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ADDITIVE FUNCTIONS MONOTONIC ON THE SET OF PRIMES II

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1. **Introduction.** Let $L\colon [1,\infty) \longrightarrow [1,\infty)$ be a nondecreasing function such that $\lim_{x\to\infty} L(x) = +\infty$. Let $f = f_L$ be a strongly additive function determined by $f(p) = L(p)$ on the set of primes. In what follows $p, p_1, p_2, \ldots, q, q_1, q_2, \ldots, P,\ Q$ stand for prime numbers, $P(n)$ denotes the largest prime divisor of n. The letters $c, c_1, c_2, \ldots$ denote suitable positive constants, not necessarily the same at each occurrence. As usual, $\pi(x)$ denotes the number of primes $p \le x$, while $\pi(x, k, \ell)$ is the number of primes $p \le x$ such that $p \equiv \ell \pmod{k}$. On the other hand, $\varphi(n)$ stands for the Euler-totient function and $\omega(n)$ is the number of distinct prime factors of n. For an integer n and real number $y \ge 1$, let

$$n_y \stackrel{\text{def}}{=} \prod_{p^\alpha \| n,\, p \le y} p^\alpha.$$

For each $0 < \Delta < 1$, let $(\mathcal{H}_\Delta)$ denote the condition

$$(\mathcal{H}_\Delta). \qquad \lim_{x\to\infty} \frac{L(x^{1-\Delta})}{L(x)} = 0.$$

Further let

$$(1.1) \qquad u(n) = u_L(n) \stackrel{\text{def}}{=} \frac{1}{L\big(P(n)\big)} \sum_{q|n,\, q<P(n)} L(q).$$

In [2], we proved that the fulfilment of condition $(\mathcal{H}_\Delta)$ for every $0 < \Delta < 1$ is necessary in order to assert that

$$(1.2) \qquad \lim_{x\to\infty} \frac{1}{x} \#\{ n \le x : u(n) \ge \varepsilon \} = 0 \quad \text{for every } \varepsilon > 0,$$

and that it is sufficient to guarantee that

$$(1.3) \qquad \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \left(e^{au(n)} - 1 \right) = 0$$

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for every $a > 0$. Now since clearly (1.3) implies (1.2), it follows that (1.2) and (1.3) are equivalent to each other and also to the assertion that $(\mathcal{H}_\Delta)$ holds for every $0 < \Delta < 1$.

Let $\mathcal{A}$ denote an infinite sequence $a_1 \leq a_2 \leq \ldots$ of positive integers such that $\lim_{n\to\infty} a_n = +\infty$ and $a_n = O(n^c)$. Furthermore, for $\delta > 0$, let

$$(1.4) \qquad d(\delta) = \limsup_{x\to\infty} \frac{1}{x} \#\{n \leq x : P(a_n) \leq x^\delta\}$$

and

$$(1.5) \qquad e(\varepsilon,\delta) = \limsup_{x\to\infty} \frac{1}{x} \#\{n \leq x : \exists p, q,\, pq | a_n,\, x^\delta < p < q < p^{1+\varepsilon}\}.$$

In [2] (Theorem 2), we proved the following result:
Assume that

$$(1.6) \qquad \lim_{\delta\to 0^+} d(\delta) = 0,$$

$$(1.7) \qquad \lim_{\varepsilon\to 0^+} e(\varepsilon,\delta) = 0 \quad \textit{for every } \delta > 0,$$

and that L satisfies condition $(\mathcal{H}_\Delta)$ *for every* $0 < \Delta < 1$. *Then*

$$(1.8) \qquad \lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : u(a_n) \geq \varepsilon\} = 0 \quad \textit{for every } \varepsilon > 0.$$

One can deduce from known sieve results that the conditions (1.6) and (1.7) are verified for the set $\mathcal{A} = \mathcal{A}_1 \stackrel{\text{def}}{=} \{p+1 : p \text{ prime}\}$, that is, the so-called set of shifted primes. The same is true for the set $\mathcal{A} = \mathcal{A}_2 \stackrel{\text{def}}{=} \{a_n = \prod_{j=1}^k (n+e_j) : n = 1, 2, \ldots\}$, where $e_1, e_2, \ldots, e_k$ are arbitrary integers. To prove the fulfilment of (1.7) for $\mathcal{A}_2$, one has to use the following result.

LEMMA 1. *For every fixed integer* $e \neq 0$ *and every* $\varepsilon > 0$, *the number of integers* $n \leq x$ *for which there exists a divisor* pq *of* $n(n+e)$, *such that* $pq > x$ *and* $p < q < p^{1+\varepsilon}$, *where p and q are prime numbers, is less than* $c\varepsilon x + o(x)$ *as* $x \to \infty$, *where c is a suitable absolute constant.*

We shall not prove this lemma, because the argument which will be used for the proof of Theorem 5 indicates how to prove it. For a further reference, see Erdös and Pomerance [3].

Our main purpose in this paper is to prove similar theorems for short intervals. More precisely, we shall prove the following two theorems.

THEOREM 1. *Assume that condition* $(\mathcal{H}_\Delta)$ *holds for every* $0 < \Delta < 1$. *Let* $\rho > 0$ *be an arbitrary positive constant, and* $z = z(x)$ *be a function such that* $x^{2/3+\rho} \leq z(x) \leq x$. *Then, if u is defined by (1.1),*

$$(1.9) \qquad \lim_{x\to\infty} \frac{1}{z} \sum_{x\leq n\leq x+z} u(n) = 0.$$

THEOREM 2. *Under the conditions of Theorem 1, for every fixed integer* $\ell \neq 0$,

$$\lim_{x\to\infty} \frac{1}{\pi(x+z)-\pi(x)} \sum_{x\leq p<x+z} u(p+\ell) = 0. \tag{1.10}$$

In Section 3, we shall formulate and prove Theorems 3–5 which, as the reader will certainly realize, can have applications elsewhere.

