

# HOMOTOPY THEORY IN ABELIAN CATEGORIES

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**Introduction.** The concept of homotopy for homomorphisms of modules, suggested by analogy with fundamental properties of the topological homotopy, was developed by Eckmann and Hilton (3; 9; 10). In the present paper, this concept of homotopy is generalized to additive categories with an additional structure, and the theory of homotopy, including in particular various exact homotopy sequences, is established in full detail. It may be helpful to the reader to realize that most of our theorems, concepts, and constructions have their analogues in the homotopy theory of topological spaces. Many of these analogues may be found in a recent paper by Eckmann and Hilton (4). Explicit references will be given at various places in the paper.

Aside from the possibility of new applications, the aim of our generalization is twofold. First, we intend a completely self-dual development of the theory. Second, we want the generalization to cover the case of the category of pairs where, already for modules (cf. (3)), a restricted class of monomorphisms and epimorphisms is distinguished.

An obvious possibility for a generalization, as already mentioned in (3), consists in replacing the category of modules by an exact category in the sense of Buchsbaum (1). Unfortunately, an exact category does not have enough structure to cover the case of the category of pairs of modules with the distinguished class of maps considered in (3). However, in the additive category of pairs, this subclass of its maps turns out to form an abelian structure in the sense of Heller (8). Thus, additive categories with an abelian structure, in short, "abelian categories," seem very appropriate for our purpose. Their use appears to have many advantages in the development of the theory (cf. Remark at the end of § 2.7) as well as in further applications (cf. (5)).

On the other hand, the concept of an abelian structure might be generalized and formulated in a way even more convenient with respect to our intention (for example, choice of two subclasses of distinguished maps dual to each other, choice of a weaker system of axioms for these structures). However, we shall use here the formulation of Heller, considering that, in doing so, we can dispose of the whole "arsenal" of propositions given in (8).

In § 1, the basic definitions concerning abelian categories are formulated. Examples are given partly to illustrate the new concepts and partly to prepare the ground for the arguments put forward in the subsequent sections.

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Section 2 is devoted to the abstract treatment of the fundamental concepts of **(3)**. In the category  $P\mathfrak{K}$  of pairs in an abelian category  $\mathfrak{K}$ , an abelian structure, depending only on that of  $\mathfrak{K}$ , is chosen, and “relative” homotopy groups are defined as particular homotopy groups in  $P\mathfrak{K}$ . The passage from the abelian category  $\mathfrak{K}$  to the abelian category  $P\mathfrak{K}$  can be iterated, thus leading to the category  $P^2\mathfrak{K}$  of double pairs in  $\mathfrak{K}$ .

In § 3, we establish various homotopy sequences, namely the two dual homotopy sequences of a pair and of a triple. Instead of verifying the exactness directly, term by term, a method is used whereby the reasoning is done in the category of pairs, and whereby the exactness follows from that of the homology sequence associated with an exact sequence of complexes. In addition, the two dual concepts of “fibration” are treated, and the corresponding “excision theorems” (cf. **(3)**, Theorems 5.2 and 5.3) are proved.

Section 4 contains three applications to special categories. Thus, special homotopy theories are obtained (homotopy theory of modules, “weak homotopy” for modules, and chain homotopy for complexes).

## SECTION 1. ABELIAN CATEGORIES

**1. Additive categories.** An *additive category*  $\mathfrak{K}$  is a category\* in which the sets  $(A, B)$  of maps from  $A$  to  $B$  are abelian groups (additively written) and the following axioms are satisfied for the terms of  $\mathfrak{K}$ :

(A) *The composition  $\alpha\beta$  of two maps  $\alpha, \beta$  of  $\mathfrak{K}$  is bilinear.*

(A0)  *$\mathfrak{K}$  has a zero-object.*

(A1) *Any two objects of  $\mathfrak{K}$  have a direct sum.*

An object  $A$  of  $\mathfrak{K}$  is a *zero-object*  $O$  if the identity map  $1_A$  of  $A$  is the zero-element  $o \in (A, A)$ . Axiom (A1) means: For any two objects  $A_1, A_2$  of  $\mathfrak{K}$ , there exists an object  $A$  of  $\mathfrak{K}$  and maps  $\iota_i \in (A_i, A)$ ,  $\pi_i \in (A, A_i)$  ( $i = 1, 2$ ) satisfying the relations (i)  $\pi_i \iota_i = 1_{A_i}$ , (ii)  $\pi_i \iota_j = o$  for  $i \neq j$ , (iii)  $\iota_1 \pi_1 + \iota_2 \pi_2 = 1_A$ . A system  $(A_1, A_2, \iota_1, \pi_1, \iota_2, \pi_2)$  of such objects and maps is called a *direct sum decomposition* of  $A$ ; the object  $A$  is said to be a *direct sum*  $A_1 + A_2$ .

