with more *ad hoc* prescriptions?

9.1.3 Differences in various definitions of entropy

Consider, for example a representative list of papers on the entropy of quantum fields, such as [Hu84, HuPav86, HuKan87, Kan88a, Kan88b, BrMuPr92, BrMuPr93, GasGio93, GasGioVen93]. We see that in some cases the entropy refers to that of the field, and is obtained by coarse graining some information of the field itself, such as making a random phase approximation, adopting the number basis, or integrating over the rotation angles. The entropy of [HuZha93b, HuZha95, AndHal93, Hal93, AnaHal95, ZuHaPa03], on the other hand, refers to that of the open system and is obtained by coarse graining the environment. Why is it that for certain generic models in some common limit (late time, high squeezing), both groups of work obtain the same result? Under what conditions would they differ? Understanding this relation could provide a more solid theoretical foundation for the intuitively argued definitions of field entropy.

At the formal level, supposing we have some system which has been decomposed into two subsystems, it is well known (e.g. [Pag93]) that between the entropies S_1, S_2 of the two subsystems, and that of the total system, S_{12}, a triangle inequality holds:

$$|S_1 - S_2| \leq S_{12} \leq S_1 + S_2 \tag{9.2}$$

In particular, if the total system is closed and so in a pure state, then it has zero entropy, so that the two subsystems necessarily have equal entropies.[2] Hence, asking for the entropy change of a system is equivalent to asking for the entropy change of the environment it couples to, if the overall closed system is in a pure state. Now consider the case of the system as a detector (or a single mode of a field) and the environment as the field. The information lost in coarse graining the field which was used to define the field entropy in the above examples is precisely the information lost as registered in the particle detector, which shows up in the calculation of entropy from the reduced density matrix. The bilinear coupling between the system and the bath as used in the simple quantum Brownian motion models also ensures that the information registered in both sectors is directly commutable. This explains the commonalities. However, not all coarse graining and coupling will lead to the same results, as we shall explicitly demonstrate in some examples below.

Another important feature of the entropy function obtained in this more rigorous open-systems definition which is not obvious in other *ad hoc* approaches is that it depends nonlocally on the entire history of the squeezing parameter. This can be seen from the fact that the rate of particle creation varies in time and its effect is history dependent [HarHu79, CalHu87]. We have seen this behavior in Chapter 4. Existing methods of calculating the entropy generation give results which only depend on the squeezing parameter at the time when a particular

[2] This could be the reason why the derivation of black hole entropy (e.g. [Bek94]) can be obtained equivalently by computing the entropy of the radiation (e.g. [FroNov93]) emitted by the black hole, or by counting the internal states (if one knows how) of the black hole (e.g. [ZurTho85, Bek83, BekMuk95, StrVaf96]). Physically one can view what happens to the particle as a probe into the state of the field.

coarse graining (or dropping the off-diagonal components of the density matrix) is implemented.

In Chapter 3 we introduced open quantum systems in terms of influence functionals, following the treatment of [HuPaZh92, HuMat94]. In Chapter 4 we introduced squeezed quantum system, using a general oscillator Hamiltonian as an example, following the treatment of [HuKaMa94, HuMat94]. Here we apply these methods to calculate the entropy and uncertainty functions and then specialize to an oscillator system, recovering en route the results of [HuZha93b, HuZha95, AndHal93, Hal93, AnaHal95, HuZha93b, HuZha95] for the uncertainty function at finite temperature, and of [ZuHaPa03] on the entropy of coherent states. These results are also useful for the consideration of entropy of particles created in the early universe (see, e.g. [KoMaHu97] for a minimally coupled scalar field mimicking a graviton field in a de Sitter universe).

9.2 Entropy of quantum fields

Our discussion in this chapter started with the posing of a deceptively simple question: Is there entropy generation in particle creation? Attempting to answer this question uncovers a host of basic issues in the statistical mechanics of quantum fields. Here we briefly describe the entropy functions obtained from two different types of considerations and operations: The first type is for particle creation in free quantum fields. The main point is the choice of representations and the specification of the initial state. The second type is for particle creation in interacting quantum fields.

To begin with, we note that for a closed system with a unitarily evolving quantum field its dynamics is governed by the quantum Liouville equation, and the von Neumann entropy constructed from the density matrix of the closed system,

$$S_{\mathrm{VN}} = -\mathrm{Tr}[\rho(t) \ln \rho(t)], \tag{9.3}$$

is exactly conserved. One can introduce approximations or assumptions to render a closed system open or effectively open (see Chapter 1). We distinguish two situations: If there is a justifiable separation of macroscopic and microscopic time-scales, one can adopt the theoretical framework of quantum kinetic field theory. If one assumes an initial factorization condition for the density matrix (as in the “molecular chaos” assumption), one obtains a relativistic Boltzmann equation. The Boltzmann entropy S_{B} defined in terms of the phase space distribution $f(k, X)$ for quasiparticles can in this case be shown to satisfy a relativistic H-theorem [GrLeWe80, CalHu88]. We want to generalize this to a correlation entropy for interacting quantum fields.

However, in the case where there does *not* exist such a separation of time-scales, how does one define the entropy of a quantum field? For nonperturbative truncations of the dynamics of interacting quantum fields, this is a nontrivial question [HuKan87]. Intuitively, one expects that any coarse graining which leads to an effectively open system with irreversible dynamics will also lead to the growth of entropy. These operations can be systematically carried out by way of the projection operator techniques. A projection operator P projects out the *irrelevant* degrees of freedom from the total system described by the density operator ρ, yielding the reduced density matrix ρ_R

$$\rho_\mathrm{R}(t) = P\rho(t) \tag{9.4}$$

There exists a well-developed formalism for deriving the equation of motion of the *relevant* degrees of freedom, and in terms of it, the behavior of the coarse-grained entropy (9.1), which will in general not be conserved [Nak58, Zwa60, Zwa61, Mor65, WilPic74, Gra82, Kam85, GoKaZi04, GorKar04, Bal75]. The projection operator formalism can be used to express the slaving of higher correlation functions in the correlation hierarchy. From it one can define an entropy in effectively open systems (see, e.g. [Ana97a, Ana97b]). (So far it has only been implemented within the framework of perturbation theory.) Another powerful method adept to field theory is the Feynman–Vernon influence functional formalism developed in Chapter 5. We shall use it to illustrate how to define the entropy functions for quantum open systems [KoMaHu97].

9.2.1 Entropy special to choice of representation and initial conditions

We begin with the simpler yet more subtle case of a free quantum field. Take for example particle creation in a time-varying background field or in an expanding universe studied in Chapter 4. Entropy is generated in the particle production process from the parametric amplification of vacuum fluctuations. The focal point is a wave equation with a time-dependent natural frequency for the amplitude function of a normal mode. (The same condition arises for an interacting field, such as the $\lambda\Phi^4$ theory in the Hartree–Fock approximation or the O(N) field theory at leading order in the large-N expansion.) Since the underlying dynamics is clearly unitary and time-reversal invariant in this case, a suitable coarse graining leading to entropy growth is not trivially evident. Hu and Pavon [HuPav86] made the observation that a coarse graining is implicitly incorporated when one chooses to depict particle numbers in the n-particle Fock (or "N") representation or to depict the phase coherence in the phase (or "P") representation. This idea has been further developed and clarified [Kan88a, Kan88b, KoMaHu97, KlMoEi98]. The source of entropy generation for free fields is very different from that of interacting fields (e.g. the growth of correlational entropy, described below) in that particle creation from parametric amplification depends sensitively on the choice of representation for the state space of the parametric oscillators, and the specificity of the initial conditions.

9.2.2 Entropy from projecting out irrelevant variables

In contrast to entropy growth resulting from parametric *particle creation* from the vacuum, entropy growth due to *particle interactions* in quantum field theory has a very different physical origin. A coarse graining scheme was proposed by Hu and Kandrup [HuKan87] for these processes. Expressing an interacting quantum field in terms of a collection of coupled parametric oscillators, their proposal is to define a reduced density matrix by projecting the full density operator onto each oscillator's single-oscillator Hilbert space in turn,

$$\varrho(\mathbf{k}) \equiv \mathrm{Tr}_{\mathbf{k}' \neq \mathbf{k}} \rho \tag{9.5}$$

and defining the reduced density operator as the tensor product Π of the projected single-oscillator density operators $\varrho(\mathbf{k})$,

$$\rho_R \equiv \Pi_{\mathbf{k}} \varrho(\mathbf{k}) \tag{9.6}$$

The coarse-grained (Hu–Kandrup) entropy by projection is then just given by equation (9.1), from which we obtain

$$S_{\mathrm{HK}} = -\sum_{\mathbf{k}} \mathrm{Tr}[\varrho(\mathbf{k}) \ln \varrho(\mathbf{k})] \tag{9.7}$$

It is interesting to observe that for a spatially translation-invariant density matrix for a quantum field theory which is Gaussian in the position basis, this entropy is just the von Neumann entropy of the full density matrix, because the spatially translation-invariant Gaussian density matrix separates into a product over density submatrices for each $\mathbf{k}$ oscillator. This projection (Hu–Kandrup) coarse graining, like the correlation-hierarchy (Calzetta–Hu) coarse-graining scheme described below, does not choose or depend on a particular representation for the single-oscillator Hilbert space. It is sensitive to the establishment of correlations through the explicit couplings.

9.2.3 Entropy from slaving of higher correlations

We presented in Chapter 6 a general procedure for obtaining coupled equations for the correlation functions at any order l in the correlation hierarchy, which involves a truncation of the *master effective action* at a finite order in the loop expansion [NorCor75, CalHu95a, CalHu00, Ber04a]. By working with an l loop-order truncation of the master effective action, one obtains a closed, time-reversal invariant set of coupled equations for the first $l+1$ correlation functions, $\hat{\phi}, G, C_3, \ldots, C_{l+1}$. In general, the equation of motion for the highest order correlation function will be linear, and thus can be formally solved using Green's function methods. The existence of a unique solution depends on supplanting this with some causal boundary conditions. When the resulting solution for the highest correlation function is back-substituted into the evolution equations for the other lower-order correlation functions, the resulting dynamics is *not* time-reversal invariant, but generically dissipative, as measured by the correlation entropy. Thus, as was described before, with the slaving of the higher-order correlation functions we have rendered a closed system (the truncated equations for correlation functions) into an *effectively open system.* This coarse-graining scheme and the associated correlation entropy defined for an interacting quantum field has the benefit that it can be implemented in a nonperturbative manner. In addition to dissipation, one expects that an effectively open system will manifest noise/fluctuations [NorCor75, CalHu95a, CalHu00, Ber04a] arising from the slaving of the four-point function to the two-point function in the symmetry-unbroken $\lambda\Phi^4$ field theory. Thus a framework exists for exploring the irreversibility and fluctuations within the context of a unitary quantum field theory, using the truncation and slaving of the correlation hierarchy. The effectively open system framework is useful for precisely those situations, where a separation of macroscopic and microscopic time-scales (which would permit an effective kinetic theory description) does *not* exist, such as is encountered in the thermalization issue.

9.3 Entropy from the (apparent) damping of the mean field

We shall discuss these two situations in more detail with examples in the following two sections. In the first case we consider entropy generation in a closed system of a free quantum field, following the treatment of [HKMP96, KlMoEi98]. In the next section we consider entropy generation in an open system interacting with an environment.

Consider the dynamics of a closed system comprising of a mean field and the fluctuation fields. The time evolution of a closed system is Hamiltonian. The general time-dependent Gaussian density matrix of the system may be parameterized by the canonical variables, as we have seen in Chapter 4. Yet, the evolution in some circumstances can manifest apparent irreversible energy flow from the coherent mean fields to the fluctuating quantum modes and give the appearance of quantum decoherence of the mean field.

So what causes the appearance of damping in the dynamics of the mean field of such systems? To highlight the essential physics we note that this process is analogous to the Landau damping of collective modes in a collisionless electromagnetic plasma described by the Vlasov equation. One can understand this damping and decoherence as the result of *dephasing* of the rapidly varying fluctuations and particle production in the time-varying mean field, as shown in Chapter 4. There, when we show the derivation of the quantum Vlasov equation for the semiclassical scalar QED following [KlMoEi98], we encounter a typical situation in nonequilibrium statistical mechanics, namely, if there is a clear separation of time-scales amongst various processes going on in a system, we can seek an effective description of a particular subsystem by coarse graining or "projecting-out" the other subsystems. In the example at hand if we are only

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interested in the behavior of the slowly varying particle number $\mathcal{N}_{\mathbf{k}}$, the fast changing correlations $\mathcal{C}_{\mathbf{k}}$ can be projected out.[3] The effect of the environment on the open system is calculated through its back-action on the subsystem of interest. Here we focus on the statistical mechanics of particle creation, highlighting the non-Markovian nature of these processes, and seek a physical definition of entropy for such quantum field systems.