Let u be defined by (1.1) and set

$$h(n;z) \stackrel{\text{def}}{=} \max_{y\geq z} u(n_y). \tag{1.11}$$

We are interested in characterizing those functions L for which

$$\lim_{z\to\infty} \limsup_{x\to\infty} \frac{1}{x}\{n \leq x : h(n;z) \geq \varepsilon\} = 0 \tag{1.12}$$

for each $\varepsilon > 0$. A function $h(n,z)$ is said to satisfy the *strong law of large numbers* if (1.12) holds.

Since $h(n;z) \geq u(n)$, then, using Theorem 1 of [2], it is easy to see that (1.12) implies that condition $(\mathcal{H}_\Delta)$ must hold for every $0 < \Delta < 1$. A necessary and sufficient condition for (1.12) can be deduced from a theorem due to Erdös [4] (see Theorem A below in Section 7). The short interval version of such a condition will be treated in Section 7.

2. Preliminary results.

2.1. *Let* $\Psi(x,y) = \#\{n \leq x : P(n) \leq y\}$. *It is known (see de Bruijn [1]) that*

$$\Psi(x,y) \leq x \exp\left(-c\frac{\log x}{\log y}\right) \tag{2.1}$$

uniformly for $x, y \geq 2$.

2.2. *Let* $\Psi([x,x+z],y) = \#\{n : n \in [x,x+z], P(n) \leq y\}$. *Then*

$$\Psi([x,x+z],y) \leq cz\frac{\log 2y}{\log z} \tag{2.2}$$

uniformly for $z \geq 2$, $1 < y \leq z$.

To see that this last result is true, we proceed as follows. Because of (2.1), we may assume that $z \leq x$. Furthermore, if $y > \sqrt{x}$, then (2.2) is obvious; therefore let $y \leq \sqrt{x}$. Define

$$f_y(n) = \sum_{q^k|n,\, q\leq y} \log q \tag{2.3}$$

Clearly $f_y(n) = \log n$ if $P(n) \le y$. For simplicity let $R = \Psi([x, x+z], y)$. First we write

$$(2.4)\qquad \begin{aligned} R\log x &\le \sum_{x\le n\le x+z} f_y(n) \\ &= \sum_{q^k\le z,\, q\le y} \log q \sum_{x\le q^k\nu\le x+z} 1 + \sum_{q^k>z,\, q\le y} \log q \sum_{x\le q^k\nu\le x+z} 1 \\ &= \sum\nolimits_1 + \sum\nolimits_2. \end{aligned}$$

Clearly

$$\begin{aligned} \sum\nolimits_1 &= \sum_{q^k\le z,\, q\le y} \log q\left(\left[\frac{x+z}{q^k}\right] - \left[\frac{x}{q^k}\right]\right) \\ &= z\sum_{q^k\le z,\, q\le y} \frac{\log q}{q^k} + O\left(\sum_{q\le z}\log q\right) \\ &\le 2z\log y + c_1 z. \end{aligned}$$

On the other hand, it is clear that $\sum_2 \le (\log y)\,\pi(y) \le c_2 y$, since for each q there exists no more than one k and one ν satisfying the stated conditions. Thus, combining the above, we obtain

$$(2.5)\qquad R\log z \le R\log x \le (2\log y + c_1)z + c_2 y,$$

which leads to (2.2) immediately.

2.3. *Let ℓ be a non-zero integer. Then*

$$(2.6)\qquad \pi(x+y, k, \ell) - \pi(x, k, \ell) < \frac{cy}{\varphi(k)\log y/k}$$

uniformly for $1 < k \le y$ and $(k, \ell) = 1$.
For a proof of this result, see Halberstam and Richert [5].

2.4. Let ℓ be a fixed non-zero integer. For each positive integer K, let

$$\Xi_{\ell,K}([x, x+z], y) = \#\left\{p \in [x, x+z],\ p \equiv -\ell \pmod K,\ P\left(\frac{p+\ell}{K}\right) \le y\right\}.$$

LEMMA 2. *Let ℓ and K be as above. Then*

$$(2.7)\qquad \Xi_{\ell,K}([x, x+z], y) < c\frac{z\log 2y}{\varphi(K)(\log z)^2}$$

uniformly for $1 < y \le z \le x$, $K \le \sqrt{z}$, where c may depend on ℓ.

PROOF. For simplicity, let $R = \Xi_{\ell,K}([x, x+z], y)$. We may assume that z (and hence also x) is large enough, and that $x + \ell > x/e$. We proceed as in the proof of (2.2). First we write

$$(2.8)\qquad \begin{aligned} R\log\frac{x}{Ke} &\le \sum_{x\le p\le x+z} f_y\left(\frac{p+\ell}{K}\right) \\ &\le \sum_{q\le y,\, q^t\le\sqrt{z}} (\log q)\big(\pi(x+z, qK, -\ell) - \pi(x, qK, -\ell)\big) \\ &\quad + \sum_{q\le y,\, q^t>\sqrt{z}} \log q \sum_{x\le p\le x+z,\, Kq^t\nu=p+\ell} 1; \end{aligned}$$

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but, assuming that $y < z^{1/8}$, this last double sum is no greater than

$$\sum_{q \le y} (\log q)\big(\pi(x+z, q^2K, -\ell) - \pi(x, q^2K, -\ell)\big).$$

Since $q^2K < z^{3/4}$, by (2.5), we have that

$$R \log \frac{x}{Ke} \le c_1 \frac{z}{\varphi(K)} \frac{\log 2y}{\log z} + c_2 \frac{z}{\varphi(K) \log z}, \tag{2.9}$$

which yields immediately (2.7). If $z^{1/8} \le y \le z$, then (2.7) is an immediate consequence of (2.6).