The terminology and notation used here are those of **(8)** with the following exceptions: The sets  $\text{Hom}(A, B; \mathfrak{K})$  are denoted by  $(A, B)$ ; the identity map  $1 : A$  of  $A$  is written as  $1_A$ . For examples of additive categories, see **(7; 8)** or § 1.4 below.

The theory of additive categories is self-dual in the sense of **(1)**. There is a rule to dualize statements and notions in  $\mathfrak{K}$ : For a statement (or a term) in  $\mathfrak{K}$  the dual statement (or dual term) is obtained by replacing the primitive terms

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\*For the concepts of category, functors, and related notions such as identity map, equivalence, etc., see **(6, chapter IV)**.

$(A, B)$  by  $(B, A)$  (that is, by reversing the maps) and  $\alpha\beta$  by  $\beta\alpha$  (that is, by reversing the composition of maps).

A statement, a system of statements, or a defined term is called *self-dual* if it coincides with its dual statement, system of dual statements, or dual term. The system of axioms of an additive category is easily seen to be self-dual. Hence, with every statement in  $\mathfrak{R}$  which can be derived from the axioms, the dual statement is true. We shall refer to this by saying that a statement follows by duality.

In the following, we sometimes have to dualize statements and notions involving two different additive categories  $\mathfrak{R}$  and  $\mathfrak{Y}$ . There we shall have to specify whether one has to apply the general rule in  $\mathfrak{R}$  or  $\mathfrak{Y}$  or in both. Now, whenever this situation arises in the sequel, the statements and notions can always be given in terms of *one* of the categories, say  $\mathfrak{R}$ . Thus, the dualization will be carried out by applying the above rule in  $\mathfrak{R}$  alone (cf. Theorems 10 and 10\*). In particular, let  $F$  be a functor from  $\mathfrak{R}$  to  $\mathfrak{Y}$ , defined in terms of  $\mathfrak{R}$ , and such that the dual definition gives again a functor. This new functor is then called the *dual functor* of  $F$  (for example, functors  $\Pi_n^i$  and  $\Pi_n^p$  of § 2.6).

In an additive category  $\mathfrak{R}$ , “exactness” and related notions may be introduced with the help of those in the category  $\mathfrak{G}$  of abelian groups, where they are assumed to be known.

Let  $\alpha \in (A, B)$  be a map of  $\mathfrak{R}$ . For any  $C$  of  $\mathfrak{R}$ , one defines a homomorphism  $\alpha_*: (C, A) \rightarrow (C, B)$  by setting  $\alpha_*\phi = \alpha\phi$  for all  $\phi \in (C, A)$ , and a homomorphism  $\alpha^*: (B, C) \rightarrow (A, C)$  by setting  $\alpha^*\phi = \phi\alpha$  for all  $\phi \in (B, C)$ . The map  $\alpha$  is a *monomorphism* or an *epimorphism* if, for all  $C$  of  $\mathfrak{R}$ ,  $\alpha_*$  or  $\alpha^*$  respectively are monomorphisms in the usual sense. Clearly, these two notions are dual.

A short sequence, that is, a sequence of the form

$$O \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \longrightarrow O,$$

is a *short exact sequence*, abbreviated “s.e.s.,” if for all  $C$  of  $\mathfrak{R}$  the two induced sequences in  $\mathfrak{G}$

$$\begin{aligned} O &\longrightarrow (C, A_1) \xrightarrow{\alpha_1^*} (C, A_2) \xrightarrow{\alpha_2^*} (C, A_3) \\ O &\longrightarrow (A_3, C) \xrightarrow{\alpha_2^*} (A_2, C) \xrightarrow{\alpha_1^*} (A_1, C) \end{aligned}$$

are exact in the usual sense.