9.3.1 Time-scales

The essential physical ingredient in passing from the quantum evolution of the particle-field system to the kinetic description by the quantum Vlasov equation is the *dephasing* phenomenon, i.e. the near exact cancellation of the rapidly varying phases of the quantum mode functions contributing to the mean electric current of the created pairs. This cancellation depends in turn upon a clean separation of the following time-scales (refer to Chapter 4 for notation): [KlMoEi98]

(1) τ_{qu}, the inverse of the *natural frequency* of a normal mode, rapidly oscillating. It is the shortest time-scale reflecting the microscopic quantum theory.
(2) τ_{cl}, the inverse of $\dot{\mathcal{N}}_{\mathbf{k}}$, measures the slowly varying *mean number* of particles in the adiabatic number basis.
(3) τ_{pl}, of the collective *plasma oscillations* of the electric current and mean electric field produced by those particles.

In the limit $\tau_{\text{qu}} \ll \tau_{\text{cl}}$ quantum coherence (reflected in the phase or correlations) between the created pairs can be neglected because of efficient dephasing and a (semi)classical local kinetic approximation to the underlying quantum theory becomes possible. In the limit $\tau_{\text{cl}} \ll \tau_{\text{pl}}$ the electric field may be treated as approximately *constant* over the interval of particle creation. Thus when both inequalities apply we can replace the true nonlocal source term which describes particle creation in field theory by one that depends only on the instantaneous value of the quasi-stationary electric field, at least over very long intervals of time.

[3] Note that projecting out or coarse graining does not mean elimination or truncation. The information of the "irrelevant" variables in the other subsystems (constituting the environment) is retained fully in the integro-differential equation for the subsystem of special interest to us (the "relevant" variable), where the nonlocal kernels retain all the information about the subsystem and the environment. One can attempt to solve it, but because of the memory functions, it requires complicated and elaborate integration procedures. Depending on what specific physical information is targeted, one can devise coarse-graining measures to describe the effect of the environment on the system thus leading to a simplification of this integro-differential equation. One extreme yet familiar example is a heat bath where the environment is so grossly coarse grained that only temperature enters into the overall effect on the system (thus making it possible to use the canonical ensemble in equilibrium statistical mechanics, and linear response theory in near-equilibrium conditions).

Making use of these local approximations, one can find [KlMoEi98] an exact analytic expression for the spontaneous pair creation rate $\frac{d}{dt}\mathcal{N}_{\mathbf{k}}(t)$ for a *constant* electric field in real time, in agreement with the Schwinger result [Sch51] in both its exponential and nonexponential factors. Then by making use of an asymptotic expansion of the exact analytic result for constant fields, uniformly valid everywhere on the real time axis, one obtains a useful *local* approximation to the spontaneous pair creation rate for the slowly varying electric fields. A numerical comparison [KlMoEi98] between the quantum and local kinetic approaches to the dynamical back-reaction problem shows remarkably good agreement, even in quite strong electric fields, $eE \simeq m^2c^3/\hbar$, over a large range of times.

9.3.2 Density matrix

After the elimination of the rapid variables $\mathcal{C}_{\mathbf{k}}$ defined in (4.113) in favor of the slow variables $\mathcal{N}_{\mathbf{k}}$ one can construct the density matrix in the adiabatic number basis easily [KlMoEi98]. In a pure state (setting $\zeta = 1$ in equation (4.48)) the only nonvanishing matrix elements of ρ are in uncharged pair states with equal numbers of positive and negative charges, $\ell_{\mathbf{k}} = n_{\mathbf{k}}^{(+)} = n_{\mathbf{k}}^{(-)}$, with $\ell_{\mathbf{k}}$ the number of pairs in the mode $\mathbf{k}$, *viz.*

$$\langle 2\ell'_{\mathbf{k}}|\rho|2\ell_{\mathbf{k}}\rangle_{\text{pure}} = e^{i({\ell_{\mathbf{k}}}'-\ell_{\mathbf{k}})\vartheta_{\mathbf{k}}(t)}\ \text{sech}^2 r_{\mathbf{k}}(t)\ \left(\tanh\, r_{\mathbf{k}}(t)\right)^{{\ell_{\mathbf{k}}}'+\ell_{\mathbf{k}}} \tag{9.8}$$

where the magnitude of the Bogoliubov transformation, $r_{\mathbf{k}}(t)$, is defined in equation (4.27) and its phase, $\vartheta_{\mathbf{k}}(t)$, is determined by

$$\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}}e^{-2i\Theta_{\mathbf{k}}} = -\sinh\, r_{\mathbf{k}}\,\cosh\, r_{\mathbf{k}}\ e^{i\vartheta_{\mathbf{k}}} \tag{9.9}$$

Hence the off-diagonal matrix elements $\ell' \neq \ell$ of ρ are rapidly varying on the time-scale τ_{qu} of the quantum mode functions, while the diagonal matrix elements $\ell' = \ell$ depend only on the adiabatic invariant average particle number via

$$\langle 2\ell_{\mathbf{k}}|\rho|2\ell_{\mathbf{k}}\rangle_{\text{pure}} \equiv \rho_{2\ell_{\mathbf{k}}} = \text{sech}^2 r_{\mathbf{k}} \tanh^{2\ell_{\mathbf{k}}} r_{\mathbf{k}} = \frac{|\beta_{\mathbf{k}}|^{2\ell_{\mathbf{k}}}}{(1+|\beta_{\mathbf{k}}|^2)^{\ell_{\mathbf{k}}+1}} = \left.\frac{\mathcal{N}_{\mathbf{k}}^{\ell_{\mathbf{k}}}}{(1+\mathcal{N}_{\mathbf{k}})^{\ell_{\mathbf{k}}+1}}\right| \tag{9.10}$$

and are therefore much more slowly varying functions of time. The average number of positively charged particles (or negatively charged antiparticles) in this basis is given by

$$\sum_{\ell_{\mathbf{k}}=0}^{\infty} \ell_{\mathbf{k}}\rho_{2\ell_{\mathbf{k}}} = \mathcal{N}_{\mathbf{k}} \tag{9.11}$$

Thus the diagonal and off-diagonal elements of the density matrix in the adiabatic particle number basis stand in precisely the same relationship to each other and contain the same information as the particle number $\mathcal{N}_{\mathbf{k}}$ and pair correlation $\mathcal{C}_{\mathbf{k}}$ respectively.

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9.3.3 Entropy generation

In the density matrix (9.10) the diagonal elements $\rho_{2\ell_{\mathbf{k}}}$ may be interpreted (for a pure state) as the independent probabilities of creating $\ell_{\mathbf{k}}$ pairs of charged particles with canonical momentum $\mathbf{k}$ from the vacuum. This corresponds to disregarding the intricate quantum phase correlations between the created pairs in the unitary Hamiltonian evolution. When physics is expressed in the adiabatic particle number basis (the Fock or N representation) the phase information is ignored. The quantum density matrix in this representation produces an entropy function which reflects that associated with particle creation but says nothing about the evolution of the quantum phase or correlation. This illustrates the crucial role played by the choice of representations in the definition of entropy associated with particle creation [HuPav86].

Results obtained from neglecting quantum phase are known to be quite accurate for long intervals of time in the back-reaction of the current on the electric field producing the pairs, because when the current is summed over all the $\mathbf{k}$ modes, the phase information in the pair correlations cancels very efficiently. Thus for practical purposes one can approximate the full Gaussian density matrix over large time intervals by its diagonal elements only, in this basis.

Let us examine the reduced von Neumann entropy constructed from the diagonal density matrix (9.10)

$$S_{\mathcal{N}}(t) = -\sum_{\mathbf{k}} \sum_{\ell_{\mathbf{k}}=0}^{\infty} \rho_{2\ell_{\mathbf{k}}} \ln \rho_{2\ell_{\mathbf{k}}} \tag{9.12}$$

Upon substituting (9.10) into this, the sums over $\ell_{\mathbf{k}}$ are geometric series which are easily performed. The von Neumann entropy of this reduced density matrix

$$S_{\mathcal{N}}(t) = \sum_{\mathbf{k}} \{(1+\mathcal{N}_{\mathbf{k}}) \ln(1+\mathcal{N}_{\mathbf{k}}) - \mathcal{N}_{\mathbf{k}} \ln \mathcal{N}_{\mathbf{k}}\} \tag{9.13}$$

is precisely equal to the Boltzmann entropy of the single particle distribution function $\mathcal{N}_{\mathbf{k}}(t)$. Hence

$$\frac{d}{dt} S_{\mathcal{N}} = \sum_{\mathbf{k}} \ln\left(\frac{1+\mathcal{N}_{\mathbf{k}}}{\mathcal{N}_{\mathbf{k}}}\right) \frac{d}{dt} \mathcal{N}_{\mathbf{k}} \tag{9.14}$$

increases if the mean particle number increases. This is always the case *on average* for bosons if one starts with vacuum initial conditions, since $|\beta_{\mathbf{k}}|^2$ is necessarily nonnegative and can only increase if it is zero initially [Kan88a, Kan88b]. Locally, or once particles are present in the initial state, particle number or the entropy (9.14) does not necessarily increase monotonically in time.

Hence the notion of entropy associated with particle creation, and the lore that it increases in time, is only valid for spontaneous production of bosons from an initial vacuum state. This function associated with fermions, and that associated with stimulated production of both boson and fermions, can decrease in time. This we have remarked in Chapter 4.

9.3.4 Decoherence functional

Decoherence is also addressable within the same framework. Consider the case, where $\omega(t)$ is a function of one external degree of freedom, the mean field $A(t)$. If only the evolution of A is of interest, then the fluctuating modes described by $f(t)$ may be treated as the "environment." To solve for the evolution of the reduced density matrix of A, we compute the influence functional of two trajectories $A_1(t)$ and $A_2(t)$ corresponding to two different evolution operators $U_1(t)$ and $U_2(t)$ defined by

$$F_{12}(t) \equiv \exp(i\Gamma_{12}(t)) \equiv \mathrm{Tr}\left(U_1(t)\rho(0)U_2^\dagger(t)\right) \tag{9.15}$$

Explicit evaluation may be carried out using (4.1.53). Restricting again to pure states with vanishing $\bar{q}$ mean fields we find

$$\Gamma_{12}\Big|_{\substack{\zeta=1\\ \bar{q}=\bar{p}=0}} = \frac{-i}{2}\ln\left\{\frac{i\hbar}{|f_1 f_2|}\left(\frac{f_1 f_2^*}{f_1\dot{f}_2^* - \dot{f}_1 f_2^*}\right)\right\} \tag{9.16}$$

in terms of the two sets of mode functions $f_1(t)$ and $f_2(t)$ which satisfy (4.54) and (4.17). This Γ_{12} is precisely the closed time path (CTP) effective action functional which generates the connected real time n-point vertices in the quantum theory [CHKMPA94]. For a pure initial state, the absolute value of F_{12} measures the overlap of the two different evolutions at some time t, beginning with the same initial $|\psi(0)\rangle$. In mean field theory, instead of evaluating Γ_{12} for two arbitrary trajectories, the evaluation is over trajectories determined by the *self-consistent* evolution of the closed system, beginning with two different initial mean fields. The intimate relation between the CTP effective action functional and the decoherence functional was pointed out by [CalHu93, CalHu95a, CalHu94, HuMat94, Ana97a, Ana97b].

9.4 Entropy of squeezed quantum open systems

In Chapter 3 we studied quantum open systems with the harmonic oscillator Brownian motion model (QBM). In Chapter 4 we studied squeezed quantum systems as exemplified by particle creation in a dynamical background (with a Lagrangian (4.233)) and squeezed quantum open system exemplified by a parametric oscillator QBM (with Lagrangian (3.133)). Now we inquire about the entropy of squeezed quantum open systems. We seek a definition of the entropy S and the uncertainty function of a squeezed system interacting with a thermal bath, and study how they change in time by following the evolution of the reduced density matrix in the influence functional formalism. As examples, we calculate the entropy of two exactly solvable squeezed systems: an inverted harmonic oscillator and a scalar field mode evolving in an inflationary universe. For the inverted oscillator with weak coupling to the bath, at both high and low temperatures, $S \to r$, where r is the squeeze parameter defined in equation (4.217). For a massless minimally coupled scalar field in the de Sitter universe, $S \to (1-c)r$ at high temperatures where $c = \gamma_0/H$, γ_0 is the coupling to the bath and H the Hubble constant. These two cases confirm previous results based on more *ad hoc* prescriptions for calculating entropy. But for such a scalar field at low temperatures, the de Sitter entropy $S \to (1/2 - c)r$ is noticeably different. This result, obtained from a more rigorous treatment, shows that factors usually ignored by the conventional approaches, i.e. the nature of the environment and the coupling strength between the system and the environment, are important. Our treatment here is based on the results obtained in Chapter 5, Section 5.4, derived from the work of Hu, Koks and Matacz [KoMaHu97, HuMat94].

