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LIMIT THEOREMS FOR SAMPLES FROM A FINITE POPULATION

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Abstract

Two theorems on limit distributions for sums of values sampled from a finite population without replacement are presented. The emphasis is on non-normal limit distributions.

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1. Main results

Let $\{a_{nk}\}$, $n = 1, 2, \ldots,$ $k = 1, \ldots, n$, be a triangular array of real numbers; we shall call them scores. Further, let m_n be a sequence of natural numbers such that $1 \leqslant m_n \leqslant n$. From each row of the array we select at random and without replacement m_n elements. The result is a triangular array $\{X_{nj}\}$ of random vector assuming any m combination of elements from $(a_{n1}, \ldots, a_{nn})$ with equal probabilities $\binom{n}{m_n}^{-1}$. We shall be interested in limit probability distributions of $X_n = \sum_{j=1}^{m_n} X_{nj}$. X_n is the statistic of a two-sample rank test with scores a_{nk}. Several methods were used within the theory of rank tests to study the asymptotic normality of X_n. One such method was devised by Erdös and Rényi in [1] (reproduced also in [3] Chap. VIII, §5). In the present paper the Erdös-Rényi method is applied to a more general situation and a large family of non-normal limit distributions is obtained.

Before we present the two main results, we must introduce three auxiliary measures C_n, M_n and D_n on R. They are defined by

$$C_n(I) = \sum_{a_{nk} \in I} 1, \qquad M_n(I) = \sum_{a_{nk} \in I} a_{nk} \quad \text{and} \quad D_n(I) = \sum_{a_{nk} \in I} a_{nk}^2.$$

Weak convergence of measures will be understood in the following sense: If μ_n are finite non-negative measures on Borel sets of the real line R, we shall say that μ_n converges weakly on a Borel set I to a measure μ if $\mu(I) < \infty$ and if $\mu_n(J) \underset{n}{\rightarrow} \mu(J)$ for any Borel $J \subset I$ which is a continuity set of μ. A complement $R - I$ of I will be denoted by $\bar{I}$.

We shall write $p_n = m_n/n$, $q_n = 1 - p_n$ and we shall assume without repeating it explicitly that $m_n \to \infty$ and that $\lim_n p_n = p$ exists; $q = 1 - p$.

THEOREM 1. *Let us assume that*

(1.1) $\qquad D_n$ *converges weakly on* R *to a measure* D *and that*

(1.2) $$\lim_n n^{-1/2} M_n(R) = \mu \text{ exists.}$$

Put $c_n = p_n \Sigma_k a_{nk} = p_n M_n(R)$ *and denote by* C *the weak limit of* C_n *on each* $\overline{[-\varepsilon, \varepsilon]}$. *Then* $X_n - c_n$ *converges in distribution and the limit characteristic function is*

(1.3) $$\Phi^{(X)}(t) = e^{(-1/2)pq(D(\{0\}) - \mu^2)t^2} \prod_x \left(qe^{-itpx} + pe^{itqx}\right)^{C(\{x\})}.$$

REMARK 1. The existence of the weak limit C on each $\overline{[-\varepsilon, \varepsilon]}$ follows from (1.1). C is well defined on $\{\bar{0}\}$ and finite on each $\overline{[-\varepsilon, \varepsilon]}$. The measure D is finite on the whole R. The product Π_x extends over all atoms of C. It is convergent uniformly with respect to t in each finite interval, so that it defines a proper characteristic function, and it converges unconditionally, that is, it is independent of the order of multiplication.

REMARK 2. The limit distribution described by (1.3) is non-trivial only if $0 < p < 1$, so that Theorem 1 is relevant only to this case. Two natural questions arise: Firstly, do there exist conditions more general than those of Theorem 1 under which

(a) $X_n - c_n$ converges in distribution for a suitable choice of c_n?

Secondly, if so,

(b) What is the most general form of the limit distribution?

Let us first assume that

(1.4) $$\sup_n D_n(R) < \infty$$

holds. Simple examples of scores satisfying (1.4) but not satisfying (1.1) may be constructed such that (a) is true for some $0 < p < 1$. However it is also easy to show that if (a) is true under (1.4), then the corresponding limit characteristic function has the form (1.3) (with $D\{0\} - \mu^2$ replaced by a suitable non-negative constant) even if (1.1) or (1.2) do not eventually hold.