2.5.

LEMMA 3. *Let ℓ be a fixed non zero integer and K be a positive integer, $K < \sqrt{x}$. Let*

$$\Xi_{\ell,K}(x,y) \stackrel{\text{def}}{=} \#\{p \le x : p \equiv -\ell \pmod{K},\ P\left(\frac{p+\ell}{K}\right) \le y\}. \tag{2.10}$$

Then

$$\Xi_{\ell,K}(x,y) \le \frac{c_1}{\varphi(K)} \frac{x}{\log x} \left(\frac{\log 2y}{\log x}\right) \tag{2.11}$$

uniformly in $1 \le y < x$, where c_1 may depend on ℓ.

PROOF. We may assume that $y \le x^{1/4}$. Furthermore it is clear that we can ignore all the primes $p \le \sqrt{x}$. Then, arguing as in the proof of Lemma 2 and using only the Brun Titchmarsh inequality instead of (2.6), we immediately obtain (2.11).

2.6. By using elementary estimates on $\pi(x)$, one can easily obtain that the two inequalities

$$\sum_{p > H} \frac{(\log p)^{-s}}{p} \le c \cdot \frac{1}{s(\log H)^s} \tag{2.12}$$

and

$$\sum_{q < H} \frac{(\log q)^s}{q} \le c \cdot \frac{(\log H)^s}{s} \tag{2.13}$$

hold uniformly for $s \ge 1,\ H \ge 2$.

2.7. *The number of solutions of the equation $p + \ell = aq$ in primes p and q, where $x - y \le p \le x$, is less than*

$$\frac{cy}{\varphi(a)(\log y/a)^2}$$

uniformly in $a < y < x$.

For a proof of this, see Halberstam and Richert [5].

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2.8. *Assume that* $(\mathcal{H}_\Delta)$ *holds for every* $0 < \Delta < 1$. *Let* C *and* ε *be arbitrary positive numbers. Then there exists a bound* $x_0 = x_0(C, \varepsilon)$ *such that*

$$\frac{L(y)}{L(x)} \leq \left(\frac{\log y}{\log x}\right)^C \tag{2.14}$$

whenever $x_0 < y < x^{1-\varepsilon}$.
This is Lemma 3 in [2].

2.9. *Let* $\ell_1 \neq \ell_2$ *be two non zero integers. Then the number of solutions of the system of equations*

$$\begin{cases} p + \ell_1 = aQ \quad , \\ p + \ell_2 = bP \quad , \end{cases}$$

in prime variables p, P, Q, *where* p *runs in the range* $2 \leq p \leq x$, *is less than*

$$\frac{cx}{\varphi(a)\varphi(b)\log^3 \frac{x}{ab}}, \tag{2.15}$$

where $c = c(\ell_1, \ell_2)$ *is a positive constant.*
For a proof of this, see Halberstam and Richert [5].

2.10. *As* $x \longrightarrow \infty$,

$$\sum_{a \leq x} \frac{1}{\varphi(a)} = c_0 \log x + c_1 + o(1)$$

and

$$\sum_{a \leq x} \frac{\log a}{\varphi(a)} = c_2(\log x)^2 + c_3(\log x) + c_4 + o(1).$$

For a proof of this, see Ward [8].

2.11. *Let* ρ *be a positive constant,* $x^{\frac{7}{12}+\rho} \leq z = z(x) \leq x$, $k \leq \log x$, $(\ell, k) = 1$. *Then*

$$\pi(x+z, k, \ell) - \pi(x, k, \ell) = \frac{1}{\varphi(k)} \int_x^{x+z} \frac{du}{\log u} + O\left(\frac{1}{k(\log x)^{10}}\right)$$

uniformly in ℓ, k, z.
For a proof of this, see Perelli, Pintz and S. Salerno [6].

3. **Sieve methods in short intervals.** In 1959, Erdös [4] proved that *almost all* integers have their factors *far one from the other* (see Theorem A in Section 7). For instance, it follows from his result that, if $\varepsilon > 0$ and $\delta > 0$ and if we let $\mathcal{N}_{\varepsilon,\delta}(x)$ stand for the set of those integers n for which there exist two primes P, Q such that $PQ|n$ and such that

$$x^\delta < P < Q < P^{1+\varepsilon}, \tag{3.1}$$

then, if $\delta > 0$ is fixed,

$$\lim_{\varepsilon \to 0} \lim_{x \to \infty} \frac{1}{x} \#\mathcal{N}_{\varepsilon,\delta}(x) = 0.$$

We shall prove analogues of that result first for short intervals of integers and then for short intervals of shifted primes. These two results can be stated as follows.

THEOREM 3. *Let $\rho > 0$ be a fixed real number and assume that $x^{\frac{2}{3}+\rho} \leq z(x) \leq x$, $x \geq 2$. Then for every choice of $0 < \varepsilon < \delta$, the number of those integers $n \in [x, x+z]$ for which there exist two prime divisors P, Q such that $x^\delta < P < Q < P^{1+\varepsilon}$ is less than $\lambda_1(\varepsilon,\delta)z$, where $\lambda_1(\varepsilon,\delta) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, for every fixed δ.*

THEOREM 4. *Let $\rho, z, \varepsilon, \delta$ be as in Theorem 3. Let $\ell \neq 0$ be an integer. Then the number of those primes p in $[x, x+z]$, for which there exist two primes P, Q such that $PQ|p+\ell$ and $x^\delta < P < Q < P^{1+\varepsilon}$ is less than $\lambda_2(\varepsilon,\delta)z/\log z$, where $\lambda_2(\varepsilon,\delta) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, for every fixed δ.*