9.4.1 Entropy from the evolutionary operator for reduced density matrix

Consider again the quantum Brownian model discussed in Chapter 3. Our system is modeled by a harmonic oscillator (with coordinate x) with time-dependent mass (M), cross-term ($\mathcal{E}$) and natural frequency (Ω) coupled bilinearly with an environment modeled by many oscillators (with coordinates q_n) of similar nature ($m_n, \varepsilon_n, \omega_n$). The total Lagrangian is given by equation (3.133).

Assume the systems are initially in the vacuum state, so that their density matrix is Gaussian. Starting with an initial Gaussian reduced density matrix in the form

$$\rho_r(x_i\ x_i'\ t_i) \propto e^{-\lambda x_i^2 + \lambda_\times x_i x_i' - \lambda^* x_i'^2} \tag{9.17}$$

it is evolved by action of the evolutionary operator $\mathcal{J}_r$ for the reduced density matrix of the parametric quantum Brownian oscillator defined in (3.49) into

$$\rho_r(x, x', t) = N e^{-Au^2 - 2iBXu - 4CX^2} \tag{9.18}$$

where $x, x' = X \pm (u/2)$ and the A, B, C functions enter into the evolutionary operators $\mathcal{J}_r$ given by (3.135). They are in turn dependent on the a_{ij}, b_k coefficients given by (4.294), which are solutions to the differential equations for the coefficients of the generalized master equation (3.150) [HuPaZh92, HuPaZh93a]. Here,

$$N = 2\sqrt{C/\pi} \tag{9.19}$$

$$A = a_{22} + \frac{1}{D}\left\{[(2\lambda_r + \lambda_\times)/4 + a_{11}]\, b_3^2 + (2\lambda_i + b_4)\, a_{12}\, b_3 - (2\lambda_r - \lambda_\times) a_{12}^2\right\} \tag{9.20}$$

$$B = -b_1/2 + \frac{1}{D}[(\lambda_i + b_4/2)\, b_2\, b_3 - (2\lambda_r - \lambda_\times) a_{12}\, b_2] \tag{9.21}$$

$$C = \frac{1}{4D}(2\lambda_r - \lambda_\times)\, b_2^2 \tag{9.22}$$

$$D = 4|\lambda|^2 - \lambda_\times^2 + 4\,(2\lambda_r - \lambda_\times) a_{11} + 4\,\lambda_i\, b_4 + b_4^2 \tag{9.23}$$

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where λ_r, λ_i are the real and imaginary parts of λ. These expressions form the basis of calculations for squeezed quantum open systems. The reduced density matrix can be obtained by using the expressions above, which depend on $\chi = \alpha + \beta$, the sum of the Bogoliubov coefficients for the effective oscillator. For more details refer to Chapter 4, Section 4.7 [HuMat94, KoMaHu97].

The entropy of a field mode has been calculated by Joos and Zeh [JooZeh85, BKLS86] and others. It can be derived from the reduced density matrix at time t by using the von Neumann entropy (9.3), and is given by

$$S = \frac{-1}{w}[w \ln w + (1-w)\ln(1-w)] \simeq 1 - \ln w \quad \text{if } w \to 0 \tag{9.24}$$

where

$$w \equiv \frac{2\sqrt{C/A}}{1+\sqrt{C/A}} \tag{9.25}$$

A simpler quantity to use is the linear entropy:

$$S_{\text{lin}} \equiv -\text{Tr}\,\rho^2 = -\sqrt{C/A} \tag{9.26}$$

and $S = 0 \to \infty$ is equivalent to $S_{\text{lin}} = -1 \to 0$, both strictly increasing. Then if $S_{\text{lin}} \to 0$ we have

$$S \to -\ln|S_{\text{lin}}| + 1 - \ln 2, \quad \text{i.e. } S_{\text{lin}} \to -\frac{1}{2}(e^{1-S}) \tag{9.27}$$

As an example, suppose we have a system in an initially pure Gaussian state ($\lambda_\times = 0$), so that noise and dissipation are absent: $\gamma_0 = 0$, defined in (3.142). In this case, we have

$$a_{11} = a_{12} = a_{22} = 0 \tag{9.28}$$

so that $C/A = 1$ and hence $S = 0$, as expected.

9.4.2 Measures of fluctuations and coherence

At this point it is useful to supplement our presentations of squeezed quantum open systems in Chapters 3–5 by a discussion of the relation between fluctuations, coherence and entropy. In some cases the description for the dynamics of a squeezed (closed) quantum system can be simplified by expressing the density matrix in terms of the so called super- and subfluctuant variables $u_{\text{SF}}, v_{\text{SF}}$ obtained as real linear combinations of the canonical variables q, p:

$$u_{\text{SF}} = -\kappa \sin\phi\, q + \cos\phi\, p \tag{9.29}$$

$$v_{\text{SF}} = \cos\phi\, q + \frac{\sin\phi}{\kappa}\, p \tag{9.30}$$

This rotation eliminates the cross-terms in the Wigner function. We fix the linear combinations such that one variable (u, the superfluctuant) grows exponentially while the other decays exponentially.

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Writing the density matrix in the $u_{\rm SF}$ basis, e.g. $\rho(u_{\rm SF}, u'_{\rm SF})$, one can then compute the fluctuations in $u_{\rm SF}$ and $v_{\rm SF}$ as (see Section IIIC of [KoMaHu97] for details)

$$\Delta u_{\rm SF}^2 = \langle u_{\rm SF}^2\rangle - \langle u_{\rm SF}\rangle^2 = \frac{\varsigma\varpi}{2C} \quad \Delta v_{\rm SF}^2 = \frac{\varsigma}{2C} \tag{9.31}$$

where $\varphi, \varsigma, \varpi$ are defined as

$$\varphi \equiv \frac{\kappa}{2}\cot\phi, \quad \varsigma \equiv \frac{\sin^2\phi}{\kappa^2}\left[4AC + (B-\varphi)^2\right] \tag{9.32}$$

$$\varpi \equiv \frac{4AC + (4\varphi\varsigma + B - \varphi)^2}{4\varsigma^2} \tag{9.33}$$

As a measure of coherence we note from (9.18) that a large A coefficient means that the density matrix is strongly peaked along its diagonal, i.e. there is very little coherence in the system. A measure of coherence was defined in [Mat94] as a squared coherence length L^2, equal to the coefficient of $-u^2$ divided by 8, so that a large L^2 means a high degree of coherence in the system. With this definition of L^2, we have

$$L_u^2 = \frac{\varsigma\varpi}{2A}, \quad L_v^2 = \frac{\varsigma}{2A} \tag{9.34}$$

We can thus relate the coherence lengths and fluctuations to the entropy of the system by

$$\frac{L_u^2}{\Delta u_{\rm SF}^2} = \frac{L_v^2}{\Delta v_{\rm SF}^2} = S_{\rm lin}^2 = \frac{C}{A} \tag{9.35}$$

(Note that linear entropy is negative by definition in order for it to increase with S. Then as $S_{\rm lin}$ increases, $S_{\rm lin}^2$ will decrease.) Also the uncertainty relation for $u_{\rm SF}, v_{\rm SF}$ becomes

$$\Delta u_{\rm SF}^2 \Delta v_{\rm SF}^2 = \frac{1}{S_{\rm lin}^2}\left[\frac{1}{4} + \frac{(4\varphi\varsigma + B - \varphi)^2}{16AC}\right] \tag{9.36}$$

For the free field the last term in the square brackets is zero while $S_{\rm lin} = -1$ (since $S = 0$), so that $\Delta u_{\rm SF}\Delta v_{\rm SF} = 1/2$.

9.4.3 Entropy and uncertainty functions of an inverted oscillator

We can now demonstrate how the previous results are used. An oscillator with time-independent frequency Ω coupled to a thermal ohmic bath of like oscillators has local dissipation (i.e. $\mathbf{D} \propto \delta'(t-t')$), and at $T \to \infty$ the noise becomes white ($\mathbf{N} \propto \delta(t-t')$). The entropy in this simple case is easily compared with known results in equilibrium statistical mechanics: the entropy at high temperature is

$$S \to 1 + \ln\frac{T}{\Omega} \tag{9.37}$$

We can also use this entropy expression to investigate the claim by Zurek, Habib and Paz [ZuHaPa03] (in the small γ_0 limit by using a Wigner function

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approach) that for large times the state of least entropy for the oscillator (with a time-independent natural frequency) is the coherent state, at least for white noise and local dissipation. Since the coherent state is the "most classical-like" quantum state, this was invoked as an indication of quantum to classical transition. Equivalently one can use the uncertainty function as a measure. This was shown by Hu and Zhang [HuZha93b, HuZha95], and Anderson, Anastopoulos and Halliwell [AndHal93, Hal93, AnaHal95].

The static inverted harmonic oscillator (IHO) is perhaps the simplest squeezed system. It has been used as a model to study quenching in a quantum phase transition (see the next section). It also models the zero mode of the inflaton field in new inflation (see Chapter 15). Its Lagrangian is:

$$L(t) = \frac{1}{2}[\dot{x}^2 + \Omega^2 x^2] \tag{9.38}$$

We touched on this case in Chapter 4, Section 4.7 as an example of a squeezed quantum system. Suppose this system is coupled to the usual environment of harmonic oscillators in a thermal state, with coupling constant $c(s) = 1$. Then the equivalent oscillator we consider has unit mass, no cross-term and frequency

$$\Omega^2_{\mathit{eff}} = -\Omega^2 - \gamma_0^2 \equiv -\kappa^2 \tag{9.39}$$

so that from (4.239) the sum of its Bogoliubov coefficients is (taking $t_i = 0$, recall $z \equiv \kappa t$, $\sigma \equiv \kappa s$)

$$\chi(t) = \cosh\, z - i\sinh\, z \tag{9.40}$$

Hence we have

$$\alpha = \cosh\, z, \quad \beta = -i\sinh\, z \tag{9.41}$$

so that at late times as $z \to \infty$, $r \to z$. To determine the entropy generated we need to calculate the various quantities in the propagator coefficients. For white noise these coefficients have analytic solutions, but for zero temperature we need to calculate them numerically.

The b_i's are independent of the temperature, and are found to be (where here and elsewhere a carat will denote division by κ)

$$b_{\{{1 \atop 4}\}} = \kappa(\pm \coth z - \hat{\gamma}_0), \quad b_{\{{2 \atop 3}\}} = \frac{\pm\kappa e^{\pm\hat{\gamma}_0 z}}{\sinh\, z} \tag{9.42}$$

High temperature

White noise is given by $\mathbf{N}(s, s') = 4\gamma_0 T\, \delta(s - s')$, or $\mathbf{N}(\sigma, \sigma') = 4\hat{\gamma}_0\kappa^2 T\delta(\sigma - \sigma')$. Using these, Kok, Matacz and Hu derived the expressions for the a_{ij} coefficients. Note that $\hat{\gamma}_0 = \gamma_0/\kappa < 1$; however if we assume small dissipation ($\hat{\gamma}_0 \ll 1$) we can write down large time limits of these quantities:

$$a_{11} \to \frac{T\hat{\gamma}_0}{1 - \hat{\gamma}_0}, \quad a_{12} \to \frac{2Te^{-(1-\hat{\gamma}_0)z}}{1 + \hat{\gamma}_0}, \quad a_{22} \to \frac{T\hat{\gamma}_0}{1 + \hat{\gamma}_0}$$
$$b_{\{{1 \atop 4}\}} \to \kappa(\pm 1 - \hat{\gamma}_0), \quad b_{\{{2 \atop 3}\}} \to \pm 2\kappa\, e^{-(1\mp\hat{\gamma}_0)z} \tag{9.43}$$

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We can then calculate large time limits of the density matrix coefficients from (9.19):

$$A \to a_{22}, \quad B \to -b_1/2, \quad C \to \frac{b_2^2}{16a_{11}} \tag{9.44}$$

These coefficients are independent of the initial conditions, which might be expected since the dissipation is acting to damp out any late time dependence on these initial conditions. We have

$$S_{\text{lin}} = -\sqrt{\frac{C}{A}} \to \frac{-\kappa^2 e^{-z}}{2\gamma_0 T} \tag{9.45}$$

From (9.27) and the fact that $r \to z$ as $z \to \infty$ we obtain

$$S \to r + 1 + \ln \frac{T\gamma_0}{\kappa^2} \tag{9.46}$$

Zero temperature

At $T = 0$, the action of the environment is due to quantum effects only. Analytic expressions for the a_{ij}, b_k coefficients in this case can be found in [KoMaHu97].