If (1.4) does not hold and $0 < p < 1$, then Remark 4 of Section 2 indicates that our method fails to give an answer to (a) or (b).

On the other hand, if $p = 0$ or $p = 1$, then our method does not require (1.4) to hold and non-trivial limit distributions can be obtained. The next Theorem 2 is formulated for $p = 0$, however it covers the case $p = 1$ too if complementary samples are used. The function h in Theorem 2 is defined by

$$h(x) = x \text{ if } |x| \leqslant 1, \qquad h(x) = \operatorname{sgn} x \text{ if } |x| \geqslant 1.$$

THEOREM 2. *Let us assume* $p_n \to 0$,

(1.5) $\quad p_n D_n$ *converges weakly on each* $\left[\,\overline{-\varepsilon, \varepsilon}\,\right]$ *to a measure* δ,

(1.6) $\quad p_n C_n$ *converges weakly on each* $\left[\,\overline{-\varepsilon, \varepsilon}\,\right]$ *to a measure* γ

and

$$\lim_n \left(\frac{1}{n} p_n\right)^{1/2} M_n([-1, 1]) = \mu \text{ exists.} \tag{1.7}$$

Put $c_n = p_n \Sigma_k h(a_{nk}) = p_n(M_n([-1, 1]) + C_n([\overline{-1, 1}]))$. *Then,* $X_n - c_n$ *converges in distribution and the limit characteristic function* $\Phi^{(X)}$ *has*

$$\log \Phi^{(X)}(t) = -\frac{1}{2}(\delta(\{0\}) - \mu^2)t^2 + \int_{-\infty}^{\infty} (e^{itx} - 1 - ith(x))\, d\gamma(x). \tag{1.8}$$

REMARK 3. It follows from (1.5) and (1.6) that, for any $\varepsilon > 0$, $\gamma([\overline{-\varepsilon, \varepsilon}]) < \infty$ and $\delta([-\varepsilon, \varepsilon]) = \int_{[-\varepsilon, \varepsilon]} x^2\, d\gamma(x) < \infty$. Hence, $\Phi^{(X)}$ is the characteristic function of an infinitely divisible distribution.

Theorem 1 will be proved in Section 2. The proof of Theorem 2 will be only outlined briefly. Section 3 contains comparison with independent sampling.

We shall conclude this section with four examples. They all concern Theorem 1. In all of them we assume $p_n \to 1/2$.

EXAMPLE 1. Let the scores in the nth row be: $\pm n$ each once, ± 1 each $(n-2)/2$ times if n is even; if n is odd, add one 0. With these scores the assumption of Theorem 1 are not satisfied, however if we rescale the scores by dividing the nth row by n, we obtain easily that $(1/n)X_n$ has the limit characteristic function $(1/4)e^{-it} + (1/2) + (1/4)e^{it}$.

EXAMPLE 2. Replace in Example 1 the scores $\pm n$ by $\pm\sqrt{n}$ and leave the rest unchanged. Then $(1/\sqrt{n})X_n$ has the limit characteristic function $e^{(-1/2)t^2}(1/4e^{-it} + (1/2) + (1/4)e^{it})$, that is, the limit distribution is a mixture of three normal distributions.

EXAMPLE 3. Let the scores in the nth row be $2, 2^2, \ldots, 2^n$. After having divided the scores by $1/2^n$ we obtain from Theorem 1 that $(1/2^n)X_n - 1$ has the limit characteristic function $\prod_{j=0}^{\infty}((1/2)e^{-i2^{-(j+1)}t} + (1/2)e^{i2^{-(j+1)}t})$. This is well known to be the characteristic function of the uniform distribution on $(-1, 1)$ ([2], p. 67(v)). Hence $1/2^{n+1}X_n$ has in the limit the uniform distribution on $(0, 1)$.

EXAMPLE 4. Let the scores in the nth row be $3, 3^2, \ldots, 3^n$. Proceeding as in Example 3 and using [2], page 67(iv) we find that the limit distribution for $2/3^{n+1}X_n$ is the well known singular distribution concentrated on the Cantor discontinuum in $(0, 1)$.