PROOF OF THEOREM 3. Let x be large. Let $\mathcal{N}(x,z)$ be the set of integers $n \in [x, x+z]$ which satisfy the conditions stated above. We will show that

$$\#\mathcal{N}(x,z) \leq \left(6\varepsilon \log\frac{1}{\delta} + c_0\rho\varepsilon\right) z, \tag{3.2}$$

thus establishing our claim. Hence let $n \in \mathcal{N}(x,z)$. Such an integer n has two prime factors P, Q with $x^\delta < P < Q < P^{1+\varepsilon}$. Let $n = PQ\nu$. Write

$$\mathcal{N}(x,z) = \mathcal{N}_1(x,z) \cup \mathcal{N}_2(x,z), \tag{3.3}$$

where in the first set on the right of (3.3), we consider those n for which $PQ \leq z$ and in the second one, we consider those n for which $PQ > z$. If $PQ \leq z$, then there exist at most $2z/PQ$ multiples of PQ in the interval $[x, x+z]$. Summing up for P and Q, we get that

$$\begin{aligned}\#\mathcal{N}_1(x,z) &\leq 2z \sum_{x^\delta \leq P \leq x} \frac{1}{P}\left(\sum_{P \leq Q < P^{1+\varepsilon}} \frac{1}{Q}\right) \\ &\leq 4z\varepsilon \sum_{x^\delta \leq P \leq x} \frac{1}{P} \leq 6z\varepsilon \log\frac{1}{\delta}.\end{aligned} \tag{3.4}$$

On the other hand, if $PQ > z$, we have

$$Q^2 > z \geq x^{\frac{2}{3}+\rho}$$

and thus

$$Q > x^{\frac{1}{3}+\frac{\rho}{2}}.$$

Furthermore

$$P\nu = \frac{PQ\nu}{Q} \leq \frac{2x}{Q} \leq 2x^{\frac{2}{3}-\frac{\rho}{2}} \leq 2zx^{-\frac{3\rho}{2}}.$$

If for some choice of P and ν at least one Q occurs, then we have that

$$x \leq PQ\nu \leq P^{2+\varepsilon}\nu \quad \text{and} \quad P^2\nu \leq 2x,$$

that is,

$$\nu \in \left[\frac{x}{P^{2+\varepsilon}}, \frac{2x}{P^2}\right]. \tag{3.5}$$

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The variable Q runs in the interval

$$\frac{x}{P\nu} \le Q \le \frac{x}{P\nu} + \frac{z}{P\nu}$$

and so, using (2.6), one easily sees that it takes on at most

$$c_1\rho \frac{z}{P\nu \log x}$$

distinct values. Therefore summing up first for all ν satisfying (3.5), and after for P, we get that

$$\#\mathcal{N}_2(x,z) \le c_2\rho \frac{z}{\log x} \sum_{x^{\frac{1}{3}} \le P < x} \frac{1}{P}\left(\sum \frac{1}{\nu}\right) < c_3\rho\varepsilon \frac{z}{\log x} \sum_{x^{\frac{1}{3}} \le P < x} \frac{\log P}{P} < c_4\rho\varepsilon z. \tag{3.6}$$

Recalling (3.3) and using (3.4) and (3.6), we obtain (3.2) and thus the theorem.

PROOF OF THEOREM 4. Let x be large and ε be small. Let $\mathcal{M}(x,z)$ be the set of primes mentioned in the statement. We will show that

$$\#\mathcal{M}(x,z) \le c_0\rho\varepsilon \cdot \frac{z}{\log x} \tag{3.7}$$

Write

$$\mathcal{M}(x,z) = \mathcal{M}_1(x,z) \cup \mathcal{M}_2(x,z), \tag{3.8}$$

where in the first set on the right of (3.8), we consider those primes p such that $PQ|p+\ell$ and $PQ < x^{\frac{2}{3}+\frac{3\rho}{4}}$, while in the second set we consider those primes p such that $PQ|p+\ell$ and $PQ \ge x^{\frac{2}{3}+\frac{3\rho}{4}}$. Using (2.6), we easily get that

$$\begin{aligned}\#\mathcal{M}_1(x,z) &\le \sum_{P,Q}\big(\pi(x+z,PQ,-\ell) - \pi(x,PQ,-\ell)\big)\\ &\le \sum_{P,Q} \frac{c\rho z}{PQ\log x} \le c_1\rho\varepsilon \log\frac{1}{\delta}\frac{z}{\log x}.\end{aligned}$$

To estimate $\#\mathcal{M}_2(x,z)$, we proceed as follows. For each $p \in \mathcal{M}_2(x,z)$, one has $p+\ell = PQ\nu$ and $PQ \ge x^{\frac{2}{3}+\frac{3\rho}{4}}$. Therefore $P\nu \le zx^{-\frac{3\rho}{2}}$ and ν is located in the interval (3.5). But for fixed ν and P, using 2.7, we deduce that the number of solutions of the equation $p+\ell = aQ$ with $a = P\nu$ is less than

$$\frac{c\rho z}{P\varphi(\nu)(\log x)^2}.$$

Summing now on the integers ν satisfying the condition (3.5) by making use of 2.10, and afterwards summing on $P \in [x^{1/3}, x]$, we obtain (3.7) and thus the theorem.

We now state another result which in a sense can be considered as a generalization of Theorem 4.

THEOREM 5. Let $\ell_1, \ell_2, \ldots, \ell_k$ be distinct non zero integers and let $F(n) = (n+\ell_1)\cdots(n+\ell_k)$. For each $\varepsilon > 0$ and $\delta > 0$, let $S_{\varepsilon,\delta}(x)$ stand for the number of primes $p \leq x$ for which there exist primes P, Q such that $PQ|F(p)$, $x^\delta \leq P < Q < P^{1+\varepsilon}$. Then

$$\limsup_{x\to\infty} \frac{S_{\varepsilon,\delta}(x)}{\pi(x)} \leq \lambda_3(\varepsilon,\delta), \tag{3.9}$$

with $\lambda_3(\varepsilon,\delta) \to 0$ as $\varepsilon \to 0$.