At $T = 0$, for weak dissipation, $\hat{\gamma}_0 \ll 1$ we have at late times,

$$A \to a_{22}, \quad B \to -b_1/2, \quad C \to \frac{b_2^2}{16a_{11}} \tag{9.47}$$

Again the coefficients are independent of the initial conditions. Since b_2 is unchanged from the high-temperature case and a_{11}, a_{22} tend toward constants, we see that

$$S_{\text{lin}} \to \frac{-\kappa e^{-z}}{2\sqrt{a_{11}a_{22}}} \tag{9.48}$$

and so again from (9.27) and since at late times, $r \to z$,

$$S \to r + 1 + \ln \frac{\sqrt{a_{11}a_{22}}}{\kappa} \tag{9.49}$$

In conclusion, approaching the problem of entropy and uncertainty from the open system viewpoint enables one to see explicitly their dependence on the coarse graining of the environment and the system–environment couplings. It also exposes the relation between quantum and classical descriptions – it is through decoherence that the quantum field becomes classical [CalHu94, AngZur96]. This is the subject of the next section.

9.4.4 Entropy from graviton production in de Sitter spacetime

We now turn to an example in cosmology, that of an inflationary universe (see Chapter 15). We want to calculate the entropy of a massless scalar field minimally coupled to gravity in a de Sitter spacetime by examining the evolution of the density matrix. As we shall see, it is a generally solvable squeezed system.

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Consider a massless minimally coupled scalar field in de Sitter space,

$$L_{\rm new}(\eta) = \sum \frac{1}{2}\left[\chi'_{\mathbf{k}}\chi_{-\mathbf{k}} + \frac{2}{\eta}\chi_{\mathbf{k}}\chi'_{-\mathbf{k}} - \chi_{\mathbf{k}}\chi_{-\mathbf{k}}\left(k^2 - \frac{1}{\eta^2}\right)\right] \tag{9.50}$$

We also use a spectral density [Wei93] of the form

$$I(\omega,\eta,\eta') = \frac{2\gamma_0}{\pi H}\frac{\omega}{\sqrt{\eta\eta'}} \tag{9.51}$$

so that $c(\eta) = 1/\sqrt{-H\eta}$. This corresponds to an ohmic bath with a time-dependent coupling to the system. Since γ_0/H is dimensionless we rewrite it as c, not to be confused with $c(\eta)$. Incorporating the bath gives the equivalent oscillator with $M = 1$, $\mathcal{E} = 1/\eta$ and frequency

$$\Omega^2_{\rm eff} = k^2 - \frac{1+c^2}{\eta^2} \tag{9.52}$$

With $z = k\eta$, $\sigma = ks$ we can write the dynamical equation for the quantity χ introduced in Chapter 4, Section 4.7 as

$$\chi''(z) + \left(1 - \frac{2+c^2}{z^2}\right)\chi = 0$$

$$\chi(z_i) = 1, \quad \chi'(z_i) = -i - 1/z_i \tag{9.53}$$

where $z < 0$. The solution of this equation can be constructed using Bessel functions whose index is a function of c; however since we are interested in small c we take the solution to be approximately that of the same equation but with c set to zero. This simplifies things greatly:

$$\chi(z) = \left(1 + \frac{i}{2z_i}\right)f(z) + \frac{i}{2z_i}f^*(z) \tag{9.54}$$

where

$$f(z) \equiv \left(1 - \frac{i}{z}\right)e^{i(z_i - z)} \tag{9.55}$$

We can further simplify χ by using a very early initial time, setting $z_i \to -\infty$. We also disregard the phase in the resulting expression for χ, since this is not expected to make any difference to physical quantities. In this case we obtain a new function which we rename χ:

$$\chi(z) \to \left(1 - \frac{i}{z}\right)e^{-iz} \tag{9.56}$$

The Bogoliubov coefficients can now be found from (4.241):

$$\alpha = \left(1 - \frac{i}{2z}\right)e^{-iz}, \quad \beta = \frac{-i}{2z}e^{-iz} \tag{9.57}$$

and so at late times

$$r \to -\ln|z| \tag{9.58}$$

This result was also obtained in [Mat94] using a different method.

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In [KoMaHu97] the expressions for a_{ij}, b_k were derived to leading order in z, and from them the authors show that the coefficients A, B, C tend to the same form as for the static oscillator.

High temperature

We begin by writing

$$\mathbf{N} = 4cc^2(s)T\delta(s - s') \tag{9.59}$$

$$= \frac{-4ck^2T}{\sigma}\,\delta(\sigma - \sigma') \tag{9.60}$$

From the expressions given in [KoMaHu97] for a_{ij}, b_k at high temperature one can obtain their behavior as $z \to 0$. Since in this case the coefficients A, B, C tend to the same form as for the static oscillator, thus

$$S_{\text{lin}} \to \frac{-|b_2|}{4\sqrt{a_{11}a_{22}}} = O|z|^{1-c}. \tag{9.61}$$

Using (9.27) and (9.58) we obtain

$$S \to (1 - c)r + \text{constant} \tag{9.62}$$

Finite temperature

In this case

$$A \to a_{22} - \frac{a_{12}^2}{4a_{11}}, \quad B \to -b_1/2, \quad C \to \frac{b_2^2}{16a_{11}} \tag{9.63}$$

and so

$$S_{\text{lin}} \to O|z|^{1/2-c} \tag{9.64}$$

Then with (9.27) and (9.58) we have

$$S \to (1/2 - c)r + \text{constant} \tag{9.65}$$

9.4.5 Discussion

In the last two sections we calculated the entropy of two physical and exactly solvable squeezed systems: an inverted harmonic oscillator and a scalar field mode evolving in a de Sitter inflationary universe. To compare these results with that obtained from the more *ad hoc* approaches, we must bear in mind that for a field mode that could be split into two independent sine and cosine (standing wave) components, the result will be twice that obtained here, namely, $S = 2r$ (rather than r in here)

For the inverted oscillator, in both temperature regimes with weak coupling, we obtained $S \to r + \text{constant}$. In the de Sitter case, the high-temperature result is $S \to (1 - c)r + \text{constant}$. In these three examples the results obtained for the entropy from the more *ad hoc* approaches comply with the first principles results

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presented here. However at lower temperatures the de Sitter entropy is $S \to (1/2 - c)r + \text{constant}$. This last result requires us to look more closely at A and C which together give the entropy.

From (9.26) and (9.27), and neglecting the added constants which are always implied, we find that in the high squeezing limit the entropy behaves as $S \to \frac{1}{2}\ln A - \frac{1}{2}\ln C$. When the system–environment coupling is weak, all of the above cases give $-1/2\ln C \to r$, which is the expected result. The dominant contribution to C always comes from b_2 in the high squeezing limit. This parameter is determined by the squeezing of the system and is essentially independent of the nature of the environment and its coupling to the system. We can therefore conclude that the $\ln C$ contribution to the entropy represents entropy intrinsic to the squeezed system itself. This should be true quite generally for squeezed systems. However these results fail to take into account the contributions to the entropy from the $\ln A$ term. This contribution is determined by the a_{ij} factors which strongly depend on the nature of the environment and its coupling to the system. There is no a priori reason to expect this contribution to be small, as illustrated by the finite temperature de Sitter example where $1/2\ln A \to -r/2$. This highlights the danger in trusting the more *ad hoc* approaches. The crucial point is that the entropy of a system depends not only on the system itself but also on the nature of the environment and how it is coupled to the system.

9.5 Decoherence in a quantum phase transition

Quantum phase transitions [Sac99] refer to phase transitions mediated by quantum fluctuations or parameters of a quantum nature, as opposed to classical fields or parameters (such as temperature or magnetic fields) in classical phase transitions. It is an area of active current research in condensed matter physics. Interestingly enough, this subject has also been the focus of theoretical cosmology – the inflationary universe proposal highlights the vital role played by phase transitions in determining the state and dynamics of the early universe. The essential quantum nature in these phase transitions comes about because the vacuum expectation value of the quantum inflaton field is what drove the universe to a period of rapid expansion and its quantum fluctuations acted as seeds to structure formation in the later universe. Topological defects [VilShe00] appearing in the field configurations, such as magnetic monopoles, cosmic strings and domain walls, may often be of quantum field origins. Unfortunately the existing theories for phase transition, structure and defect formation have largely been built on classical field models. Such existing classical theories may not be naively adaptable for the description of these quantum phenomena without careful scrutiny. Overall, we know that any reliable investigation of these processes should entail both the quantum field and the nonequilibrium (dynamical) aspects. A number of basic issues common to them need be addressed from both the conceptual and the technical levels. Foremost is the question of how the quantum field comes to

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behave like a classical field, and how the quantum fluctuations become classical stochastic sources. These are the issues of decoherence and noise of quantum fields respectively. We will discuss in this section the issue of decoherence and quantum-to-classical transitions in the context of a second-order phase transition for an interacting quantum field, and revisit both issues in the context of structure formation from quantum fluctuations in the early universe in Chapter 15.

The key question we wish to seek an answer to is the emergence of a classical order parameter field after a second-order phase transition described in quantum field theory language [Cal89]. The system field can be the long-wavelength modes of a quantum field and the environment field can be its own short-wavelength modes, or a different set of quantum fields. We have given a thorough treatment of these two cases in Chapter 5, with a derivation of the influence action, the master equations, and an analysis of the dissipation and noise kernels. Here we show how those results can be of use for tackling this problem. The goal here is to compute the decoherence times for the system-field modes and place them in relation to the other time-scales in the model. If it is shorter than all the other relevant physical time-scales then it may provide some justification for viewing the system quantum field as a classical order parameter field, thus providing a justification for the common practices in existing theories of classical phase transitions. If not, then one has to work out the theory of quantum phase transitions from first principles to highlight the differences from their classical counterparts.

Criteria for decoherence

Correlations peaking around the classical trajectory in the phase space, as indicated by the Wigner function showing such behavior (for a long time being perceived as the closest analog to a classical distribution function), were once believed to be a sufficient criterion of classicality [Hal89], but was shown to be inadequate by Habib and Laflamme [HabLaf90]. As we mentioned in Chapter 5, the Wigner function contains just as much information as the density matrix, and thus one needs to demonstrate by some mechanism the diminishing of the phase information in the quantum system to begin to possess some classical attributes. Since a quantum system almost always interacts with its environment, according to the environment-induced decoherence viewpoint, one can use the diminishing or vanishing of the off-diagonal elements of the reduced density matrix in a suitable basis (such as the "pointer basis" of Zurek) as an indication of, or criterion for, decoherence and the transition to classicality. Likewise one needs to do this for the Wigner function.

Models for quench transition

We focus on quenching which is a second-order quantum phase transition. For a quantum field ϕ with infinite degrees of freedom undergoing a continuous transition, the field ordering after the transition begins is due to the growth in amplitude of its unstable long-wavelength modes. A quench transition can in general be characterized by the quench transition time-scale t_{q}. Physically this is the time by which the order parameter field has sampled the degenerate ground states. One can take the field to be classical by the time it is ordered as such. This has implications [RiLoMa02] for the formation of the defects that are a necessary by-product of transitions.

A simple model for quench transition is an inverted harmonic oscillator (IHO) which we studied in some detail in an earlier section in this chapter. This is also the simplest model which depicts the evolution of the inflaton field and the growth of inhomogeneities in the early universe. To see why, recall that the normal modes of a massless free scalar field propagating in a Friedmann–Lemaitre–Robertson–Walker universe satisfy the equation

$$\phi_k'' + \left(k^2 - \frac{a''}{a}\right)\phi_k = 0. \tag{9.66}$$

For sufficiently long wavelengths ($k^2 \ll a''/a$), this equation describes an unstable oscillator.

