ON LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER HAVING GEODESIC SOLUTIONS

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Introduction. The coefficients of the second derivatives in an elliptic or hyperbolic differential equation of the second order determine a Riemannian metric on the space of the independent variables. A Riemannian space has been called harmonic if the Laplace equation corresponding to it has a solution which is a function only of geodesic distance in that space. Harmonic spaces have been studied in some detail **(1; 3)**. In this note are examined the circumstances under which a non-parabolic second order linear equation may have a "geodesic" solution of the type described. It will be shown that the equation must be self-adjoint, that the Riemann space corresponding to the equation must be harmonic and that the coefficient of the dependent variable must be a constant. Conversely, if these conditions are satisfied, the equation has two geodesic solutions, one of which is an elementary solution. In the case of elliptic equations, the second solution is connected with a certain mean value property which is valid in harmonic spaces.

1. Riemannian metric. Any homogeneous linear partial differential equation of the second order, in N independent variables, and which is not parabolic, can be written in the form

$$L[u] \equiv \Delta u + \mathbf{b} \cdot \nabla u + cu = 0. \tag{1.1}$$

Here

$$(\nabla u)_i = \frac{\partial u}{\partial x^i} \tag{1.2}$$

is the gradient vector of the dependent variable u.
The Laplacian operator

$$\Delta u = a^{-\frac{1}{2}} \frac{\partial}{\partial x^i}\left(a^{\frac{1}{2}} a^{ik} \frac{\partial u}{\partial x^k}\right) \tag{1.3}$$

is that based on the metric form

$$ds^2 = a_{ik}\, dx^i\, dx^k, \tag{1.4}$$

the a_{ik} being defined in terms of the coefficients a^{ik} in (1.1) by

$$a_{ik}\, a^{kj} = \delta_i^j. \tag{1.5}$$

The absolute value of the determinant of the a_{ik} will be denoted by the letter a. In (1.1) also, $\mathbf{b}$ is a contravariant vector and c a scalar invariant, both given

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in advance. The differential equation is self-adjoint, in the invariantive sense, if and only if the vector **b** is zero identically.

The equation (1.1) is elliptic if and only if the metric (1.4) is (positive) definite. Similarly (1.1) is normal hyperbolic if and only if the signature of (1.4) is $\neq (N - 2)$. While we have in mind mainly these two cases, we need not for the moment restrict the signature. It will be supposed that all coefficients appearing in the differential operator L are four times continuously differentiable in the given coordinate system.

If both **b** and c are zero, the equation is Laplace's equation $\Delta u = 0$. It is well known that in a flat space (constant coefficients a^{ik}), Laplace's equation has an elementary solution of the form $\log r$ $(N = 2)$, r^{2-N} $(N > 2)$, where r is the distance function in the space. Thus a flat space is harmonic. A Riemannian space is said to be completely harmonic if the base point from which the geodesic distance $s = s(P,Q)$ is measured can be any point of the space. We shall assume that the base point is arbitrary, in our theorem. Completely harmonic spaces form a rather special class; for instance, they are all Einstein spaces **(1)**.

We remark that Thomas and Titt **(5)** have investigated the conditions under which an equation (1.1) has a solution which is a power of the geodesic distance. In particular they have shown that Laplace's equation in a space of definite signature has such a solution only if the space is flat and $N > 2$.

2. A coordinate system. For convenience we shall use the squared geodesic distance

$$\Gamma = \Gamma(P, Q) = s^2(P, Q), \tag{2.1}$$

which is always real. It is well known **(2**, p. 433**)** that

$$(\nabla\Gamma)^2 = a^{ik}\frac{\partial\Gamma}{\partial x^i}\frac{\partial\Gamma}{\partial x^k} = 4\Gamma \tag{2.2}$$

and that, for any function $F(\Gamma)$ of Γ alone,

$$\Delta F(\Gamma) = F'(\Gamma)\Delta\Gamma + 4F''(\Gamma)\Gamma. \tag{2.3}$$

(Here derivatives of $F(\Gamma)$ with respect to Γ are indicated by dashes.) It follows from (2.3) that a space V_N is harmonic if and only if $\Delta\Gamma$ is expressible as a function of Γ alone **(1)**; thus, V_N is harmonic if and only if