PROOF. It is enough to prove the theorem for $k=2$. Hence let $F(n) = (n+\ell_1)(n+\ell_2)$. Let S_i, $i = 1,2$, be the number of primes $p \leq x$ for which there exist two primes P, Q such that $PQ|p+\ell_i$, $x^\delta \leq P < Q < P^{1+\varepsilon}$. Let T stand for the number of primes $p \leq x$ for which there exist primes P, Q satisfying condition (3.1) and such that $P|p+\ell_i$ and $Q|p+\ell_j$, $i \neq j$. Then clearly

$$S_{\varepsilon,\delta}(x) \leq S_1 + S_2 + T.$$

We first estimate T. Let ε_1 be a small positive number to be determined later. By using the Brun-Titchmarsh inequality and observing that those primes $p \leq x$, for which the corresponding expression PQ does not exceed $x^{1-\varepsilon_1}$, belong to two arithmetic progressions mod PQ, we deduce that their number is

$$\ll \frac{1}{\varepsilon_1} \cdot \frac{x}{PQ\log x}.$$

Summing for P and Q, we shall get at most

$$\frac{1}{\varepsilon_1}\frac{x}{\log x}\sum_P \frac{1}{P}\Big(\sum_Q \frac{1}{Q}\Big) \ll \frac{\varepsilon}{\varepsilon_1}\log\frac{1}{\delta}\frac{x}{\log x} \tag{3.10}$$

distinct primes. Let $\varepsilon < \varepsilon_1$. If

$$x^{1-\varepsilon_1} \leq PQ \leq x^{1+\varepsilon_1},$$

then

$$P \leq x^{\frac{1}{2}+\frac{\varepsilon_1}{2}}, \qquad Q \geq x^{\frac{1}{2}-\frac{\varepsilon_1}{2}},$$

and so

$$x^{\frac{1}{2}-\frac{\varepsilon_1}{2}} \leq Q < P^{1+\varepsilon} \leq x^{(\frac{1}{2}+\frac{\varepsilon_1}{2})(1+\varepsilon)}.$$

Since $Q|p+\ell_1$ or $Q|p+\ell_2$, then, by the Brun-Titchmarsh inequality, we get that the contribution of these numbers to T is less than

$$c\Big(\sum_Q \frac{1}{Q}\Big)\frac{x}{\log x} \leq c_1\varepsilon_1\frac{x}{\log x}. \tag{3.11}$$

It remains to consider those primes p for which $PQ > x^{1+\varepsilon_1}$. Let $p+\ell_i = \nu P, p+\ell_j = \mu Q$. If there exists a solution for some ν, μ then ν, μ are close to each other, that is, for every large x, we have

$$\frac{\mu}{2} \leq \nu < \mu^{1+c\varepsilon}, \tag{3.12}$$

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where c is an absolute constant. Furthermore $\nu\mu \le c_1 \frac{x^2}{PQ}$, and so $\nu\mu \le c_1 x^{1-\varepsilon_1}$. For fixed ν, μ, making use of 2.9, we obtain that the number of solutions of the system

$$\begin{cases} p + \ell_i = \nu P \\ p + \ell_j = \mu Q \end{cases}$$

is less than

$$c \frac{x}{\varphi(\nu)\varphi(\mu)\log^3 \frac{x}{\nu\mu}} \le \frac{c_1}{\varepsilon_1^3} \frac{x}{\varphi(\nu)\varphi(\mu)\log^3 x}. \tag{3.13}$$

Summing for those ν which satisfy (3.12) and afterwards for μ, we obtain, with the help of 2.10, the upper bound

$$c_2 \frac{\varepsilon}{\varepsilon_1^3} \frac{x}{\log x}. \tag{3.14}$$

Taking into account (3.10), (3.11) and (3.14), we get

$$T \le c\left(\frac{\varepsilon}{\varepsilon_1^3} + \frac{\varepsilon \log \frac{1}{\delta}}{\varepsilon_1} + c_1\varepsilon_1\right)\frac{x}{\log x}. \tag{3.15}$$

The estimation of S_i is more simple. We use the Brun-Titchmarsh inequality directly to estimate the contribution of the divisors PQ of $p + \ell_i$ which are such that $PQ < x^{1-\varepsilon_1}$. Since $PQ < x$, for all the others, there exists a prime divisor P such that $x^{\frac{1}{2}-2\varepsilon_1} < P < x^{\frac{1}{2}}$, if $\varepsilon / \varepsilon_1$ is small enough. From this it follows that

$$S_i \le c\left(\frac{\varepsilon \log \frac{1}{\delta}}{\varepsilon_1} + \varepsilon_1\right)\frac{x}{\log x}. \tag{3.16}$$

Choosing $\varepsilon_1 = \varepsilon^{\frac{1}{4}}$ and using (3.15) and (3.16), our theorem follows.

4. **Proof of Theorem 1.** Let

$$S(x,z) \stackrel{\text{def}}{=} \sum_{x \le n \le x+z} \sum_{\substack{q \mid n \\ q < P(n)}} \frac{L(q)}{L\big(P(n)\big)} = \sum_{x \le n \le x+z} u(n). \tag{4.1}$$

We must prove that

$$\lim_{x \to \infty} \frac{S(x,z)}{z} = 0. \tag{4.2}$$

Let $\varepsilon > 0$ be small and denote by C a large number which will be specified later.