Guth and Pi [GutPi85] used the IHO model to study the evolution of the inflaton field. They assumed that at the onset of inflation the universe was in a Gaussian quantum state centered on the maximum of the potential. It is easy to show from the solution of the (functional) Schrödinger's equation that the initial wave packet spreads quickly in time but maintains its Gaussian shape (due to the linearity of the model). The initial Gaussian state becomes highly squeezed and indistinguishable from a classical stochastic process. Since the wavefunction is Gaussian, the Wigner function is positive for all times and peaks on the classical trajectories in phase space as the wavefunction spreads. In these situations the Wigner function can be interpreted as a classical probability distribution for coordinates and momenta, showing sharp classical correlations at long times. But the harmonic oscillator is a special case where this condition holds. As remarked above, this criterion based on correlations in phase space is not sufficient to prove the transition from quantum to classical. One needs to also show how the phase information in the quantum system disappears, such as by invoking an environment-induced decoherence process.

Open systems

Guth and Pi did not expound the decoherence and quantum to classical transition issues in depth, but simply invoked the uncertainty principle as an indication of such a transition. Uncertainty principle at a finite temperature was studied by Hu and Zhang [HuZha93b, HuZha95] (see also Anderson, Anastopoulos and Halliwell [AndHal93, Hal93, AnaHal95, HuZha93b, HuZha95]) using a harmonic oscillator bath at finite temperature as the environment. They showed explicitly how a quantum oscillator system evolves from a quantum- to a thermal-dominated state which marks such a transition. Independently Zurek, Habib and Paz [ZuHaPa03] showed that a quantum system interacting with a high-temperature ohmic bath will most likely evolve to a coherent state, which is known as the quantum state with the most classical features. This was invoked as a criterion for classicality. In an earlier section we have shown how these two criteria, i.e. uncertainty at finite temperature and a quantum state evolving to a coherent state, are actually two sides of the same coin in the environment-induced decoherence perspective.

Interacting fields

The feature of a Gaussian wavefunction maintaining its Gaussian nature in evolution are special to linear systems and the linear instabilities described above are valid only for free fields. For example in an inverted anharmonic oscillator, it has been shown [LoMaMo00] by numerically evolving the Schrödinger equation that an initially Gaussian wavefunction becomes non-Gaussian, the Wigner function develops negative parts, and its interpretation as a classical probability breaks down.[4] A similar argument for quantum mechanics, but for *open systems*, was also presented in [LoMaMo00]. Coupling an inverted oscillator with an anharmonic potential to a high-temperature environment, the authors showed that it becomes classical very quickly, even before the wavefunction probes the nonlinearities of the potential. Being an early time event, the quantum-to-classical transition can be studied perturbatively. Lombardo, Mazzitelli and Rivers (LMR) [LoMaRi01] have extended this to field theory by considering a system field comprising the long wavelengths of the order parameter interacting with a large number of environmental fields, including its own short wavelengths. Assuming weak coupling and high critical temperature, they showed that decoherence is a short time event, shorter than the quench transition time $t_{\rm sp}$. As a result, perturbative calculations are justified. Subsequent dynamics can be described by a stochastic Langevin equation, the details of which are only known for early times.[5]

[4] In this connection we mention numerical computations of quantum mechanical models and of different approximations to interacting field theory (see Chapter 12). In such calculations, classical correlations do appear in some field theory models [CHKMPA94, BoVeHo99]. However, since such decoherence (in a time-averaged sense) takes place at long times after the transition has been achieved initially, when the mean field approximation has broken down, this may be an artifact of the Gaussian-like approximations [LoMaMo00].

[5] A remark on the relation with thermal field theory is in place here. As pointed out by [LoMaRi01] there are similarities and differences between this quantum open system approach and the well-established classical behavior of thermal scalar field theory [AarBer02, AarSmit98] at high temperature. It is known that at high temperatures, the behavior of long-wavelength modes is determined by classical statistical field theory. The effective classical theory is obtained after integrating out the hard modes with $k \geq gT$. The "classical behavior" in this soft thermal mode analysis is defined through the coincidence of the quantum and the statistical correlation functions. Thermal equilibrium is assumed to hold at all times and the cut-off that divides system and environment depends on the temperature, which is externally fixed. In phase transitions, the quantum-to-classical transition is defined by the effective diagonalization of the reduced density matrix, which is not assumed to be thermal and the separation between long and short wavelengths is determined by their stability, which depends on the parameters of the potential.

Using a model with biquadratic coupling between the system and the environment, LMR first [LoMaRi01] considered the case of an instantaneous quench, then a slow quench [RiLoMa02]. The consideration of slow quenches is very important since the Kibble–Zurek mechanism [Kib80, Kib88, Zur85, Zur96] predicts the relation between the subsequent domain structure and the quench time (by indirectly counting defects [LagZur97, LagZur98, YatZur98]). The authors of [LoMaRi03] worked out the theories for other couplings but show that the biquadratic coupling is the most relevant for the quantum-to-classical transition. Also, since all relevant time-scales depend only logarithmically on the parameters of the theory, they also showed the necessity of keeping track of $O(1)$ prefactors carefully. In the next section we illustrate a quench quantum transition following their treatment.

9.6 Spinodal decomposition of an interacting quantum field

The model we discuss contains a real system field ϕ, which undergoes a transition, coupled biquadratically to other scalar fields χ_a $(a = 1, 2, \ldots, N)$, which constitute the external part of the environment (the internal environment is provided by the short-wavelength modes of the field ϕ itself). The influence functional and the master equation obtained from integrating out the environmental fields have been derived in Chapter 5. We focus on the diffusion coefficients central to the process of decoherence and evaluate upper bounds on the decoherence time t_{D} for slow quenches. The general conclusion is that the decoherence time is typically shorter than the quench transition time.

The model

We consider for the system a self-interacting scalar ϕ-field which describes the order parameter, whose $\mathcal{Z}_2$ symmetry is broken by a double-well potential, and an environment comprising N free scalar fields χ_{a} with classical action

$$S[\phi, \chi] = S_{\mathrm{syst}}[\phi] + S_{\mathrm{env}}[\chi] + S_{\mathrm{int}}[\chi_{\mathrm{a}}, \phi] \tag{9.67}$$

where (with μ^2, $m^2 > 0$)

$$S_{\mathrm{syst}}[\phi] = \int d^4x \left\{ -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right\}$$

$$S_{\mathrm{env}}[\chi_{\mathrm{a}}] = \sum_{\mathrm{a}=1}^{N} \int d^4x \left\{ -\frac{1}{2}\partial_\mu\chi_{\mathrm{a}}\partial^\mu\chi_{\mathrm{a}} - \frac{1}{2}m_{\mathrm{a}}^2\chi_{\mathrm{a}}^2 \right\}$$

The most important interactions will turn out to be of the biquadratic form

$$S_{\mathrm{int}}[\chi_{\mathrm{a}}, \phi] = S_{\mathrm{qu}}[\phi, \chi] = -\sum_{\mathrm{a}=1}^{N} \frac{g_{\mathrm{a}}}{8} \int d^4x \ \phi^2(x)\chi_{\mathrm{a}}^2(x) \tag{9.68}$$

Physical conditions

To keep our calculations tractable, we need a significant part of the environment to have a strong impact upon the system field, but not vice versa, from which we can bound $t_{\rm D}$. The simplest way to implement this is to take a large number $N \gg 1$ of scalar χ_a fields with comparable masses $m_a \simeq \mu$ weakly coupled to the ϕ, with λ, $g_a \ll 1$. Thus, at any step, there are N weakly coupled environmental fields influencing the system field, but only one weakly self-coupled system field to back-react upon the explicit environment.

For one-loop consistency at second order in our calculation of the diffusion coefficient (that enforces classicality) it is sufficient, at order of magnitude level, to take identical $g_a = g/\sqrt{N}$. Further, at the same order of magnitude level, we take $g \simeq \lambda$.[6]

For small g the model has a continuous transition at a temperature $T_{\rm c}$. The environmental fields χ_a reduce $T_{\rm c}$ and, in order that $T_{\rm c}^2/\mu^2 = 24/(\lambda + \sum g_a) \gg 1$, we must take $\lambda + \sum g_a \ll 1$, whereby $1 \gg 1/\sqrt{N} \gg g$. Further, with this choice the dominant hard loop contribution of the ϕ-field to the χ_a thermal masses (see Chapter 10) is

$$\delta m_T^2 = O(gT_{\rm c}^2/\sqrt{N}) = O(\mu^2/N) \ll \mu^2 \tag{9.69}$$

Similarly, the two-loop (setting sun) diagram which is the first to contribute to the discontinuity of the χ-field propagator is of magnitude

$$g^2 T_{\rm c}^2/N = O(g\mu^2/N^{3/2}) \ll \delta m_T^2 \tag{9.70}$$

in turn. That is, the effect of the thermal bath on the propagation of the environmental χ-fields is ignorable. In particular, the infinite N limit does not exist. Dependence on N is implicit through $T_{\rm c}$ as well as through the couplings, for initial temperatures $T_0 = O(T_{\rm c})$. $\eta = \sqrt{6\mu^2/\lambda}$ determines the position of the minima of the potential and the final value of the order parameter. As has been shown in [LoMaRi01] this choice of coupling and environments gives the hierarchy of scales necessary for establishing a reliable approximation scheme.

We shall assume that the initial states of the system and environment are both thermal, at a high temperature $T_0 > T_{\rm c}$. We then imagine a change in the global environment (e.g. expansion in the early universe) that can be characterized by a change in temperature from T_0 to $T_{\rm f} < T_{\rm c}$. That is, we do not attribute the transition to the effects of the environment fields. As initial conditions of the

[6] This is very different from the more usual large-N $O(N+1)$-invariant theory with one ϕ-field and N $\chi_{\rm a}$ fields, dominated by the $O(1/N)$ $(\chi^2)^2$ interactions, that has been the standard way to proceed for a *closed* system. With our choice there are no direct χ^4 interactions, and the indirect ones, mediated by ϕ loops, are depressed by a factor $g/\sqrt{N}$. In this way the effect of the external environment qualitatively matches the effect of the internal environment provided by the short-wavelength modes of the ϕ-field, but in a more calculable way.

open system we take a factorized density matrix at temperature T_0 of the form $\hat{\rho}[T_0] = \hat{\rho}_\phi[T_0] \times \hat{\rho}_\chi[T_0]$.[7]

Provided the change in temperature is not too slow the exponential instabilities of the ϕ-field grow so fast that the field has populated the degenerate vacua well before the temperature has dropped significantly below T_c. Since the temperature T_c has no particular significance for the environment fields, for these early times we can keep the temperature of the environment fixed at $T_\chi \approx T_\mathrm{c}$. For simplicity the χ_a masses are fixed at the common value $m \simeq \mu$.

9.6.1 The quench transition time

To describe the physics of the quenching transition we show the estimation of the quench transition time t_sp defined from $\langle\phi^2\rangle_{t=t_\mathrm{sp}} \sim \eta^2$. We assume that the quench begins at $t = 0$ and ends at time $t = 2\tau_\mathrm{q}$, with $\tau_\mathrm{q} \gg t_\mathrm{r} \sim \mu^{-1}$. At the qualitative level at which we are working it is sufficient to take $m_\phi^2(T_0) = \mu^2$ exactly. Most simply, we consider a quench linear in time, with temperature $T(t)$, for which the mass function is of the following form [BowMom98][8]

$$m^2(t) = m_\phi^2(T(t)) = \begin{cases} \mu^2 & \text{for} \quad t \le 0 \\ \mu^2 - \frac{t\mu^2}{\tau_q} & \text{for} \quad 0 < t \le 2\tau_q \\ -\mu^2 & \text{for} \quad t \ge 2\tau_q \end{cases} \tag{9.72}$$

The field behaves as a free field in an inverted parabolic potential for an interval of approximately t_sp [KarRiv97], where

$$\langle\phi^2\rangle_{t_\mathrm{sp}} \sim \eta^2 \tag{9.73}$$

The equation of motion for the mode $u_k(t)$, with wavenumber k is, in the quench period,

$$\left[\frac{d^2}{ds^2} + k^2 + \mu^2 - \frac{\mu^2 s}{\tau_\mathrm{q}}\right] u_k(s) = 0 \tag{9.74}$$

[7] Given our thermal initial conditions it is not the case that the full density matrix has ϕ and χ fields uncorrelated initially, since it is the interactions between them that lead to the restoration of symmetry at high temperatures. Rather, incorporating the hard thermal loop "tadpole" diagrams of the χ (and ϕ) fields in the ϕ mass term leads to the effective action for ϕ quasi-particles,

$$S_\mathrm{syst}^\mathrm{eff}[\phi] = \int d^4x \left\{-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m_\phi^2(T_0)\phi^2 - \tfrac{\lambda}{4!}\phi^4\right\} \tag{9.71}$$

where $m_\phi^2(T) \propto (T/T_\mathrm{c} - 1)$ for $T \approx T_\mathrm{c}$. As a result, we can take an initial factorized density matrix at temperature T_0 of the form $\hat{\rho}[T_0] = \hat{\rho}_\phi[T_0] \times \hat{\rho}_\chi[T_0]$, where $\hat{\rho}_\phi[T_0]$ is determined by the quadratic part of $S_\mathrm{syst}^\mathrm{eff}[\phi]$ and $\hat{\rho}_\chi[T_0]$ by $S_\mathrm{env}[\chi_a]$. That is, the many χ_a fields have a large effect on ϕ, but the ϕ-field has negligible effect on the χ_a.