$$\Delta\Gamma = f(\Gamma). \tag{2.4}$$

Let $x^k = \sigma p_0^k$ be Riemannian coordinates with pole at Q, an arbitrary but fixed point **(5)**. Here σ is a normalized parameter on the geodesics issuing from Q; σ takes the values zero at Q and unity on a suitable surface S enclosing Q. It is convenient to choose this surface S in such a way that any geodesic line through Q meets S in two points which are at the same geodesic distance from Q. This distance is not necessarily the same for all rays through Q. If the metric is positive definite, however, we may take σ proportional to s, so that S is a geodesic sphere with centre Q. The components a^{ik} can now be assumed twice differentiable in this coordinate system.

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Now we have

(2.5) $$\Gamma(P, Q) = a_{ik}(Q)\, x^i x^k = \sigma^2 a_{ik}(Q)\, p_0^i p_0^k.$$

The gradient vector $\Delta\Gamma$ has components

(2.6) $$(\nabla\Gamma)_i = 2\sigma a_{ik}(P)\, p_0^k = 2\sigma a_{ik}(Q)\, p_0^k,$$

in this Riemannian coordinate system (**4**, p. 87). Let a be the modulus of the determinant $|a_{ik}|$ in Riemannian coordinates, then in view of (1.3)

(2.7) $$\Delta\Gamma = \Delta_p\Gamma(P, Q) = 2N + \sigma\frac{\partial \log a}{\partial\sigma}.$$

The second term on the right of (2.7) is $O(\sigma^2)$ as $\sigma \to 0$.

3. Geodesic solutions. If the coefficient c is zero, the equation (1.1) has the trivial solution u = const., which solution we shall exclude in the hypotheses of our theorem. In order to allow such solutions as are singular for $\Gamma = 0$, we shall suppose only that the solution function is twice continuously differentiable for $\Gamma \neq 0$ (and is defined for Γ sufficiently small). We write $\Gamma = \Gamma(P,Q)$, Q being arbitrary.

THEOREM I. *There exists a solution* $u = F(\Gamma)$ *of* (1.1), *of class* C^2 *for* $\Gamma \neq 0$ *and which is not a constant in any neighbourhood of* $\Gamma = 0$, *if and only if*

(a) *the coefficient c is a constant;*

(b) *the vector* **b** *vanishes identically;*

(c) *the Riemann space* V_N *is completely harmonic.*

Conversely, if (a), (b), (c) *hold, there exist two independent solutions of the form* $u = F(\Gamma)$, *one of which is an elementary solution, the other being regular at the origin* $\Gamma = 0$.

Suppose, to prove the first part of the result, that a solution $u = F(\Gamma)$ exists. Then from (1.1) and (2.3) we have

(3.1) $$LF(\Gamma) = 4F''(\Gamma)\Gamma + F'(\Gamma)\{\Delta\Gamma + \mathbf{b}\cdot\nabla\Gamma\} + cF(\Gamma) = 0.$$

Now from (2.6)

(3.2) $$\mathbf{b}\cdot\nabla\Gamma = 2a_{ik}(Q)\, x^i b^k = 2\sigma a_{ik} p_0^i\, b^k = 2\sigma b_\sigma,$$

say, where b_σ is thus defined. We may assume that $F(\Gamma)$ is not identically zero for Γ small, or else the solution u would be zero, and so we may divide (3.1) by $F(\Gamma)$ and let Γ tend to zero. In view of (2.7) and (3.2), we see that

(3.3) $$c = c(Q) = -\lim_{\Gamma\to 0}\left\{\frac{4F''(\Gamma)\Gamma + 2NF'(\Gamma)}{F(\Gamma)}\right\}.$$

The limit on the right exists, because, by hypothesis, $F(\Gamma)$ is a solution function and is of class C^2 for $\Gamma \neq 0$; and also $c(P)$ is continuous and tends to the limit $c(Q)$. However, the right-hand side of (3.3) depends only on the function $F(\Gamma)$ and not at all on Q. Thus $c(Q)$ is independent of Q and so is a constant. This shows that condition (a) is necessary.