A sequence

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \dots \longrightarrow A_n \xrightarrow{\alpha_n} A_{n+1}$$

is *exact*, if there are s.e.s.

$$O \longrightarrow I_j \xrightarrow{\mu_j} A_j \xrightarrow{\epsilon_j} I_{j+1} \longrightarrow O \quad \text{for } j = 1, 2, \dots, n,$$

an epimorphism  $\epsilon_0: A_0 \rightarrow I_1$ , and a monomorphism  $\mu_{n+1}: I_{n+1} \rightarrow A_{n+1}$  such that  $\alpha_j = \mu_{j+1}\epsilon_j$ . The term s.e.s., and thus the notion of “exactness,” is self-dual.

An s.e.s.

$$O \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \longrightarrow O$$

splits, if there exist maps  $\alpha_1' \in (A_2, A_1)$  and  $\alpha_2' \in (A_3, A_2)$  such that  $(A_1, A_3, \alpha_1, \alpha_1', \alpha_2', \alpha_2)$  is a direct sum decomposition of  $A_2$ .

If for a map  $\alpha \in (A, B)$  there exist s.e.s.

$$O \rightarrow K \xrightarrow{\iota} A \xrightarrow{\epsilon} I \rightarrow O$$

and

$$O \rightarrow I \xrightarrow{\mu} B \xrightarrow{\pi} C \rightarrow O$$

such that the diagram

$$(1.1) \quad \begin{array}{ccccccc} O & \rightarrow & K & \xrightarrow{\iota} & A & \xrightarrow{\epsilon} & I \rightarrow O \\ & & & & \alpha \downarrow & & \downarrow 1_I \\ O & \leftarrow & C & \xleftarrow{\pi} & B & \xleftarrow{\mu} & I \leftarrow O \end{array}$$

is commutative, these s.e.s. are unique up to equivalence. We therefore refer to  $\alpha = \mu\epsilon$  as the *canonical factorization* of the map  $\alpha$  (if it exists). The objects  $K, I,$  and  $C$  are called *kernel, image,* and *cokernel* of  $\alpha$ ;  $\iota$  and  $\pi$  are called *canonical monomorphism* from the kernel to  $A$  and *canonical epimorphism* from  $B$  to the cokernel. We shall use the following abbreviations:  $K = \text{Ker } \alpha,$   $I = \text{Im } \alpha,$   $C = \text{Coker } \alpha,$  and  $\text{Ker } \alpha \rightarrow A$  resp.  $B \rightarrow \text{Coker } \alpha$  for  $\iota$  resp.  $\pi$ .

A map  $\alpha$  is said to *split* if there exists a canonical factorization of  $\alpha$  such that the s.e.s. of diagram (1.1) split.

**2. Abelian structures.** Let  $\mathfrak{R}$  be an additive category and  $\mathfrak{s}$  a subclass of its maps. The class  $\mathfrak{s}$  is called an *abelian structure* in  $\mathfrak{R}$ , if the following axioms are satisfied for its elements, called *proper maps*:

- (P0) *Every identity map is proper.*
- (P1) *If  $\alpha$  is a proper epimorphism,  $\beta$  is proper, and  $\beta\alpha$  is defined, then  $\beta\alpha$  is proper.*
- (P1\*) *If  $\alpha$  is a proper monomorphism,  $\beta$  is proper, and  $\alpha\beta$  is defined, then  $\alpha\beta$  is proper.*
- (P2) *If  $\beta\alpha$  is a proper monomorphism, then  $\alpha$  is proper.*
- (P2\*) *If  $\alpha\beta$  is a proper epimorphism, then  $\alpha$  is proper.*

An s.e.s. is called *proper s.e.s.* if its maps are proper.

- (P3) *If  $\alpha \in (A, B)$  is proper, there are proper s.e.s.  $O \rightarrow K \rightarrow A \rightarrow I \rightarrow O$  and  $O \rightarrow I \rightarrow B \rightarrow C \rightarrow O$  such that the diagram*

$$(2.1) \quad \begin{array}{ccccccc} O & \rightarrow & K & \rightarrow & A & \rightarrow & I \rightarrow O \\ & & & & \alpha \downarrow & & \downarrow 1_I \\ O & \leftarrow & C & \leftarrow & B & \leftarrow & I \leftarrow O \end{array}$$

*is commutative.*

We briefly say, there exists a *proper canonical factorization* of the map  $\alpha$ ; the objects  $K$  and  $C$  will be called *proper kernel* and *proper cokernel* of  $\alpha$  respectively.