Since $u(n) \le \omega(n)$ and $\sum_{x \le n \le n+z} \omega^2(n) \ll (\log\log x)^2 z$, it follows, by using the Cauchy-Schwarz inequality, that

$$\sum_{\substack{x \le n \le x+z \\ P(n) < \exp(\sqrt{\log x})}} u(n) \le \Bigg(\sum_{\substack{x \le n \le x+z \\ P(n) < \exp(\sqrt{\log x})}} 1\Bigg)^{1/2} \Big(\sum_{x \le n \le x+z} \omega^2(n)\Big)^{1/2},$$

and thus, because of (2.2), that

$$\sum_{\substack{x \le n \le x+z \\ P(n) < \exp(\sqrt{\log x})}} u(n) \ll \frac{z \log\log x}{(\log x)^{1/4}} = o(z).$$

From (2.14), one can deduce easily that the contribution to $S(x,z)$ of the terms $\frac{L(q)}{L\left(P(n)\right)}$ satisfying the conditions $q \le \log x$ and $P(n) \ge \exp(\sqrt{\log x})$ is $o(z)$.

We now appeal to Theorem 3 with $\delta = 1/10$ and obtain, using (2.14), that

$$\sum_{x \le n \le x+z} \sum_{x^{1/10} \le q < P(n)} \frac{L(q)}{L\left(P(n)\right)} \le c_1 \lambda_1(\varepsilon, 1/10) z. \tag{4.3}$$

Assume that x is large enough so that $\log x > x_0$. Hence, using (2.14), we obtain

$$\sum_{\substack{q \mid n \\ q < P(n)^{1-\varepsilon} \\ q > \log x}} \frac{L(q)}{L\left(P(n)\right)} \le \sum \left(\frac{\log(q)}{\log\left(P(n)\right)} \right)^C \le (1-\varepsilon)^{C-1} \left(\frac{\log 2x}{\log P(n)} \right). \tag{4.4}$$

Let

$$S_1(x,z) \stackrel{\text{def}}{=} \sum_{\substack{x \le n \le x+z \\ P(n) > x^{1/4}}} u(n).$$

Taking into account (4.3), the earlier estimations and the fact that the right hand side of (4.4) is less than $5(1-\varepsilon)^{C-1}$ if $P(n) > x^{1/4}$, we obtain that

$$S_1(x,z) \le \left(c_1 \lambda_1(\varepsilon, 1/10) + 5(1-\varepsilon)^{C-1} \right) z + o(z). \tag{4.5}$$

Let $S_2(x,z)$ be the sum of $\frac{L(q)}{L(p)}$ for all those integers n for which $P(n) = p$, $q \mid n$, with $\log x < q < p^{1-\varepsilon}$ and $p \le x^{1/4}$. Using again (2.14), we obtain that

$$S_2(x,z) \le \sum_{q < p \le x^{1/4}} \left(\frac{\log q}{\log p} \right)^C \Psi\left(\left[\frac{x}{qp}, \frac{x+z}{qp} \right], p \right),$$

whence, by (2.2),

$$S_2(x,z) \le \frac{cz}{\log z} \sum_{q < p \le x^{1/4}} \left(\frac{\log q}{\log p} \right)^C \frac{\log q}{\log p}.$$

Summing on the right hand side first for q, then for p, we obtain, using (2.12) and (2.13),

$$S_2(x,z) \le \frac{c_3 z}{C}. \tag{4.6}$$

There are still some possible factors p, q which have not been considered yet. Hence let $pq \mid n$, $p = P(n)$, $p^{1-\varepsilon} \le q < x^{1/4}$, and denote by $S_3(x,z)$ the sum of $\frac{L(q)}{L(p)}$ over these q, p, n. Since trivially $\frac{L(q)}{L(p)} < 1$, it follows that $S_3(x,z)$ is not greater than the number of solutions of the inequalities

$$x \le pq\nu \le x+z, \tag{4.7}$$

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where p, q runs over those primes satisfying $p^{1-\varepsilon} \leq q < p < x^{1/4}$, and ν runs over those integers for which $P(\nu) \leq p$. Since $pq < \sqrt{x} < z$, we can use (2.2) to estimate $S_3(x, z)$. Hence we obtain

$$S_3(x,z) \leq \sum_{p,q} \Psi\left(\left[\frac{x}{pq}, \frac{x+z}{pq}\right], p\right) \leq \frac{c_1 z}{\log z} \sum_p \frac{\log p}{p}\Big(\sum_{p^{1-\varepsilon}<q<p} \frac{1}{q}\Big) \leq c_2 \varepsilon z. \tag{4.8}$$

Clearly

$$S(x,z) \leq S_1(x,z) + S_2(x,z) + S_3(x,z) + o(z)$$

and thus, collecting our inequalities (4.5), (4.6) and (4.8), we obtain that

$$\limsup_{x\to\infty} \frac{S(x,z)}{z} \leq c_1\lambda_1(\varepsilon, 1/10) + 5(1-\varepsilon)^{C-1} + \frac{c_3}{C} + c_2\varepsilon. \tag{4.9}$$

Fixing ε and letting C tend to infinity, we get that the right hand side of (4.9) is not greater than $c_1\lambda_1(\varepsilon, 1/10) + c_2\varepsilon$. Letting ε tend to zero, we obtain, by Theorem 3, that the left hand side of (4.9) must be zero. This completes the proof of Theorem 1.

5. **Proof of Theorem 2.** Since the proof of Theorem 2 can be obtained essentially along the same lines as the one of Theorem 1, we shall not proceed to give a complete proof: we shall indicate only the necessary changes.