2. Proofs

Our method is based on a relation between the triangular array $\{X_{nk}\}$ and an auxiliary triangular array of two-dimensional random vectors $\{W_{nk}\} = \{(U_{nk}, V_{nk})\}$, $n = 1, 2, \ldots,$ $k = 1, 2, \ldots, n$, such that, for each n, $W_{n1}, \ldots, W_{nn}$ are independent and

$$P((U_{nk}, V_{nk}) = (0, 0)) = q_n, \qquad P((U_{nk}, V_{nk}) = (1, a_{nk})) = p_n.$$

Denote by $\Phi_{nk}^{(W)}$ the characteristic function of W_{nk} and by $\Phi_n^{(W)}$ the characteristic function of $W_n = (U_n, V_n)$ where $U_n = \sum_k U_{nk}$ and $V_n = \sum_k V_{nk}$. Clearly

$$\Phi_{nk}^{(W)}(s, t) = q_n + p_n e^{i(s + ta_{nk})} \quad \text{and} \quad \Phi_n^{(W)}(s, t) = \prod_k \Phi_{nk}^{(W)}(s, t). \tag{2.1}$$

Finally, if we denote by $\Phi_n^{(X)}$ the characteristic function of X_n, we have

$$\Phi_n^{(X)}(t) = \frac{1}{2\pi B_n} \int_{-\pi}^{\pi} e^{-ism_n}\Phi_n^{(W)}(s, t)\, ds. \tag{2.2}$$

where $B_n = \binom{n}{m_n}p_n^{m_n}q_n^{n-m_n}$. The formula (2.2) can be derived by a simple combinatorial argument and it is the starting point for the Erdös-Rényi method mentioned in Section 1, although its interpretation in terms of W_{nk} is not mentioned explicitly in [1] or [3].

Put $d_n = \sqrt{np_nq_n}$. Substituting s/d_n for s in (2.2) we obtain

$$\Phi_n^{(X)}(t) = \frac{A_n}{\sqrt{2\pi}} \int_{-d_n\pi}^{d_n\pi} \mathrm{ex}\left(-\frac{ism_n}{d_n}\right)\Phi_n^{(W)}\left(\frac{s}{d_n}, t\right) ds \tag{2.3}$$

where $A_n \to 1$ and $\mathrm{ex}(\alpha) = e^\alpha$; later we shall also use the notation $\mathrm{ex}_1(\alpha) = e^\alpha - 1$, $\mathrm{ex}_2(\alpha) = e^\alpha - 1 - \alpha$ and $\mathrm{ex}_3(\alpha) = e^\alpha - 1 - \alpha - \alpha^2/2$.

The characteristic function of V_{nk} will be denoted by $\Phi_{nk}^{(V)}(t) = \Phi_{nk}^{(W)}(0, t)$ and the characteristic function of $V_n = \sum_k V_{nk}$ will be denoted by $\Phi_n^{(V)} = \prod_k \Phi_{nk}^{(V)}$.

Further, for any n, k we shall write

$$\varphi_{nk}(s, t) = q_n \operatorname{ex}\left(-ip_n\left(\frac{s}{d_n} + ta_{nk}\right)\right) + p_n \operatorname{ex}\left(iq_n\left(\frac{s}{d_n} + ta_{nk}\right)\right)$$

and $\psi_{nk}(t) = \varphi_{nk}(0, t)$.

If φ is a characteristic function, $\log \varphi(t)$ will denote its natural continuous logarithm defined uniquely in a sufficiently small neighborhood of 0 by $\log \varphi(0) = 0$.

In the proofs, G_l will denote a function of several variables. For simplicity reasons, these variables or parameters will not be written explicitly; L_l will denote a constant not depending, unless said otherwise, on the variables or parameters occurring in the relation, as long as these variables and parameters are kept within the indicated limits.

LEMMA 1. *Let us assume that* (1.2) *and* (1.4) *hold. Further, let there exist constants* c_n *such that* $V_n - c_n$ *converges in distribution. Denote the limit characteristic function by* $\Phi^{(V)}$. *Then* $X_n - c_n$ *converges in distribution and the limit characteristic function is*

$$\Phi^{(X)}(t) = \operatorname{ex}\left(\frac{1}{2}pq\mu^2t^2\right)\Phi^{(V)}(t).$$

REMARK 4. Under the additional assumption $0 < p < 1$, the condition (1.4) is redundant. It is possible to prove that the convergence of $V_n - c_n$ implies (1.4) under $0 < p < 1$, however we shall not prove this statement as (1.4) follows from (1.1) anyway.