(P4) *If in the commutative diagram*

$$\begin{array}{ccccccc}
 & & O & & O & & O \\
 & & \downarrow & & \downarrow & & \downarrow \\
 O & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & A_3 \rightarrow O \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (2.2) & & O & \rightarrow & B_1 & \rightarrow & B_2 \rightarrow B_3 \rightarrow O \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & O & \rightarrow & C_1 & \rightarrow & C_2 \rightarrow C_3 \rightarrow O \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & O & & O & & O
 \end{array}$$

*all columns and the second two rows are proper s.e.s., then the first row is also a proper s.e.s.*

If the class of all maps of  $\mathfrak{R}$  defines an abelian structure, it is called an *exact structure*. For an exact structure, it is sufficient that axiom (P3) holds, since (P0-2) are obviously redundant, and since it can be shown that (P4) too is redundant (see (1), Theorem 5.5).

The notion of an additive category  $\mathfrak{R}$  with a subclass  $\mathfrak{s}$  of its maps satisfying (P0-4) is due to Heller (8) who calls it an abelian category. It is a generalization of the notion of an exact category (with direct sum axiom) in the sense of Buchsbaum (1), where  $\mathfrak{s}$  is the class of *all* maps of  $\mathfrak{R}$ . It should be noted that the terminology varies in the literature: exact categories of (1) and (8) are called abelian categories by Grothendieck (7). To avoid any misunderstanding, we prefer to use the explicit expression “additive category with abelian structure” instead of “abelian category.” For examples, see § 1.4.

In view of (P3), an abelian structure in  $\mathfrak{R}$  is completely characterized by the class of proper s.e.s.. A class  $\mathfrak{S}$  of s.e.s. is called *abelian* if it is the class of proper s.e.s. of an abelian structure. The axioms (P0, 1, 2, 4) can be translated for an abelian class  $\mathfrak{S}$  (see (8, Proposition 3.2)); axiom (P3) simply becomes a characterization of the class  $\mathfrak{s}$  of proper maps: a map  $\alpha$  belongs to  $\mathfrak{s}$  if  $\alpha = \mu\epsilon$  for an epimorphism  $\epsilon$  of an s.e.s. in  $\mathfrak{S}$  and a monomorphism  $\mu$  of an s.e.s. in  $\mathfrak{S}$ .

As one notices, the dual statement (P4\*) of axiom (P4) is not listed among the axioms of an abelian structure. However, it can be shown that (P4\*) follows from the other axioms (see (8, Proposition 4.1)). This result plus the fact that axiom (P3) is self-dual show that the theory of abelian categories is again completely self-dual.

Finally, let us introduce two notions concerning additive subcategories of an additive category  $\mathfrak{R}$  with abelian structure  $\mathfrak{s}$ .

An additive subcategory  $\mathfrak{R}'$  of  $\mathfrak{R}$  is called *complete*, if for every map  $\alpha \in (A, B)$

of  $\mathfrak{R}$ ,  $A$  and  $B$  in  $\mathfrak{R}'$  imply  $\alpha$  in  $\mathfrak{R}'$ . Thus, a complete additive subcategory  $\mathfrak{R}'$  of  $\mathfrak{R}$  is determined by the class of its objects.

If  $\mathfrak{s}$  is an abelian structure in  $\mathfrak{R}$ , the class  $\mathfrak{s}'$  of those maps of  $\mathfrak{s}$  belonging to  $\mathfrak{R}'$  is called the *induced structure* in  $\mathfrak{R}'$ .  $\mathfrak{s}'$  does not necessarily form an abelian structure in  $\mathfrak{R}'$ ; in order for it to be an abelian structure, it is sufficient that axiom (P3) holds.

**3. Injectives and projectives.** Let  $\mathfrak{R}$  be an additive category and  $\mathfrak{S}$  be a class of short sequences in  $\mathfrak{R}$ . An object  $X$  of  $\mathfrak{R}$  is *injective with respect to  $\mathfrak{S}$*  or  *$\mathfrak{S}$ -injective* if for every sequence

$$O \rightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \rightarrow O$$

in  $\mathfrak{S}$  the induced sequence in  $\mathfrak{G}$

$$(3.1) \quad O \rightarrow (A_3, X) \xrightarrow{\alpha_2^*} (A_2, X) \xrightarrow{\alpha_1^*} (A_1, X) \rightarrow O$$

is exact. If  $\mathfrak{S}$  is abelian,  $\mathfrak{s}$  the abelian structure given by  $\mathfrak{S}$ ,  $X$  is sometimes called  *$\mathfrak{s}$ -injective* or, simply, *injective* (if no confusion is possible).