By hypothesis, $F'(\Gamma)$ is different from zero for a sequence of values of Γ tending to zero. For such values of Γ, we may divide (3.1) by $F'(\Gamma)$. Noting

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that c is a constant, we see that the quantity

$$\Delta\Gamma + \mathbf{b}\cdot\nabla\Gamma = 2N + 2\sigma b_\sigma + O(\sigma^2) \tag{3.4}$$

then depends only on Γ.

On any non-null geodesic through Q, let P_1 and P_2 be points lying in the order P_1QP_2 on the geodesic, and such that $\sigma(P_1) = \sigma(P_2)$. It follows that

$$x_1^i = -x_2^i, \quad p_{01}^i = -p_{02}^i.$$

Also, from (2.5) we see that $\Gamma(P_1,Q) = \Gamma(P_2Q)$. Now let (3.4) be written down for $P = P_1$, and $P = P_2$; and let the two resulting equations be subtracted. Thus

$$2\sigma b_\sigma(P_1) = 2\sigma b_\sigma(P_2) + O(\sigma^2), \tag{3.5}$$

for any value of Γ in the above-mentioned sequence. Dividing (3.5) by 2σ we have

$$b_\sigma(P_1) = b_\sigma(P_2) + O(\sigma),$$

and the terms appearing in this relation are continuous near $\sigma = 0$. Thus from (3.2),

$$\begin{aligned} a_{ik}(Q)p_{01}^i b^k(P_1) &= a_{ik}(Q)p_{02}^i b^k(P_2) + O(\sigma) \\ &= -a_{ik}(Q)p_{01}^i b^k(P_2) + O(\sigma). \end{aligned} \tag{3.6}$$

Now let $\sigma \to 0$, so that P_1, $P_2 \to Q$. Then the functions $b^k(P_1)$ and $b^k(P_2)$, being continuous, tend to their limits $b^k(Q)$. Thus

$$a_{ik}(Q)p_{01}^i b^k(Q) = -a_{ik}(Q)p_{01}^i b^k(Q) = 0 \tag{3.7}$$

holds for any non-null direction p_{01}^i at Q. Since we can find N independent non-null directions, and since $|a_{ik}(Q)|$ is not zero, we conclude that $b^k(Q) = 0$ $(k = 1, \ldots, N)$. Since Q is an arbitrary point, the vector $\mathbf{b}$ must vanish identically. Thus condition (b) is necessary.

From (3.1), in which the term $\mathbf{b}\cdot\nabla\Gamma$ is now absent, we see that $\Delta\Gamma$ is defined as a function of Γ provided that $F'(\Gamma) \neq 0$. If $F'(\Gamma)$ does not vanish throughout any interval, $\Delta\Gamma$ is defined by continuity. On the other hand, if $F'(\Gamma)$ is zero throughout any interval it follows that the product $cF(\Gamma)$ must be zero. We show that this possibility is excluded by our hypotheses. If c is zero, then

$$4\Gamma F''(\Gamma) + \Delta\Gamma\, F'(\Gamma) = 0,$$

and $F'(\Gamma)$ has a zero. Thus $F'(\Gamma)$ must be identically zero since it is also a solution of this homogeneous differential relation. If c is not zero then

$$4\Gamma\, F''(\Gamma) + \Delta\Gamma\, F'(\Gamma) + cF(\Gamma) = 0,$$

and $F(\Gamma)$, $F'(\Gamma)$ are simultaneously zero. Again, it follows that $F(\Gamma)$ must vanish identically. According to our hypothesis, therefore, $F'(\Gamma)$ is not zero throughout any interval. Hence $\Delta\Gamma$ is defined as a function of Γ for all (sufficiently small) values of Γ, and for any base point Q. That is, V_N is completely harmonic, which is condition (c). This establishes the necessity of the three conditions.