[8] Note that the τ_q of [LoMaRi03] is the inverse quench rate $T_\mathrm{c}^{-1}dT/dt|_{T=T_\mathrm{c}}$, and so differs from that of [BowMom98] by a factor of 2.

subject to the boundary condition $u_k(t) = e^{-i\omega t}$ for $t \leq 0$, where $\omega^2 = \mu^2 + k^2$. Instead of the simple exponentials of the instantaneous quench, $u_k(t)$ has solution [BowMom98]

$$u_k(t) = a_k \mathrm{Ai}\left(\frac{\Delta_k(t)}{\bar{t}}\right) + b_k \mathrm{Bi}\left(\frac{\Delta_k(t)}{\bar{t}}\right) \tag{9.75}$$

with Ai(s), Bi(s) the Airy functions; $\Delta_k(t) = t - \omega^2 \bar{t}^3$ and $\bar{t} = (\tau_{\mathrm{q}}/\mu^2)^{1/3}$. Note that $\Delta_0(t) = t - \tau_q$, the time since the onset of the transition. In the causal analysis of Kibble [Kib80] $\bar{t}$ ($\mu^{-1} \ll \bar{t} \ll \tau_{\mathrm{q}}$) is the time at which the adiabatic field correlation length collapses at the speed of light, the earliest time in which domains could have formed. The analysis of [LoMaRi03] suggests that this earliest time is not $\bar{t}$, but t_{sp}.

It is straightforward to establish a relationship between $\bar{t}$ and $t_{\mathrm{sp}} > \bar{t}$. The constants of integration in (9.75) are

$$\begin{aligned} a_k &= \pi[\mathrm{Bi}'(-\omega^2\bar{t}^2) + i\omega\bar{t}\,\mathrm{Bi}(-\omega^2\bar{t}^2)] \\ b_k &= -\pi[\mathrm{Ai}'(-\omega^2\bar{t}^2) + i\omega\bar{t}\,\mathrm{Ai}(-\omega^2\bar{t}^2)] \end{aligned} \tag{9.76}$$

It follows that, when $\Delta_k(t)/\bar{t}$ is large, then

$$\begin{aligned} |u_k(t)|^2 &\approx \omega\bar{t}\left(\frac{\bar{t}}{\Delta_k(t)}\right)^{1/2} \exp\left[\frac{4}{3}\left(\frac{\Delta_k(t)}{\bar{t}}\right)^{3/2}\right] \\ &\approx \mu\bar{t}\left(\frac{\bar{t}}{\Delta_0(t)}\right)^{1/2} \exp\left[\frac{4}{3}\left(\frac{\Delta_0(t)}{\bar{t}}\right)^{3/2}\right] e^{-k^2/\bar{k}^2} \end{aligned} \tag{9.77}$$

where $\bar{k}^2 = \bar{t}^{-3/2}(\Delta_0(t))^{-1/2}/2$.

For large initial temperature $T_0 = O(T_c)$, the power spectrum for field fluctuations peaked around $\bar{k}$, and

$$\langle\phi^2\rangle_t \approx \frac{T_0}{2\pi^2\mu^2}\int k^2\,dk\,|u_k(t)|^2 \approx \frac{CT_0}{\mu\bar{t}^2}\left(\frac{\Delta_0(t)}{\bar{t}}\right)^{-5/4} \exp\left[\frac{4}{3}\left(\frac{\Delta_0(t)}{\bar{t}}\right)^{3/2}\right] \tag{9.78}$$

The prefactor C is included to show that terms, nominally $O(1)$, can in fact be large or small (in this case $C = (64\sqrt{2}\pi^{3/2})^{-1} = O(10^{-3})$). Note that, although the unstable modes have a limited range of k-values, increasing in time, this is effectively no restriction when $\Delta_0(t)/\bar{t}$ is significantly larger than unity.

Finally, we obtain

$$\frac{\eta^2}{C'} \simeq \frac{T_{\mathrm{c}}}{\mu\bar{t}^2}\exp\left[\frac{4}{3}\left(\frac{\Delta_0(t_{\mathrm{sp}})}{\bar{t}}\right)^{3/2}\right] \tag{9.79}$$

where $C' = C[\ln(\mu\bar{t}^2\eta^2/CT_{\mathrm{c}})^{-5/6}]$. Since the effect on t_{sp} only arises at the level of "ln ln" terms, $C' \approx C$ is a good estimation in all that follows. (Since this choice underestimates t_{sp} it only strengthens the claim that $t_{\mathrm{sp}} > t_{\mathrm{D}}$.)

9.6.2 Decoherence time

We now turn to the question of whether decoherence proceeds faster than spinodal decomposition. Rather than attempt a full estimate of the decoherence time $t_{\rm D}$ (see [LoMaRi03]), we shall run a simple test.

As we have already remarked, at early times the system field may be described as an inverted harmonic oscillator. The evolution is then well approximated by an ensemble of classical trajectories, but there remains the question of whether two different classical histories are consistent in the Gell-Mann–Hartle sense [RivLom05, LoRiVi07].

A classical history displays spatial structure as well as time evolution. We are helped by the observation that the ordering of the field is due to the growth of the long-wavelength unstable modes. Unstable long-wavelength modes start growing exponentially as soon as the quench is performed, whereas short-wavelength modes will oscillate. As a result, the field correlation function rapidly develops a peak (Bragg peak) at wavenumber $k = \bar{k} \ll \mu$. Specifically [KarRiv97], initially as $\bar{k}^2 = \mathcal{O}(\mu/\sqrt{t\tau_{\rm q}})$, where $\tau_{\rm q}^{-1}$ is the quench rate. Assuming that a classical description can be justified *post hoc*, a domain structure forms quickly with a characteristic domain size $O(\bar{k})$, determined from the position of this peak. (As an example, see the numerical results of [LagZur97, LagZur98, YatZur98], where this classical behavior has been assumed through the use of stochastic equations – see later.) With this in mind, we adopt an approximation in which the system-field contains only one Fourier mode with $\mathbf{k} = \mathbf{k}_0 = O(\bar{k})$, characteristic of the domain size. For simplicity, we shall further assume $\mathbf{k} = \mathbf{k}_0 = 0$ (we refer the reader to [LoMaRi03] for a more complete analysis).

We may simplify the issue further by considering only trajectories which begin from $\phi = 0$ at $t = \tau_{\rm q}$. Two such trajectories are distinguished by the value of $\dot{\phi}$ at the initial time. We shall ask what is the minimum speed difference at the initial time that ensures consistency by $t = t_{\rm sp}$. If this minimum difference is much smaller than the natural spread $\sim (T_{\rm c}/V)^{1/2}$, then the conclusion that decoherence is faster than spinodal decomposition is upheld [RivLom05].

We will calculate the decoherence functional to lowest nontrivial order (two vertices) for large N. Again, we assume weak coupling $\lambda \simeq g \ll 1$, where we have defined g by the order of magnitude relations $g_a \simeq g/\sqrt{N}$. As such we may expand the logarithm of the decoherence functional up to second nontrivial order in coupling strengths.

As "trial" classical solutions, we take

$$\phi(\mathbf{x}, s) = \dot{\phi}\, u(s), \quad \phi'(\mathbf{x}, s) = \dot{\phi}' u(s) \tag{9.80}$$

where $u(s)$ is the solution of the mode equation with boundary conditions $u(\tau_q) = 0$, $\dot{u}(\tau_q) = 1$. Since we are neglecting the self-interaction term, our conclusions are only trustworthy for $t \leq t_{\rm sp}$.

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The solution of the equations of motion for the mode functions is given by

$$u(t) = \frac{\pi \bar{t}}{3^{2/3}\Gamma(2/3)} \left[\sqrt{3}\mathrm{Ai}\left(\frac{t-\tau_{\mathrm{q}}}{\bar{t}}\right) + \mathrm{Bi}\left(\frac{t-\tau_{\mathrm{q}}}{\bar{t}}\right)\right] \tag{9.81}$$

where $\mathrm{Ai}(s)$ and $\mathrm{Bi}(s)$ are the Airy functions, and we have used $\Delta_0(t) = t - \tau_{\mathrm{q}}$.

The procedure outlined above is quite general and applies to a range of couplings [LoMaRi03]. We now specialize to biquadratic coupling. The modulus of the decoherence functional is given by

$$|\mathcal{D}[\phi^1, \phi^2]|^2 \sim \exp\left\{-\frac{g^2}{32}\int d^4x \int d^4y\ \phi_-^{(2)}(x) N_{\mathrm{q}}(x,y)\phi_-^{(2)}(y)\right\} \tag{9.82}$$

where $N_{\mathrm{q}}(x-y) = \mathrm{Re}G_{\mathrm{F}}^2(x,y)$ is the noise (diffusion) kernel. G_{F} is the relevant Feynman propagator of the χ-field at temperature T_0. We have defined $\phi_-^{(2)} = (\phi^1)^2 - (\phi^2)^2$. For our chosen classical histories, and at times $t \sim t_{\mathrm{sp}}$ it becomes

$$|\mathcal{D}[\phi^1, \phi^2]|^2 \sim \exp\left\{-\frac{g^2}{32}\left[(\dot{\phi}^1)^2 - (\dot{\phi}^2)^2\right]^2 DV\right\} \tag{9.83}$$

where V is the volume of space and

$$D = 2\int_0^{t_{\mathrm{sp}}} dt \int_0^t ds\ u^2(t) F(t-s) u^2(s) \tag{9.84}$$

$$F(t) = \frac{27}{512}\mathrm{Re}G_{\mathrm{F}}^2(0;t) \tag{9.85}$$

where $G_{\mathrm{F}}^2(k,t)$ is the Fourier transform of the square of the Feynman χ propagator. It is only in $u(s)$ that the slow quench is apparent.

In the high-temperature limit ($T \gg \mu$), LMR obtain the explicit expression for the kernels

$$\mathrm{Re}G_{\mathrm{F}}^2(0;t) = \frac{T_{\mathrm{c}}^2}{64\pi^2}\int_0^\infty dp\ \frac{p^2}{(p^2+\mu^2)^2}\cos\left(2\sqrt{p^2+\mu^2}\,t\right) \tag{9.86}$$

where μ is the thermal χ-field mass at temperature $T \sim T_{\mathrm{c}}$. In this scheme, it is approximately the cold χ mass. Because the χ-field propagator is unaffected by the ϕ-field interactions one can obtain the detail of the expression in (9.86).

We see that, for times $\mu t \geq 1$, the behavior of D is dominated by the exponential growth of $u(s)$, and the integral in equation (9.84) by the interval $s \approx t$. We will assume large $\Delta_0(t)$ (and $\Delta_0(s)$), which means $\Delta_0(t), \Delta_0(s) \gg \bar{t}$. This condition is satisfied provided s is larger than and not too close to $\omega_0^2\tau_{\mathrm{q}}/\mu^2$, and allows us to use the asymptotic expansions of the Airy functions and their derivatives for the evaluation of $u(s)$. This will be justified *post hoc*. In particular,

$$u(t) = \left(\frac{\sqrt{\pi}\bar{t}}{3^{2/3}\Gamma[2/3]}\right)\left(\frac{\bar{t}}{t}\right)^{1/4}\exp\left[\frac{2}{3}\left(\frac{(t-\tau_q)}{\bar{t}}\right)^{\frac{3}{2}}\right] \tag{9.87}$$

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Keeping only the parametric dependence, we obtain

$$|\mathcal{D}[\phi^1,\phi^2]|^2 \sim \exp\left\{-g^2 V\bar{t}^5 \left[(\dot{\phi}^1)^2 - (\dot{\phi}^2)^2\right]^2 \frac{T_c^2\bar{t}^2}{\mu t_{\rm sp}} \exp\left[\frac{8}{3}\left(\frac{t_{\rm sp}-\tau_{\rm q}}{\bar{t}}\right)^{3/2}\right]\right\} \tag{9.88}$$

Now we can use the relations

$$\left[(\dot{\phi}^1)^2 - (\dot{\phi}^2)^2\right]^2 \sim \frac{T_c}{V}(\dot{\phi}^1 - \dot{\phi}^2)^2 \tag{9.89}$$

$$\exp\left[\frac{4}{3}\left(\frac{\Delta_0(t_{\rm sp})}{\bar{t}}\right)^{3/2}\right] \sim \frac{\mu\bar{t}^2\eta^2}{T_c}, \tag{9.90}$$

to get

$$|\mathcal{D}[\phi^1,\phi^2]|^2 \sim \exp\left\{-g^2 T_c \frac{\bar{t}^{11}}{t_{sp}}\,\mu\eta^4(\dot{\phi}^1 - \dot{\phi}^2)^2\right\} \tag{9.91}$$

Unless the self-coupling is exceedingly small or the space volume too big (in which case it is not appropriate to disregard the spatial structure of the relevant classical evolutions), strong enough decoherence follows from the observation that $\tau_{\rm q}/t_{\rm sp} \gg 1$.