The *injective closure*  $\text{Cl}^*\mathfrak{S}$  of a class  $\mathfrak{S}$  of short sequences is the class of all short sequences  $O \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow O$  such that for every  $\mathfrak{S}$ -injective  $X$ , the sequence (3.1) is exact. The class  $\mathfrak{S}$  is *injectively closed* if  $\mathfrak{S} = \text{Cl}^*\mathfrak{S}$ .

The notions of  *$\mathfrak{S}$ -projectives*, the *projective closure*  $\text{Cl}_*\mathfrak{S}$ , and *projectively closed* classes  $\mathfrak{S}$  are defined dually. A class  $\mathfrak{S}$  of short sequences is, simply, *closed*, if  $\mathfrak{S} = \text{Cl}^*\mathfrak{S} \cap \text{Cl}_*\mathfrak{S}$ .

This terminology is taken over from (8), where many propositions concerning these notions are found. Let us state only one of them that will be used in the sequel: *if  $\mathfrak{S}$  is an abelian class of s.e.s. in  $\mathfrak{R}$  and  $\mathfrak{T}$  a closed class of short sequences in  $\mathfrak{R}$ , then  $\mathfrak{S} \cap \mathfrak{T}$  is abelian (8, Proposition 7.2).*

Let  $\mathfrak{R}$  be an additive category with abelian structure  $\mathfrak{s}$  and  $\mathcal{S}$  a class of objects of  $\mathfrak{R}$ . The complete additive subcategory  $\mathfrak{R}'$  of  $\mathfrak{R}$  determined by  $\mathcal{S}$  is called a *proper subcategory relative to  $\mathfrak{s}$* , and the objects of  $\mathcal{S}$ , *proper objects*, if the following axioms are satisfied for  $\mathfrak{R}'$ :

- (H) *There is an abelian structure  $\mathfrak{s}'$  in  $\mathfrak{R}'$  contained in the structure induced by  $\mathfrak{s}$ .*
- (H1) *All  $\mathfrak{s}'$ -injectives of  $\mathfrak{R}'$  are  $\mathfrak{s}$ -injectives of  $\mathfrak{R}$ .*
- (H1\*) *All  $\mathfrak{s}'$ -projectives of  $\mathfrak{R}'$  are  $\mathfrak{s}$ -projectives of  $\mathfrak{R}$ .*
- (H2) *There are enough injectives, that is, for every object of  $\mathfrak{R}'$  there is a proper monomorphism to an  $\mathfrak{s}'$ -injective object.*
- (H2\*) *There are enough projectives, that is, for every object of  $\mathfrak{R}'$  there is a proper epimorphism from a  $\mathfrak{s}'$ -projective object.*

If  $\mathcal{S}$  is the class of all objects of  $\mathfrak{R}$ , then  $\mathfrak{R}$  is called a *proper category*. A proper category is an abelian category with enough injectives and projectives. Hence, the notion of an abelian category with a class  $\mathcal{S}$  of proper objects is a

generalization of the notion of an abelian category with enough injectives and projectives (the latter being too restrictive for our purpose).

**4. Examples.** The following examples of abelian structures will not only illustrate the notions introduced above, but will also be used later on.

(a) *The category  $\mathfrak{M}$  of  $R$ -modules.* Let  $R$  be a ring with unit element. For the notion of the category  $\mathfrak{M}$  of  $R$ -modules, see (2, chapter I).

The class  $\mathfrak{t}$  of all homomorphisms of  $\mathfrak{M}$  is known to define an exact structure in  $\mathfrak{M}$ ; furthermore, it is known that there are enough injectives and projectives. Another obviously abelian structure is given by the abelian class  $\mathfrak{S}_0$  of all splitting s.e.s. in  $\mathfrak{M}$ . Relative to this structure, all objects are injective as well as projective.