PROOF. The proof will consist of a number of steps. In the whole proof, $t \neq 0$ is fixed.

a) For any n, k, s

(2.4) $$|\psi_{nk}(t) - 1| \leq \frac{1}{2}p_nq_nt^2a_{nk}^2,$$

(2.5) $$\varphi_{nk}(s, t) = \psi_{nk}(t) \to G_1$$

where $|G_1| \leq (1/d_n)p_nq_n|s|$,

(2.6) $$\varphi_{nk}(s, t) = \psi_{nk}(t) - \frac{1}{2n}s^2(1 + G_2) + G_3$$

where $|G_2| \leq (1/3d_n)|s|$ and $|G_3| \leq (1/d_n)p_nq_n|sta_{nk}|$,

(2.7) $$\varphi_{nk}(s, t) = \psi_{nk}(t) - \frac{1}{2n}s^2(1 + G_4) - \frac{1}{d_n}p_nq_nsta_{nk} + G_5$$

where $|G_4| \leq (1/3d_n)|s| + |ta_{nk}|$, $|G_5| \leq (1/2d_n)p_nq_n|s|t^2a_{nk}^2$.

PROOF OF a): Put $\sigma_1 = -ip_n s/d_n$, $\sigma_2 = iq_n s/d_n$, $\tau_1 = -ip_n ta_{nk}$, $\tau_2 = iq_n ta_{nk}$. Then

$$\begin{aligned}
G_1 &= q_n \operatorname{ex}_1(\sigma_1)\operatorname{ex}(\tau_1) + p_n \operatorname{ex}_1(\sigma_2)\operatorname{ex}(\tau_2),\\
G_2 &= -(2n/s^2)(q_n \operatorname{ex}_3(\sigma_1) + p_n \operatorname{ex}_3(\sigma_2)),\\
G_3 &= q_n \operatorname{ex}_1(\sigma_1)\operatorname{ex}_1(\tau_1) + p_n \operatorname{ex}_1(\sigma_2)\operatorname{ex}_1(\tau_2),\\
G_4 &= G_2 - \frac{2n}{s^2}(q_n \operatorname{ex}_2(\sigma_1)\operatorname{ex}_1(\tau_1) + p_n \operatorname{ex}_2(\sigma_2)\operatorname{ex}_1(\tau_2)),\\
G_5 &= i(p_n q_n/d_n)s\left[-\operatorname{ex}_2(\tau_1) + \operatorname{ex}_2(\tau_2)\right].
\end{aligned}$$

b) For $0 < \varepsilon \leqslant \min\{1, t^{-2}\}$, $0 < \lambda < 1$, $|a_{nk}| \leqslant \varepsilon$ and $|s| \leqslant d_n\lambda$,

$$\log \varphi_{nk}(s, t) = (\varphi_{nk}(x, t) - 1)(1 + G_6)$$

where $|G_6| \leqslant \varepsilon + \lambda$.

PROOF OF b): By (2.4) and (2.5), $|\varphi_{nk}(s, t) - 1| < (1/8)\varepsilon + (1/4)\lambda + 1/2$. Then use $\log(1 + y) = y(1 + z)$ with $|z| < |y|$ if $|y| < 1/2$.

In the rest of the proof, $\Sigma_{\leqslant\varepsilon}$ or $\Sigma_{>\varepsilon}$ will denote a sum extending over all k such that $|a_{nk}| \leqslant \varepsilon$ or $|a_{nk}| > \varepsilon$ respectively. The same rule will apply to $\Pi_{\leqslant\varepsilon}$ and $\Pi_{>\varepsilon}$.

c) For ε, λ, s satisfying the conditions of b),

$$\sum_{\leqslant\varepsilon} \log \varphi_{nk}(s, t) = \sum_{\leqslant\varepsilon} \log \psi_{nk}(t) - \frac{1}{2}s^2\left[\frac{1}{n}C_n([-\varepsilon, \varepsilon]) + G_7\right] - s\left[\frac{1}{d_n}p_n q_n t M_n([-\varepsilon, \varepsilon]) + G_8\right] + G_9$$

where $|G_l| \leqslant L_1 \cdot (\varepsilon + \lambda)$ for $l = 7, 8, 9$.