When these bounds are satisfied the minimum wavelength for which the modes decohere by time $t_{\rm sp}$ can be shown [RiLoMa02] to be shorter than that which characterizes domain size at that time. Although one can talk loosely, but sensibly, about a classical domain structure at time $t_{\rm sp}$ one cannot yet talk about classical defects on their boundaries, as the naive picture might suggest. Defects (in this case, walls) are described by shorter wavelength modes ($k \leq \mu$). Nonetheless, the classical domain structure is sufficient to determine their density [RiLoMa02].

The emphasis has been on the many weak environments because of the control that this gives us on establishing a robust upper bound on $t_{\rm D}$. However, LMR noted that their total contribution at one loop was qualitatively that of the short-wavelength modes of the ϕ field alone without assuming the action of the environmental fields. So it seems that rapid decoherence is a general feature.

In Chapter 5 we have also shown how for a general class of system–environment interactions (such as the $\phi^2\chi^m$ types studied), the effect of the environment is largely equivalent to the presence of a stochastic source term in the dynamics of the classical system field, with the correlation functions obeyed by the noise $\xi_{\rm m}(x)$ corresponding to the specific type of couplings. In particular, for the linear interaction with the environment (to the exclusion of self-interaction) LMR recovered the *additive* noise that has been the basis for stochastic equations in relativistic field theory that confirm the scaling behavior of Kibble's and Zurek's analysis. For times later than $t_{\rm sp}$, neither perturbation theory nor more general non-Gaussian methods are valid. Also LMR found that the role initially attributed by Kibble (and subsequently by others, e.g. [BraMag99]) to the Ginzburg regime is just not present.

9.7 Decoherence of the inflaton field

As another example of the application of the coarse-grained effective action and the influence functional formalism, we consider the decoherence of the inflaton field in the early universe. The key ingredient in this consideration is the noise associated with quantum fluctuations. We have seen how it is defined from the influence action for an interacting field in Chapter 5. Some background material on inflationary cosmology can be gleaned from the first part of Chapter 15, which the uninitiated readers may wish to consult before reading this section.

As noted earlier, an inverted harmonic oscillator model was used by Guth and Pi [GutPi85] to describe the dynamics of the inflaton field. Though useful for intuitive reasoning, it is over-simplistic in addressing the quantum-to-classical transition issue. This model has also been used by many authors to describe the appearance of classical inhomogeneities from quantum fluctuations in the inflationary era [PolSta96, LePoSt97]. Due to the linearity of the model and the Gaussian form of the wavefunction [Hab04] the quantum–classical correspondence is straightforward. In more general circumstances, the Wigner function can be negative and the simple identification with the one-particle classical distribution function no longer holds. One needs to consider decoherence of the (reduced) Wigner function by an environment [Hab90, HabLaf90], just as we have done for the reduced density matrix in similar considerations.

Turning our attention briefly to cosmology, the proposal to view the long-wavelength sector quantum field as classical, such as demanded by stochastic inflation (in fact, commonly assumed in most theories of structure formation), can only be justified by showing that some decoherence mechanism applies to the inflaton field. Interaction of a quantum system with an environment may bring about decoherence, as we have seen in model problems (such as the QBM) discussed in Chapters 3 and 5. The effectiveness of an environment to bring about quantum-to-classical transition depends on many factors, such as the type of coupling (bilinear, nonlinear), the nature of the bath (spectral density, temperature) and how the interaction determines the pointer basis. Quantitatively, decoherence is usually described by the diagonalization of the reduced density matrix, but this is only meaningful (since a symmetric matrix can always be diagonalized) by specifying or, better yet, showing the likely existence of a pointer basis, which is a physical rather than a mathematical issue. There is by now a huge literature on decoherence (see, e.g. the reviews [GKJKSZ96, Paz00, Zur03]), both in terms of conceptual discussions and model calculations. Here we will limit our discussion only to some attributes of decoherence, and in the context of quantum processes in the early universe.

What in a realistic situation could play the role of the environment field? One can consider either one interacting field partitioned into two sectors, the low-frequency sector as the system and the high-frequency sector as the environment, as in the stochastic inflation scheme for the inflaton field; or two separate self-interacting scalar fields coupled biquadratically, each assuming a full spectrum of modes. Both cases have been treated in Chapter 5 using the CTP CGEA in flat space. The environment field can also be referring to other fields present besides the inflaton field. Only the quantum fluctuations of such fields need be present to generate the noise which seeds the galaxies. Even if one assumes nothing, there is always the gravitational field itself which the inflaton field is coupled to, and the vacuum gravitational fluctuations can also seed the structures in our universe [MuFeBr92, CalHu95, CalGon97, Mat97a, Mat97b]. (Note that in such cases the coupling is of a derivative form rather than a polynomial form. Noise arising from a derivative type of coupling has been studied before in the context of minisuperspace quantum cosmology [SinHu91].)

We add a cautionary note that the simple criterion of classicality derived from the study of linear systems (e.g. free fields) fails when interactions are taken into account. Indeed, as shown in simple quantum mechanical models (e.g. the anharmonic inverted oscillator [LoMaMo00]), an initially Gaussian wavefunction becomes non-Gaussian when evolved under the Schrödinger equation. The Wigner function will develop negative parts, and its interpretation as a classical probability breaks down.

Assuming weak self-coupling constant (a nearly flat inflaton potential) Lombardo and Nacir [LomNac05] have shown that decoherence is an event shorter than the time $t_{\rm end}$, which is a typical time-scale for the duration of inflation.

9.7.1 Noise from interacting quantum fields

From the influence functional for an interacting field in a de Sitter universe given in Chapter 5 for the Minkowski spacetime, or the conformally related theory in de Sitter spacetime, we learned how to identify the noise (in both cases our treatment follows [Zha90, HuPaZh93b, LomMaz96, CaHuMa01]). Now we use it to consider decoherence and structure formation in stochastic inflation.

For illustrative purposes, in discussing the issue of decoherence, we shall derive the master equation from this influence functional only for a special case. This equation and its associated Langevin or Fokker–Planck equation will enable us later to calculate the fluctuation spectrum as a problem in classical stochastic dynamics.

Consider a real, gauge singlet, massive, $\lambda\Phi^4$ self-interacting scalar field in a de Sitter spacetime. In the inflationary regime of interest, the scale factor $a(t)$ expands exponentially in cosmic time t

$$a(t) = a_0 \exp Ht \tag{9.92}$$

We split the classical action of the inflaton field $\Phi(x)$ as

$$S[\Phi] = S_0[\Phi] + S_I[\Phi] \tag{9.93}$$

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where $S_0[\Phi]$ is that part of the classical action describing a free, massless, conformally coupled scalar field, and

$$S_I[\Phi] = \int d^n x \sqrt{-g(x)} \Big\{ -\frac{1}{2} \left[m^2 \Phi^2 + (\xi_n - \xi_c) R(x) \Phi^2 \right] - \frac{1}{4!} \lambda \Phi^4 \Big\} \tag{9.94}$$

contains the remaining (interactive) terms with contributions from nonzero mass m, self-interaction λ, and ξ_c, the coupling between the field and the spacetime curvature scalar R. Here, $\xi_c = 1/6$ for conformal coupling and $\xi_c = 0$ for minimal coupling in four dimensions, $\xi_n = (n-2)/4(n-1)$ is a constant equal to 1/6 in four dimensions, and $\sqrt{-g(x)} = a^{n-1}(t) = a^n(\eta)$.

In the stochastic inflation scheme, one makes a system–bath field splitting

$$\Phi(\mathbf{x}, t) = \phi(\mathbf{x}, t) + \psi(\mathbf{x}, t) \tag{9.95}$$

such that the system field is defined by

$$\phi(\mathbf{x}, t) = \int_{|\mathbf{k}|<\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} \, \Phi(\mathbf{k}, t) \, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{9.96}$$

and the bath field is defined by

$$\psi(\mathbf{x}, t) = \int_{|\mathbf{k}|>\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} \, \Phi(\mathbf{k}, t) \, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{9.97}$$

where Λ is the cut-off wavenumber determined by the horizon size. The system field $\phi(x)$ contains the long-wavelength modes, which undergo a slow roll-over phase transition in the inflation period, while the bath field ψ contains the short-wavelength modes, which are the quantum fluctuations.

With this splitting, we find the following effective action from expanding the influence action for $\chi = \phi a$, $\chi' = \phi' a$ to one-loop order in $\hbar$ and second order in S_I. We consider only the biquadratic coupling here, which corresponds to the limit where the system field is homogeneous.

The computation of the effective action follows the lines of Chapter 5 with conformal time here replacing cosmic time there. The dissipation is of a nonlinear nonlocal type, and there is a multiplicative (nonlinearly coupled) colored noise. The fluctuation–dissipation theorem for this field model in de Sitter space has the same form as that in Minkowski space.

9.7.2 Decoherence in two interacting fields model

The functional quantum master equation for this field-theoretical model with general nonlinear nonlocal dissipation and nonlinearly coupled colored noise has a complicated form in cosmic time t. However, in conformal time η it has the same form as in Minkowski spacetime, derived in Chapter 5, following the work of [Zha90, HuPaZh93a, HuPaZh93b, Paz94]. We will consider a simpler case here,

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where one can get an explicit form of the functional quantum master equation, i.e. by making a local truncation in the effective action. Setting

$$V(x-x') = v_0(\eta)\delta^4(x-x') \tag{9.98}$$

$$\mu(x-x') = \frac{\partial}{\partial(\eta-\eta')}\Big\{\gamma_0(\eta)\delta^4(x-x')\Big\} \tag{9.99}$$

$$\nu(x-x') = \nu_0(\eta)\delta^4(x-x') \tag{9.100}$$

and using the same procedure as outlined in Chapter 5, we can derive the functional quantum master equation in the local truncation approximation [Zha90]:

$$i\frac{\partial}{\partial\eta}\,\rho_r[\chi^1,\chi^2,\eta] = \hat{H}_\rho[\chi^1,\chi^2,\eta]\,\rho_r[\chi^1,\chi^2,\eta] \tag{9.101}$$

where

$$\begin{aligned} H_\rho[\chi^1,\chi^2,\eta] = \int d^3\mathbf{x}\Bigg\{&\hat{h}_r(\chi^1) - \hat{h}_r(\chi^2) + 3\lambda^2\gamma_0(\eta)\Big[(\chi^1(\mathbf{x}))^4 - (\chi^2(\mathbf{x}))^4\Big] \\ &+ 2\lambda^2\gamma_0(\eta)\Big[(\chi^1(\mathbf{x}))^2 - (\chi^2(\mathbf{x}))^2\Big]\Big[\chi^1(\mathbf{x})\frac{\delta}{\delta\chi^1(\mathbf{x})} - \chi^2(\mathbf{x})\frac{\delta}{\delta\chi^2(\mathbf{x})}\Big] \\ &- (i/2)\lambda^2\nu_0(\eta)\Big[(\chi^1(\mathbf{x}))^2 - (\chi^2(\mathbf{x}))^2\Big]\Bigg\} \end{aligned} \tag{9.102}$$

and

$$\begin{aligned} \hat{h}_r(\phi) = &-\frac{1}{2}\frac{\delta^2}{\delta\chi^2(\mathbf{x})} + \frac{1}{2}\big[\nabla\chi(\mathbf{x})\big]^2 + \frac{1}{2}a^2(\eta)\Big[m_r^2 + \frac{1+\xi_r}{6}R(\eta)\Big]\chi^2(\mathbf{x}) \\ &+ \frac{1}{4!}\lambda_r\chi^4(\mathbf{x}) + \delta m^2(\eta)a^2(\eta)\chi^2(\mathbf{x}) - \frac{1}{2}\lambda^2 v_0(\eta)\chi^4(\mathbf{x}) \end{aligned} \tag{9.103}$$

This functional quantum master equation and its associated Langevin equation or Fokker–Planck–Wigner equation can be used to analyze the dynamics of the system field (long-wavelength modes in the stochastic inflation scheme) for studying the decoherence and structure formation processes in the early universe [HuPaZh93b]. Instead of solving these equations in detail, we can get some qualitative information on how the system decoheres by analyzing the behavior of the diffusion term in the master equation.