Let  $R$  and  $S$  be two rings (with unit element) and  $\phi$  a unitary ring-homomorphism from  $S$  to  $R$ .  $\mathfrak{M}_R$  and  $\mathfrak{M}_S$  denoting the categories of  $R$ -modules and  $S$ -modules respectively,  $\phi$  may be used to define an exact covariant functor  $\Phi: \mathfrak{M}_R \rightarrow \mathfrak{M}_S$ . Using the abelian class  $\mathfrak{S}_0$  of splitting s.e.s. in  $\mathfrak{M}_S$  and the abelian class  $\mathfrak{T}$  of all s.e.s. in  $\mathfrak{M}_R$ , the class  $\mathfrak{S}_\phi = \mathfrak{T} \cap \Phi^{-1}\mathfrak{S}_0$  in  $\mathfrak{M}_R$  is abelian (see (8, Proposition 7.3)). The so-defined abelian structure  $\mathfrak{s}_\phi$  in  $\mathfrak{M}_R$  is known as the  *$\phi$ -relative structure*. There are enough  $\mathfrak{s}_\phi$ -injectives and  $\mathfrak{s}_\phi$ -projectives (cf. (2, II.6)).

(b) *The category  $D\mathfrak{M}$  of complexes in  $\mathfrak{M}$ .* The objects of  $D\mathfrak{M}$  are complexes

$$A = (\dots \rightarrow A_i \xrightarrow{\alpha_i} A_{i+1} \rightarrow \dots)$$

of  $R$ -modules  $A_i$  and  $R$ -homomorphisms  $\alpha_i$  ( $\alpha_i\alpha_{i-1} = 0$ ); the maps  $\phi \in (A, A')$  are sequences  $(\phi_i)_{i \in Z}$  of  $R$ -homomorphisms  $\phi_i \in (A_i, A'_i)$  such that  $\phi_{i+1}\alpha_i = \alpha'_i\phi_i$ . For two maps  $\phi = (\phi_i)_{i \in Z}$ ,  $\phi' = (\phi'_i)_{i \in Z}$  of  $D\mathfrak{M}$ , composition  $\phi\phi'$  is defined as  $(\phi_i\phi'_i)_{i \in Z}$ , and sum  $\phi + \phi'$  as  $(\phi_i + \phi'_i)_{i \in Z}$ . Then  $D\mathfrak{M}$  is an additive category, and the class  $\mathfrak{t}$  of all maps of  $D\mathfrak{M}$  can be shown to define an exact structure in  $D\mathfrak{M}$  (see (7, 1.7)).

A further abelian structure in  $D\mathfrak{M}$  is given by the class  $\mathfrak{s}_1$  of all maps  $\phi = (\phi_i)_{i \in Z}$  of  $D\mathfrak{M}$  such that for all  $i \in Z$  the homomorphisms  $\phi_i$  split (cf. § 1.1). It is easy to verify that the axioms (P0-4) hold.

In order to show that  $D\mathfrak{M}$  with abelian structure  $\mathfrak{s}_1$  has enough injectives, for a complex

$$A = (\dots \rightarrow A_i \xrightarrow{\alpha_i} A_{i+1} \rightarrow \dots),$$

a complex

$$X(A) = (\dots \rightarrow X_i \xrightarrow{\xi_i} X_{i+1} \rightarrow \dots)$$

and a map  $\mu = (\mu_i)_{i \in Z} \in (A, X(A))$  is constructed as follows. Put  $X_i = A_{i+1} + A_i$  with direct sum decomposition  $(A_{i+1}, A_i, \iota'_i, \pi'_i, \iota_i, \pi_i)$ , and set  $\xi_i = \iota_{i+1}\pi'_i$  and  $\mu_i = \iota'_i\alpha_i + \iota_i$ . Then  $\xi_i\xi_{i-1} = 0$ , that is,  $X(A)$  is a complex;  $\mu_{i+1}\alpha_i = \xi_i\mu_i$  and  $\pi_i\mu_i = 1_{A_i}$ , that is,  $\mu$  is a proper monomorphism. The proof that  $X(A)$  is  $\mathfrak{s}_1$ -injective (and  $\mathfrak{s}_1$ -projective) is left to the reader. To show that the dual property holds, a complex

$$Y(A) = (\dots \rightarrow Y_i \xrightarrow{\eta_i} Y_{i+1} \rightarrow \dots)$$

and a map  $\epsilon = (\epsilon_i)_{i \in \mathbb{Z}} \in (Y(A), A)$  is given as follows. Put  $Y_i = A_{i-1} + A_i$  with direct sum decomposition  $(A_{i-1}, A_i, \iota_i', \pi_i', \iota_i, \pi_i)$  and set  $\eta_i = \iota_{i+1}'\pi_i$  and  $\epsilon_i = \alpha_{i-1}\pi_i' + \pi_i$ . Then  $\epsilon$  is a proper epimorphism and  $Y(A)$  is  $\mathfrak{s}_1$ -projective.