First of all, as we did in the previous proof, we shall omit the terms $\frac{L(q)}{L(p+1)}$ satisfying the conditions $q < \log x$ or $L(p+1) > \exp\left(\sqrt{\log x}\right)$: this introduces an error $o(z/\log z)$. The contribution of the terms $L(p+1) > z^{1/10}$ can be estimated by using Theorem 4, the inequality (4.4) and by observing that $\pi(x+z) - \pi(x) \ll z/\log z$; their contribution is less than

$$(1-\varepsilon)^{C-1} c_1 \frac{z}{\log z} + c_1\lambda_2(\varepsilon, 1/10)\frac{z}{\log z}.$$

Now the sum of the remaining terms is thus

$$\sum_{Q,P} \frac{L(Q)}{L(P)} \#\{p \in [x, x+z] : P(p+\ell) = P\}$$

This last expression can be split into two parts according to $Q \leq P^{1-\varepsilon}$ or $Q > P^{1-\varepsilon}$. By using Lemma 2 and (2.14), we observe that the first sum is less than

$$\frac{cz}{\log^2 z} \sum_{Q<P<x^{1/4}} \left(\frac{\log Q}{\log P}\right)^C \frac{\log P}{PQ} \leq \frac{c_2 z}{C\log z}.$$

It remains to consider those Q, P, p for which $P^{1-\varepsilon} \leq Q < P \leq x^{1/4}$, $p \in [x, x+z]$, $p + \ell = QP\nu$, $P(\nu) \leq P$. Since $PQ \leq \sqrt{z}$, we may apply Lemma 2 with $K = PQ$, in which case we get

$$\Xi_{\ell,PQ}([x, x+z], P) < c\frac{z\log 2P}{PQ\log^2 z}.$$

Summing first on Q and then on P, we obtain that the second $\sum_2$ is less than $c_3\varepsilon z/\log z$. Then the proof of Theorem 2 can be completed along the lines of the proof of Theorem 1.

6. **Application to shifted primes.** The following result can be proven similarly as Theorem 2, by using Theorem 5 and Lemma 3.

THEOREM 6. *Let $\ell_1, \ell_2, \ldots, \ell_k$ be nonzero distinct integers and set $F(n) = \prod_{i=1}^{k}(n + \ell_i)$. Assume that condition $(\mathcal{H}_\Delta)$ holds for every $0 < \Delta < 1$. Then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} u\big(F(p)\big) = 0.$$

7. **A condition on the strong law of large numbers.** In 1959, Erdös [4] obtained the following result.

THEOREM A (ERDÖS [4]). *Let $\varepsilon_p > 0$, $\delta_p = \min(1, \varepsilon_p)$. The divergence of $\sum_p \delta_p/p$ is a necessary and sufficient condition in order that almost all integers have two prime factors p, q satisfying*

$$p < q < p^{1+\varepsilon_p}. \tag{7.1}$$

Let L be a continuous and strictly monotonic function. Define $u(n) = u_L(n)$ by (1.1) and $h(n, w)$ by (1.11). We will characterize those functions L for which (1.12) holds.

Before doing so, we introduce a sequence of functions $t_1(z), t_2(z), \ldots$. First let $t(z)$ be defined by the relation

$$L\big(t(z)\big) = 2L(z).$$

Next, for each integer $k \ge 1$, let $t_k(z)$ be the k-fold iterate of $t(z)$, that is, $t_1(z) = t(z)$, $t_k(z) = t\big(t_{k-1}(z)\big)$.

THEOREM 7. *A necessary and sufficient condition for L to satisfy (1.12) for every $\varepsilon > 0$ is that*

$$\sum_p \min\left(1, \frac{\log t_k(p)}{\log p} - 1\right) p^{-1} < +\infty \tag{7.2}$$

for every k.

PROOF. Assume that (7.2) does not hold for some positive integer k. Then, by Theorem A, almost all integers n have two prime factors p, q such that

$$w < p < q < t_k(p) \tag{7.3}$$

(here w is an arbitrary fixed positive integer). But then $\dfrac{L(p)}{L(q)} \ge \dfrac{1}{2^k}$ and so

$$\limsup_{x\to\infty} \frac{1}{x} \#\left\{n \le x : h(n, w) \ge \frac{1}{2^k}\right\} = 1.$$

Since this relation is true for every w, it follows that (1.12) does not hold.

Conversely, assume that (7.2) holds. Let $d(w)$ be the upper density of the integers n having a divisor pq satisfying (7.3). Then $d(w) \longrightarrow 0$ as $w \longrightarrow \infty$. Let w_1 be defined by

$$L(w_1) = 2^k \sum_{p \leq w} L(p).$$

Write

$$n = n_w q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r},$$

where

$$w < q_1 < \ldots < q_{i-1} \leq w_1 < q_i < \cdots < q_r.$$

Assume that n is not a multiple of any pq satisfying (7.3). Then

$$L(q_\ell) < \frac{1}{2^k} L(q_{\ell+1}) \qquad (\ell \geq 1)$$

and

$$f(n_w) < \frac{1}{2^k} L(q_i).$$

Thus

$$h(n, w_1) \leq \frac{2}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}} + \cdots \leq \frac{1}{2^{k-2}}$$

and so

$$\limsup_{x \longrightarrow \infty} \frac{1}{x} \# \Big\{ n \leq x : \, h(n, w_1) > \frac{1}{2^{k-2}} \Big\} \leq d(w).$$

Since $d(w) \longrightarrow 0$ as $w \longrightarrow \infty$, it follows that (1.12) is true if $\varepsilon = \frac{1}{2^{k-2}}$. Since (7.2) holds for every k, the result follows.