PROOF OF c): As $\psi_{nk}(t) = \varphi_{nk}(0, t)$, we have by b) $\Sigma_{\leqslant\varepsilon} \log \varphi_{nk}(s, t) - \Sigma_{\leqslant\varepsilon} \log \psi_{nk}(t) = G^* + G^{**}$ where $G^* = \Sigma_{\leqslant\varepsilon} \varphi_{nk}(s, t) - \Sigma_{\leqslant\varepsilon} \psi_{nk}(t)$ and $|G^{**}| \leqslant 2(\varepsilon + \lambda)\Sigma_{\leqslant\varepsilon}|\varphi_{nk}(s, t) - 1|$. Apply (2.7) to G^* and use (2.4) and (2.6) to estimate $\Sigma|\varphi_{nk}(s, t) - 1|$. At one stage, the inequality $n^{-1/2}\Sigma_k|a_{nk}| \leqslant (D_n(R))^{1/2}$ must be used.

d) Let ε, λ, s satisfy the conditions of b). Then there exists $n_1(\varepsilon)$ (depending only on ε) such that for any $n \geqslant n_1(\varepsilon)$

$$\prod_{\leqslant\varepsilon} \varphi_{nk}(s, t) = \prod_{\leqslant\varepsilon} \psi_{nk}(t) \operatorname{ex}\left(-\frac{1}{2}s^2(1 + G_{10}) - s\left(t(pq)^{-1/2}\mu + G_{11}\right) + G_{12}\right)$$

where $|G_l| \leqslant L_2 \cdot (\varepsilon + \lambda)$ for $l = 10, 11, 12$.

Proof of d): Under the assumption of Lemma 1, $(1/n)C_n([-\varepsilon, \varepsilon]) \underset{n}{\to} 1$ and $(1/d_n)p_nq_nM_n([-\varepsilon, \varepsilon]) \underset{n}{\to} (pq)^{-1/2}\mu$. Use this in c).

e) For any $\varepsilon > 0$, $\lambda > 0$, $|s| \leqslant d_n\lambda$ and any n, k

$$\prod_{<\varepsilon} \varphi_{nk}(s, t) = \prod_{<\varepsilon} \psi_{nk}(t) + G_{13}$$

where $|G_{13}| \leqslant L_{3,\varepsilon} \cdot \lambda$ with $L_{3,\varepsilon}$ depending only on ε.

Proof of e): By (2.5), $|\varphi_{nk}(s, t) - \psi_{nk}(t)| < \lambda$ for all n, k.

Hence, the two products differ by less than $\sup_n C_n(\overline{[-\varepsilon, \varepsilon]}) \cdot \lambda$.

f) Let ε, λ satisfy the conditions of b) and be so small that $L_2 \cdot (\varepsilon + \lambda) < 1/4$. Then for any $|s| \leqslant d_n\lambda$ and $n \geqslant n_1(\varepsilon)$

$$\left|\mathrm{ex}\left(-is\frac{m_n}{d_n}\Phi_n^{(W)}\left(\frac{s}{d_n}, t\right) - \mathrm{ex}\left(-\frac{1}{2}s^2 - st\mu\right)\Phi_n^{(V)}(t)\right)\right|$$
$$\leqslant \mathrm{ex}\left(-\frac{1}{4}s^2 + L_{3,\varepsilon} \cdot |s|\right) \cdot (s^2 + |s| + 1) \cdot L_4 \cdot \left[(L_{3,\varepsilon} + 1)\lambda + \varepsilon\right].$$

Proof of f): The expression on the left-hand side equals

$$\left|\prod_k \varphi_{nk}(s, t) - \mathrm{ex}\left(-\frac{1}{2}s^2 - st\mu\right)\prod_k \psi_{nk}(t)\right|$$
$$= \mathrm{ex}\left(-\frac{1}{2}s^2 - st\mu\right)\left|\prod_k \psi_{nk}(t)\, \mathrm{ex}_1(G_{14}) + \prod_{<\varepsilon} \psi_{nk}(t)\, \mathrm{ex}(G_{14}) \cdot G_{13}\right|$$

where $G_{14} = (-1/2)s^2G_{10} - sG_{11} + G_{12}$. Apply d) and e).

g) To an arbitrary $\eta > 0$ there exists $\lambda > 0$ and $n_3(\eta)$ such that for all $n \geqslant n_3(\eta)$

$$\int_{-d_n\lambda}^{d_n\lambda}\left|\mathrm{ex}\left(-i\left(s\frac{m_n}{d_n} + tc_n\right)\right)\Phi_n^{(W)}\left(\frac{s}{d_n}, t\right) - \mathrm{ex}\left(-\frac{1}{2}s^2 - st\mu\right)\Phi^{(V)}(t)\right| ds < \eta.$$