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Turning to the converse statement, consider the equation

$$\Delta u + cu = 0, \tag{3.8}$$

in a completely harmonic space V_N, where c is a constant. From (2.3) and (2.4) we see that $u = F(\Gamma)$ is a solution if and only if

$$4\Gamma F''(\Gamma) + f(\Gamma) F'(\Gamma) + cF(\Gamma) = 0. \tag{3.9}$$

Setting $F = y$, $\Gamma = x$, we see that (3.9) is the ordinary differential equation

$$4xy'' + f(x) y' + cy = 0. \tag{3.10}$$

We shall refer to (3.10) as the fundamental equation. If the conditions of the theorem are met, the fundamental equation is defined, and any solution of it yields a solution $u = F(\Gamma)$ of the partial differential equation. The fundamental equation has two linearly independent solutions.

4. Nature of the solutions. The origin is a regular singular point of the fundamental equation, since from (2.4) and (2.7) we see that

$$f(x) = 2N + O(x^2), \qquad x \to 0. \tag{4.1}$$

Supposing that the solutions can be developed in the form

$$y = x^p y_1(x) \tag{4.2}$$

where $y_1(0) \neq 0$ and $y_1(x)$ can be expressed as a Taylor series with remainder about the origin, we find the indicial equation for p to be

$$4p(p-1) + 2Np = 0. \tag{4.3}$$

The roots are $p = 0$, $p = -\frac{1}{2}N + 1$. Corresponding to $p = 0$ is a solution finite and continuous at the origin. The root $p = -\frac{1}{2}N + 1$ leads to a solution which is singular at the origin of the order indicated. If the function $f(x)$ is analytic, and N is odd, the singular solution has the form of a power series, multiplied by $x^{-\frac{1}{2}N+1}$. If N is even, $N \geqslant 4$, the roots of the indicial equation differ by an integer, and the solution which is singular at the origin will in general contain a logarithmic term. If $N = 2$, the roots are equal, and the singularity is that of $\log x$. In all these cases the solution which is singular at the origin leads to an elementary solution of the partial differential equation. This establishes the converse part of Theorem 1.

If $f(x) = 2N$, the fundamental equation is a form of Bessel's equation of order $\pm\frac{1}{2}(N-2)$ (**2**, p.227). This corresponds to Riemann spaces of the type known as simply harmonic. A simply harmonic space of elliptic or normal hyperbolic type is necessarily flat (**3**; **5**). It follows that the only elliptic or normal hyperbolic equations with elementary solutions of the Bessel function type given in (**2**) are the classical equations $\Delta u + cu = 0$ with constant coefficients, constant, that is, when Cartesian coordinates are used. In particular, if the power Γ^p ($p = -\frac{1}{2}N + 1$) is to be a solution, we must have $c = 0$; i.e. the equation is $\Delta u = 0$ in a flat space (**5**).

Returning to the general case, we see that if $c = 0$ the fundamental equation can be integrated explicitly. The solutions are $y =$ const. and

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(4.4) $$y = C\int_a^x t^{-\frac{1}{2}N} \exp\left[-\frac{1}{4}\int_b^t (f(\tau) - 2N)\frac{d\tau}{\tau}\right] dt,$$

where C, a, b are constants. The solution finite at the origin is in this case a constant, while the other is the Ruse elementary solution (**1**, p. 118).

5. A mean value theorem. Consider the (elliptic) equation

(5.1) $$\Delta u + cu = 0$$

in a completely harmonic space of positive definite metric, and where c is a constant. Let Q be an arbitrary point, let x^i be Riemannian coordinates with pole at Q such that

(5.2) $$a_{ii} = 1, \quad a_{ij} = 0 \qquad (i \neq j).$$

Then

(5.3) $$\Gamma = s^2(P, Q) = \sum_{i=1}^{N} (x^i)^2.$$

Let θ^α ($\alpha = 1 \ldots N - 1$) be angular coordinates, forming, together with s, a system of geodesic polar coordinates with origin at Q.