Let $0 < \rho < 1/3$ be fixed and $z(x)$ be an arbitrary function such that $x^{2/3+\rho} \leq z(x) \leq x$. Let $F(n)$ be the function defined in Theorem 6. Further assume that ℓ is a nonzero integer and that $\varepsilon > 0$. Set

$$A_1(\varepsilon) = \limsup_{w \longrightarrow \infty} \limsup_{x \longrightarrow \infty} \frac{1}{z(x)} \# \{ n \in [x, x+z] : \, h(n, w) \geq \varepsilon \}, \tag{7.4}$$

$$A_2(\varepsilon) = \limsup_{w \longrightarrow \infty} \limsup_{x \longrightarrow \infty} \frac{1}{\pi(x+z) - \pi(x)} \# \{ p \in [x, x+z] : \, h(p+\ell, w) \geq \varepsilon \}, \tag{7.5}$$

$$A_3(\varepsilon) = \limsup_{w \longrightarrow \infty} \limsup_{x \longrightarrow \infty} \frac{1}{\pi(x)} \# \Big\{ p \leq x : \, h\big(F(p), w\big) \geq \varepsilon \Big\}, \tag{7.6}$$

$$A_4(\varepsilon) = \limsup_{w \longrightarrow \infty} \limsup_{x \longrightarrow \infty} \frac{1}{x} \# \Big\{ n \leq x : \, h\big(F(n), w\big) \geq \varepsilon \Big\}, \tag{7.7}$$

THEOREM 8. *If L is such that (7.2) holds for every positive integer k, then $A_j(\varepsilon) = 0$ for each $\varepsilon > 0$ and $j = 1, 2, 3, 4$. On the other hand, if (7.2) fails to hold for some positive integer k, then for a suitable $\varepsilon_0 > 0$, $A_j(\varepsilon_0) = 1$ $(j = 1, 2, 3, 4)$.*

PROOF. We shall prove only the assertion for $j = 2$. The other cases can be treated in a similar way. Clearly it is enough to prove that Theorem A remains valid for the subsequence of shifted primes located in a short interval $[x, x+z]$ satisfying the conditions above.

Assume that

$$\sum_p \frac{\delta_p}{p} < +\infty, \tag{7.8}$$

where $\delta_p = \min(1, \varepsilon_p)$. Let $B(x, w)$ be the number of those $p + \ell$ for which there exists PQ, $PQ | p + \ell$, such that

$$w < P < Q < P^{1+\varepsilon_P}. \tag{7.9}$$

One can estimate $B(x, w)$ using (2.5) and Theorem 4. If $PQ|p + \ell$, then $P \leq 2\sqrt{x}$ for every large x. Let $\varepsilon^{(1)}$ be a small positive number. The number of $p + \ell$ having a prime divisor P satisfying $\varepsilon_P \geq \varepsilon^{(1)}$, $P > w$, is less than

$$\sum_{w<P<2\sqrt{x}} \left(\pi(x+z, P, -\ell) - \pi(x, P, -\ell)\right) \leq c_1 \frac{z}{\log z} \sum_{p>c_1, \varepsilon_p>\varepsilon^{(1)}} \frac{1}{p}.$$

The number of those $p + \ell$ for which $PQ|p + \ell$ and $x^{1/3} \leq P < Q < p^{1+\varepsilon^{(1)}}$ is less than $\lambda_2(\varepsilon^{(1)}, 1/3)$ (see Theorem 4). The contribution of the other terms is less than

$$\sum_{P,Q} \left(\pi(x+z, PQ, -\ell) - \pi(x, PQ, -\ell)\right),$$

where the above double sum runs through those primes P, Q such that $w < P \leq Q \leq P^{1+\varepsilon_P}$, $PQ \leq x^{2/3}$. Clearly this last double sum is bounded by

$$c_3 \frac{z}{\log z} \sum_{w<P<x} \frac{1}{P} \left(\sum_{P<Q<P^{1+\varepsilon_P}} \frac{1}{Q} \right) < c_4 \frac{z}{\log z} \sum_{P>w} \frac{\varepsilon_P}{P}.$$

Collecting the above estimates, we have

$$\limsup_{x\to\infty} \frac{B(x, w)}{z/\log z} \leq c_5 \left(\sum_{P>w} \frac{\varepsilon_P}{P} + \sum_{P>w, \varepsilon_P\geq\varepsilon^{(1)}} \frac{1}{P} \right) + \lambda_2(\varepsilon^{(1)}, 1/3). \tag{7.10}$$

The convergence of the second sum on the right hand side of (7.10) is a consequence of (7.8). Let us consider the limit superior of the left hand side of (7.10). It is not greater than $\lambda_2(\varepsilon^{(1)}, 1/3)$. Hence, letting $\varepsilon^{(1)} \to 0$, we obtain that

$$\lim_{w\to\infty} \limsup_{x\to\infty} \frac{B(x, w)}{z/\log z} = 0. \tag{7.11}$$

At this point, one can proceed as in the proof of Theorem 7 and obtain that $A_2(\varepsilon) = 0$.

Assume now that (7.8) does not hold. Let $S(p + \ell)$ denote the number of divisors of $p + \ell$ of the form PQ satisfying (7.9) and $P < \log\log x$. We will see that $S(p + \ell) \to \infty$ for all $p \in [x, x + z]$ with the exception of at most $o(z/\log z)$ primes.

Clearly, we can assume that $\varepsilon_p \leq 1$. By using a Hoheisel-type theorem with small modules (see 2.11), Turan's method [7] can be applied and leads to the inequality

$$\sum_{x\leq p<x+z} \Big(S(p+\ell)-A_w(x)\Big)^2 \leq c_1 A_w(x)\frac{z}{\log z}, \tag{7.12}$$

with

$$A_w(x)=\sum_{w<P<\log\log x}\frac{1}{P-1}\sum_{P<Q<P^{1+\varepsilon_p}}\frac{1}{Q-1}. \tag{7.13}$$

But $A_w(x)\longrightarrow\infty$ as $x\longrightarrow\infty$; furthermore, from (7.12), it follows that

$$\#\{p\in[x,x+z] : S(p+\ell)<\frac{A_w(x)}{2}\}\leq\frac{c_2}{A_w(x)}\frac{z}{\log z}.$$

This proves that the left hand side of (7.11) is 1 and thus ends the proof of Theorem 8.

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