Proof of g): To any $\varepsilon > 0$, there exists $n_2(\varepsilon)$ such that

$$|\mathrm{ex}(-itc_n)\Phi_n^{(V)}(t) - \Phi^{(V)}(t)| < \varepsilon \quad \text{for all } n > n_2(\varepsilon).$$

Then for any ε, λ satisfying the conditions of f) and any $n > \max\{n_1(\varepsilon), n_2(\varepsilon)\}$, the above integral is less than $\int_{-\infty}^{\infty} \mathrm{ex}((-1/4)s^2 + L_5|s|) \cdot (s^2 + |s| + 1)\, ds \cdot L_6 \cdot (2\varepsilon + (L_{3,\varepsilon} + 1)\lambda) = L_7\varepsilon + L_{8,\varepsilon}\lambda$ where $L_{8,\varepsilon}$ depends on ε. To a given η, choose first an ε satisfying the conditions of f) and such that $L_7\varepsilon < \eta/2$. Then choose a λ satisfying the conditions of f) and such that $L_{8,\varepsilon}\lambda < \eta/2$. The assertion of g) holds for $n_3(\eta) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$.

h) For any $0 < \lambda < \pi$

$$\int_{d_n\lambda < |s| < d_n\pi} |\Phi_n^{(W)}(s, t)|\, ds \underset{n}{\to} 0.$$

Proof of h): For $|s| \geqslant d_n\lambda$ and k such that $|ta_{nk}| \leqslant \lambda/2$, $|\Phi_{nk}^{(W)}(s/d_n, t)| \leqslant \mathrm{ex}(-p_n q_n(1 - \cos(\lambda/2)))$. Hence, the above integral is less than $2\pi d_n\, \mathrm{ex}(-d_n^2(1 - \cos(\lambda/2))\, \mathrm{ex}(C_n[\overline{-\lambda/(2t), \lambda/(2t)}])$. This expression tends to 0 as $d_n \to \infty$.

Finally, it follows from (2.3), g) and h) that

$$\mathrm{ex}(-itc_n)\Phi_n^{(X)}(t) \underset{n}{\to} \Phi^{(V)}(t)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(-1/2)s^2 - st\mu}\, ds.$$

Lemma 2. *Let* (1.1) *hold and let* c_n *be defined as in Theorem* 1. *Then* $V_n - c_n$ *converges in distribution and the limit characeristic function* $\Phi^{(V)}$ *is given by the right-hand side of* (1.3) *with* $\mu = 0$.

Proof. For any atom x of C we shall write

$$\psi_x(t) = q\, \mathrm{ex}(-itpx) + p\, \mathrm{ex}(itqx).$$

The symbols $\Sigma_{<\varepsilon}$ or $\Sigma_{>\varepsilon}$ if applied to $\psi_{nk}(t)$ will have the same meaning as in the proof of Lemma 1. If applied to $\psi_x(t)$, they will denote sums extending over all atoms x of C such that $|x| \leqslant \varepsilon$ or $|x| > \varepsilon$ respectively. The same rule will apply to products. The proof will consist again of several steps.

a) For any $t_0 > 0$, the product $\Pi_x[\psi_x(t)]^{C(\{x\})}$ converges uniformly with respect to $t \in [-t_0, t_0]$ and unconditionally (see Remark 1 in Section 1).

Proof of a): For any $\varepsilon > 0$ and $|t| \leqslant t_0$

$$\sum_{>\varepsilon} |\psi_x(t) - 1| C(\{x\}) = \lim_n \sum_{>\varepsilon} |\psi_{nk}(t) - 1| \leqslant \frac{1}{2} t_0^2 \sup_n D_n(R) < \infty$$

by (2.4). Hence $\sup_{|t| \leqslant t_0} \Sigma_x |\psi_x(t) - 1| C(\{x\}) < \infty$.