The volume element in Riemannian coordinates is

(5.4) $$dV = a^{\frac{1}{2}}\, dx^1 \ldots dx^N,$$

where $a = |a_{ij}|$ has its previous meaning. Now

(5.5) $$dV = ds\, dS,$$

where dS is the surface element on the geodesic sphere of radius s about Q. Let $d\Omega$ denote the element of solid angle in terms of the angle variables θ^α (which may be defined from the x^i exactly as they are in Euclidean space). Thus

(5.6) $$dx^1 \ldots dx^N = s^{N-1}\, ds\, d\Omega,$$

so that

(5.7) $$dS = s^{N-1}\, a^{\frac{1}{2}}\, d\Omega.$$

We shall use ω_N to denote the total solid angle at a point. In view of (2.4) and (2.7), we have

(5.8) $$a = a(P) = a(Q) \exp\left[\int_0^s (f(t^2) - 2N)\frac{dt}{t}\right],$$

so that a is a function of s alone, and not of the angle variables.

Let $v(Q,s) = v(s)$ denote the mean value

(5.9) $$v(s) = \frac{1}{\omega_N}\int_\Omega u(s, \theta^\alpha)\, d\Omega,$$

of a function $u(P)$ over the geodesic sphere of radius s about Q.

THEOREM II. *If u (P) is a solution of* (5.1) *in an elliptic completely harmonic Riemann space, then $v(s)$, defined by* (5.9), *is that solution of the fundamental equation which is equal to one at the origin.*

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To prove this we have (**2**, p. 411).

$$\frac{dv}{ds} = \frac{1}{\omega_N}\int_\Omega \frac{\partial u}{\partial s}\,d\Omega = \frac{1}{\omega_N}\int_\Omega \frac{\partial u}{\partial n}\,d\Omega, \tag{5.10}$$

since

$$\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n} = \nabla u \cdot \nabla s = \frac{\partial u}{\partial s}.$$

From (5.7),

$$\frac{dv}{ds} = \frac{1}{\omega_N a^{\frac{1}{2}} s^{N-1}}\int_S \frac{\partial u}{\partial n}\,dS, \tag{5.11}$$

where S is the surface of the geodesic sphere. By Green's formula and (5.1):

$$\frac{dv}{ds} = \frac{1}{\omega_N\, a^{\frac{1}{2}}\, s^{N-1}}\int_K \Delta u\,dV = \frac{-c}{\omega_N\, a^{\frac{1}{2}}\, s^{N-1}}\int_K u\,dV, \tag{5.12}$$

where K denotes the interior of the sphere. We note that if $c = 0$ the constancy of $v(s)$ follows. Otherwise, we have from (5.5) and (5.12)

$$\frac{d^2v}{ds^2} = c\,\frac{N-1+s\partial(\log a^{\frac{1}{2}})/\partial s}{\omega_N a^{\frac{1}{2}} s^{N-1}}\int_K u\,dV - \frac{c}{\omega_N a^{\frac{1}{2}} s^{N-1}}\int_S u\,dS. \tag{5.13}$$

In view of (5.9) and (5.12), (5.13) becomes

$$\frac{d^2v}{ds^2} + \left(\frac{N-1}{s} + s\,\frac{\partial \log a^{\frac{1}{2}}}{\partial s}\right)\frac{dv}{ds} + cv = 0. \tag{5.14}$$

From (2.4) and (2.7) with $\sigma = s$ in this elliptic case, it follows that if we set $x = \Gamma = s^2$, we find

$$4xv'' + f(x)\,v' + cv = 0. \tag{5.15}$$

which is just the fundamental equation. Clearly $v(0)$ has the value unity. This completes the proof of Theorem II.

This mean value property is well known in Euclidean space; we see that it holds whenever the partial differential equation has a geodesic solution. In particular, the mean value over a geodesic sphere of a harmonic function in a completely harmonic space is equal to its value at the centre (**5**, Theorem 1).

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