Diffusive effects are generated by the last term in the effective action, the variation of which produces the following contribution on the right-hand side of the master equation for $\rho[\chi^1,\chi^2]$:

$$\dot{\rho}[\chi^1,\chi^2,\eta] \propto -\big[(\chi^1)^2 - (\chi^2)^2\big] * \nu(\eta) * \big[(\chi^1)^2 - (\chi^2)^2\big] \times \rho[\chi^1,\chi^2,\eta] \tag{9.104}$$

Here the symbol $*$ denotes the convolution product and χ represents a configuration of the scalar field in a surface of constant conformal time. The diffusion

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"coefficient" ν is therefore a nonlocal kernel that can be written in terms of its spatial Fourier transform as

$$\nu(\mathbf{x},\mathbf{y},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\nu_{\mathbf{k}}(\eta)\; e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{9.105}$$

To justify treating the long-wavelength modes classically, a minimal check is to see if the diffusive effects are stronger for long-wavelength modes than they are for short ones. To do so, note that [HuPaZh93b] the coefficient in (9.104) can be written in terms of the product of the Fourier transform (9.105) and that of the field ϕ^2:

$$\left[(\phi^1)^2-(\phi^2)^2\right] * \nu(t) * \left[(\phi^1)^2-(\phi^2)^2\right] = \int d\mathbf{k}\,[(\phi^1)^2-(\phi^2)^2]_{\mathbf{k}} D_{\mathbf{k}}[(\phi^1)^2-(\phi^2)^2]_{\mathbf{k}} \tag{9.106}$$

We want to examine the dependence on $k = |\mathbf{k}|$ of the function $D_{\mathbf{k}}$ entering in (9.106). This function can be written in terms of the physical wave vector $\mathbf{p} = \mathbf{k}/a$ as

$$D_{\mathbf{k}}(\eta) = \frac{a^3}{4\pi}\lambda^2\left[1 - \frac{H}{p}f\left(\frac{p}{H}\right) + g\left(\frac{p}{H}\right)\right] \tag{9.107}$$

where

$$f(x) = \frac{1}{2\pi}\int_0^{2x} dx[-\sin x\,\mathrm{Ci}(x) + \cos x\,\mathrm{Si}(x)] \tag{9.108}$$

$$g(x) = \frac{1}{2\pi}\int_0^{2x} dx\,[\cos x\,\mathrm{Ci}(x) + \sin x\,\mathrm{Si}(x)] \tag{9.109}$$

and $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$ are the usual integral trigonometric functions. A plot of $D_{\mathbf{k}}(\eta)$ for a fixed value of the conformal time as a function of p/H, i.e. the ratio between the horizon size and the physical wavelength can be found in [HuPaZh93b]. The function has a strong peak in the infrared region of the spectrum suggesting that diffusion effects (decoherence is one of them) are indeed more pronounced for long-wavelength modes and weaker for wavelengths shorter than the horizon size.

We learned from earlier discussions that noise of quantum origin arising from nonlinear fields is under general circumstances both multiplicative and colored (see, e.g. [HuPaZh93a]). Noise could generate fluctuations which could give rise to non-Gaussian galaxy distributions (NGD).[9]

As for the present scheme, since the value of λ is restricted to be very small ($<10^{-12}$) in the standard inflationary models (so that the magnitude of the

[9] There are, of course, simpler ways to generate NGD. A changing Hubble rate $H = \dot{a}/a$ as in a "slow-roll" transition, or an exponential potential $V(\phi)$ [LucMat85] will do. However, such mechanisms only generate NGD at very long wavelengths, much longer than the horizon size to be relevant to the observable spectrum. See, e.g. Proceedings of ICTP meeting, July 2006 [SelCre06].

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density contrast is compatible with the observed value $\delta\rho/\rho \approx 10^{-4}$ when the fluctuation mode enters the horizon), the constituency of the colored portion of the noise is accordingly small. The effect of nonlinear coupling on the generation of inhomogeneities is an active research topic at the accumulation of increasingly detailed observational data. Details of galaxy formation analysis from the stochastic equations of motion derived here with different types of colored noise and realistic physical parameters will come from solutions to these stochastic equations for galaxy formation considerations. We will have more discussions on the effect of quantum noise on structure formation in Chapter 15.

9.7.3 Partitioning one interacting field: noise from high frequency modes

In an earlier section we have discussed the appearance of classical features in a quantum phase transition. There the separation between long and short wavelengths is determined by their stability, which depends on the parameters of the potential. For our present consideration of quantum-to-classical transition in inflationary cosmology, this separation is conveniently set by the existence of the Hubble radius. Modes crossing the horizon during their evolution are usually treated as classical. The rationale for it can only come from a detailed study of decoherence, such as identifying the conditions whereby the behavior of a quantum fluctuation field can be adequately described by a classical stochastic field. We now discuss this issue.

The influence functional and the density matrix

For this case, we consider a massless quantum scalar field minimally coupled to a de Sitter spacetime. We choose the initial time η_i to be when $a(\eta_i) = 1(\eta_i = -H^{-1})$. Perform a system–environment field splitting [LomMaz96]

$$\chi = \chi_< + \chi_> \tag{9.110}$$

where the system field $\chi_<$ contains the modes with wavelength longer than the partition scale $\ell_c \equiv 2\pi/\Lambda$, while the environment field $\chi_>$ contains modes with wavelength shorter than ℓ. As we set $a(\eta_i) = 1$, a physical length $\ell_{\rm phys} = a(\eta)\ell$ coincides with the corresponding comoving length ℓ_i at the initial time. Therefore, the splitting between the system and the environment defines a system sector containing all the modes with physical wavelengths longer than the partition scale ℓ_c at the initial time η_i.

The influence functional for a similar problem has been computed in Chapter 5, Section 5.1, except that here a is a function of time. If there is a natural separation of the real and imaginary terms in this functional (as illustrated in the QBM model discussed in Chapter 3) one can then identify a noise and dissipation kernel related by a categorical fluctuation–dissipation relation. Assuming that the initial state $\hat{\rho}_>[\eta_i]$ is the Bunch–Davies vacuum state, the real and

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imaginary parts of the influence action are given by

$$\begin{aligned}\mathrm{Re}S_{IF} &= -\lambda \int d^4x_1 \left\{\chi_-^{(4)}(x_1) - 6\chi_-^{(2)}(x_1)G_F^\Lambda(x_1,x_1)\right\} \\ &+ \lambda^2 \int d^4x_1 \int d^4x_2\ \theta(\eta_1-\eta_2)\left\{32\chi_+^{(3)}(x_1)\mathrm{Im}G_F^\Lambda(x_1,x_2)\chi_-^{(3)}(x_2)\right. \\ &\left. -144\chi_+^{(2)}(x_1)\mathrm{Im}[G_F^\Lambda(x_1,x_2)]^2\chi_-^{(2)}(x_2)\right\}, \qquad (9.111)\end{aligned}$$

$$\begin{aligned}\mathrm{Im}S_{IF} &= \lambda^2 \int d^4x_1 \int d^4x_2 \left\{8\chi_-^{(3)}(x_1)\mathrm{Re}G_F^\Lambda(x_1,x_2)\chi_-^{(3)}(x_2)\right. \\ &\left. + 36\chi_-^{(2)}(x_1)\mathrm{Re}[G_F^\Lambda(x_1,x_2)]^2\chi_-^{(2)}(x_2)\right\}, \qquad (9.112)\end{aligned}$$

$\theta(x)$ is the Heaviside step function, and the integrations in time run from η_i to η. $G_{\mathrm{F}}^\Lambda(x_1,x_2) \equiv \langle T\chi_>^1(x_1)\chi_>^1(x_2)\rangle_0$ is the relevant short-wavelength closed time path correlator (it is proportional to the Feynman propagator of the environment field, where the integration over momenta is restricted by the presence of the partition momentum Λ), and we have defined

$$\chi_-^{(n)} = (\chi_<^1)^n - (\chi_<^2)^n, \quad \chi_+^{(n)} = \frac{1}{2}[(\chi_<^1)^n + (\chi_<^2)^n] \qquad (9.113)$$

with $n = 1, 2, 3$.

Master equation and diffusion coefficients

As we learned by the QBM model (Chapter 3) and the field theory example (Chapter 5) once one obtains the evolutionary operator $\mathcal{J}_r$ for the reduced density matrix one can derive the master equation for the reduced density matrix. These expressions for a quantum scalar field in the de Sitter universe were obtained by Zhang [Zha90].

To get a qualitative idea of decoherence, as noted earlier, one could just focus on the behavior of the diffusion "coefficients" (actually nonlocal functions) related to the noise kernel obtained from the imaginary part of the influence action. Making the further simplification that the system field contains only one mode k_0, Lombardo and Nacir showed that the terms in the master equation relevant to decoherence are [LomNac05]

$$\begin{aligned} i\partial_\eta\rho_r[\phi_{<f}^1|\phi_{<f}^2;\eta] &= \langle\phi_{<f}^1|[\hat{H}_{\mathrm{ren}},\hat{\rho}_r]|\phi_{<f}^2\rangle \\ &- i\left[\Gamma_3 D_3(\mathbf{k_0},\eta,\Lambda) + \Gamma_2 D_2(\mathbf{k_0},\eta,\Lambda)\right]\rho_r[\phi_{<f}^1|\phi_{<f}^2;\eta] + \dots\end{aligned}$$

where $\Gamma_2 = \frac{\lambda^2 V}{4}[(\phi_{<f}^1)^2 - (\phi_{<f}^2)^2]^2$ and $\Gamma_3 \equiv \frac{\lambda^2 V}{H^2}[(\phi_{<f}^1)^3 - (\phi_{<f}^2)^3]^2$. (The subscripts 2, 3 refer to the order of the system field $\phi_{<f} = \chi_{<f}/a(\eta_f)$.) The ellipsis denotes additional terms coming from the time derivative that do not contribute to the diffusive effects.

This equation contains time-dependent diffusion coefficients D_j. Up to one loop, only D_2 and D_3 survive. Coefficient D_2 is related to the interaction term

$\phi_<^2 \phi_>^2$ while D_3 is related to $\phi_<^3 \phi_>$. These coefficients can be (formally) written as

$$D_2(\mathbf{k_0}, \eta, \Lambda) = 36 \int_{\eta_i}^{\eta} d\eta'\, a^2(\eta) a^2(\eta') F_{\rm cl}^2(\eta, \eta', k_0) \times \left\{ \mathrm{Re}[G_F^{>}(\eta, \eta', 2\mathbf{k_0})]^2 + 2\, \mathrm{Re}[G_F^{>}(\eta, \eta', 0)]^2 \right\}, \tag{9.114}$$

and

$$D_3(\mathbf{k_0}, \eta, \Lambda) = -\frac{H^2}{2} \int_{\eta_i}^{\eta} d\eta'\, a^3(\eta) a^3(\eta') F_{\rm cl}^3(\eta, \eta', k_0) \times \mathrm{Re} G_F^{>}(\eta, \eta', 3\mathbf{k_0})\, \theta(3k_0 - \Lambda) \tag{9.115}$$

with the function $F_{\rm cl}$ defined by

$$F_{\rm cl}(\eta, \eta_i, k_0) = \frac{\sin[k_0(\eta - \eta_i)]}{k_0 \eta} + \frac{\eta_i \cos[k_0(\eta - \eta_i)]}{\eta} \tag{9.116}$$

Note that only the effect of normal diffusion terms are included in our considerations here. It is known from QBM studies [Zha90, HuPaZh92, HuPaZh93a, PaHaZu93, Paz94, HalYu96] that anomalous diffusion terms can also be relevant at zero temperature.

Using these expressions for the two diffusion functions and placing the physical parameters relevant to successful inflationary models, Lombardo and Nacir [LomNac05] calculated the decoherence times t_{d_2} and t_{d_3} associated with D_2 and D_3. They conclude that if one sets $\Lambda \leq H$, the decoherence time-scale for the system field is shorter than the minimal duration of inflation for all the wavevectors in the system sector. This is by far the most detailed and thorough study of the decoherence of the inflaton.