In the rest of the proof, $t \neq 0$ and $\eta > 0$ are fixed.

b) There exists an $\varepsilon_0 > 0$ such that

$$\left| \prod_x [\psi_x(t)]^{C(\{x\})} - \prod_{>\varepsilon} [\psi_x(t)]^{C(\{x\})} \right| < \eta \quad \text{for all } 0 < \varepsilon \leqslant \varepsilon_0.$$

Proof of b): Follows from a).

c) There exist $0 < \varepsilon \leqslant \varepsilon_0$ and n_4 such that

$$\left| \sum_{<\varepsilon} \log \psi_{nk}(t) + \frac{1}{2} pqD(\{0\})t^2 \right| < \eta \quad \text{for all } n \geqslant n_4.$$

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PROOF OF c): Using methods similar to those in the proof of Lemma 1 we can show that, for any $\varepsilon \leqslant \min\{1, t^{-2}\}$ and any n

$$\left|\sum_{<\varepsilon} \log \psi_{nk}(t) + \frac{1}{2} p_n q_n D_n([-\varepsilon, \varepsilon]) t^2\right| < L_9 \varepsilon.$$

Choose an $\varepsilon > 0$ which is a continuity point of D and such that $\varepsilon \leqslant \min\{1, t^{-2}, \varepsilon_0\}$, $L_0\varepsilon < \eta/2$ and $t^2/2|D([-\varepsilon, \varepsilon]) - D(\{0\})| < \eta$. Finally, choose n_4 so that $t^2/2|D_n([-\varepsilon, \varepsilon]) - D([-\varepsilon, \varepsilon])|\eta$ for all $n \geqslant n_4$.

d) To the ε of c), there exists n_5 such that

$$\left|\prod_{>\varepsilon} \psi_{nk}(t) - \prod_{>\varepsilon} [\psi_x(t)]^{C(\{x\})}\right| < \eta \quad \text{for all } n \geqslant n_5.$$

PROOF OF d): Follows from $C_n \underset{n}{\rightarrow} C$ weakly on $\overline{[-\varepsilon, \varepsilon]}$.

Combining b), c), and d) we have for $n > \max\{n_4, n_5\}$

$$\left|\prod_k \psi_{nk}(t) - \mathrm{ex}\left(-\frac{1}{2} pqD(\{0\}) t^2 \prod_x \psi_x(t)\right)\right| < 5\eta.$$

Theorem 1 follows from the combination of Lemma 1 and Lemma 2. Similarly, Theorem 2 follows from the combination of the following two lemmas.

LEMMA 3. *Let* (1.7) *hold and let there exist constants c_n such that $V_n - c_n$ converges in distribution. Denote the limit characteristic function by $\Phi^{(V)}$. Then $X_n - c_n$ converges in distribution and the limit characteristic function is*

$$\Phi^{(X)}(t) = \mathrm{ex}\left(\frac{1}{2}\mu^2 t^2\right)\Phi^{(V)}(t).$$

LEMMA 4. *Let $p_n \to 0$ and let* (1.5) *and* (1.6) *hold. Further, let c_n be defined as in Theorem* 2. *Then $V_n - c_n$ converges in distribution and the logarithms of the limit characteristic function $\Phi^{(V)}$ is given by the right-hand side of* (1.8) *with $\mu = 0$.*

Lemma 3 can be proved in a similar way to Lemma 1, although some technical changes are necessary, for example in the definition of φ_{nk} and ψ_{nk} the scores a_{nk} must be truncated. On the other hand, Lemma 4 does not need any proof. Under $p_n \to 0$, the triangular array of independent random variables $\{V_{nk}\}$ satisfies the null (uniform negligibility) condition and the result follows from the general theory for such arrays.

3. Comparison with independent sampling

If the sampling described in Section 1 is independent, that is with replacement, the result is a triangular array $\{Y_{nj}\}$ of random variables such that, for each n, $Y_{n1}, \ldots, Y_{nm_n}$ are independent, equally distributed with $\mathcal{P}(Y_{nj} = a_{nk}) = 1/n$ for all n, j, k. Put $Y_n = \sum_{j=1}^{m_n} Y_{nj}$.

If the assumptions of Theorem 1 hold, then $Y_n - c_n$ converges in distribution and the limit characteristic function $\Phi^{(Y)}$ has

$$\log \Phi^{(Y)}(t) = -\frac{1}{2}p(D(\{0\}) - \mu^2)t^2 + p\sum_x (e^{itx} - 1 - itx)C(\{x\}).$$

Comparing this formula with (1.3) we see that the dependent sampling reduces the variance of the normal component and converts the Poisson components into Bernoulli distributions.

Under the assumptions of Theorem 2, $Y_n - c_n$ has in the limit the same distribution as $X_n - c_n$. This is not surprising; the assumption $p_n \to 0$ means that X_{nj} are asymptotically uncorrelated.

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