(c) *The category  $P\mathfrak{R}$  of pairs in an additive category  $\mathfrak{R}$ .* The objects of  $P\mathfrak{R}$  are *pairs in  $\mathfrak{R}$* , that is, maps  $\alpha \in (A_1, A_2)$  of  $\mathfrak{R}$ , denoted by  $[\alpha; A_1, A_2]$  (often by  $\alpha$  only) if they are considered as such objects; the maps  $f \in (\alpha, \alpha')$  are pairs  $\{\phi_1, \phi_2\}$  of maps  $\phi_i \in (A_i, A_i')$  such that  $\alpha'\phi_1 = \phi_2\alpha$ . The maps  $\phi_1$  and  $\phi_2$  are called the *components* of  $f = \{\phi_1, \phi_2\}$ . Composition and sum of maps in  $P\mathfrak{R}$  being defined in terms of the corresponding operations in  $\mathfrak{R}$ ,  $P\mathfrak{R}$  is an additive category.

Let  $\mathfrak{s}$  be an abelian structure in  $\mathfrak{R}$ . Then the *class  $P\mathfrak{s}$  of the maps  $\{\phi_1, \phi_2\}$* , whose components  $\phi_1$  and  $\phi_2$  belong to  $\mathfrak{s}$ , defines an abelian structure in  $P\mathfrak{R}$ . Since the composition in  $P\mathfrak{R}$  is defined by that of  $\mathfrak{R}$ , the axioms (P0-2) are obviously satisfied. Axiom (P3) follows from (P3) in  $\mathfrak{R}$  and from the properties of an s.e.s.. Axiom (P4) is easily verified by the application of (P4) in  $\mathfrak{R}$  to the components.

In the special case where  $\mathfrak{s}$  is an exact structure in  $\mathfrak{R}$ ,  $P\mathfrak{s}$  is an exact structure in  $P\mathfrak{R}$ . Furthermore, if  $\mathfrak{R}$  is a proper category (cf. § 1.3), then so is  $P\mathfrak{R}$ .

Another abelian structure in  $P\mathfrak{R}$ , important for the sequel, is given as follows. Let  $\mathcal{Q}$  be the class of all pairs  $[\xi; X_1, X_2]$  where  $\xi$  splits,  $X_1$  and  $X_2$  are  $\mathfrak{s}$ -injective,  $\mathcal{P}$  the class of all pairs  $[\eta; Y_1, Y_2]$  where  $\eta$  splits,  $Y_1$  and  $Y_2$  are  $\mathfrak{s}$ -projective. Let  $\mathcal{Q}$  denote the class of all short sequences such that all  $\xi$  in  $\mathcal{Q}$  are  $\mathcal{Q}$ -injective, and  $\mathcal{P}$  the class of all short sequences such that all  $\eta$  in  $\mathcal{P}$  are  $P$ -projective. Clearly,  $\mathcal{Q} \cap \mathcal{P}$  is closed. Let  $\mathcal{S}$  denote the abelian class of proper s.e.s. in  $\mathfrak{R}$  with respect to the abelian structure  $\mathfrak{s}$ . Hence, by the proposition on relatively closed structures mentioned in § 1.3 (8, Proposition 7.2), the *class  $P^*\mathcal{S} = P\mathcal{S} \cap \mathcal{Q} \cap \mathcal{P}$  is abelian*. The corresponding class of proper maps will be denoted by  $P^*\mathfrak{s}$ .

The abelian structure  $P^*\mathfrak{s}$  has the following unpleasant features: (i) *The class  $P^*\mathfrak{s}$  need not be an exact structure, even if  $\mathfrak{s}$  is exact.* (ii) *If  $\mathfrak{s}$  is not an exact structure,  $P\mathfrak{R}$  need not be a proper category relative to  $P^*\mathfrak{s}$ , (cf. § 1.3) even if this is the case for  $\mathfrak{R}$  relative to  $\mathfrak{s}$ .* An example to verify (i) is easily found. For an example to verify (ii), let us consider an exact category with an additional non-exact abelian structure  $\mathfrak{s}$  which is proper relative to  $\mathfrak{s}$ . Then one shows that a pair  $\alpha$  admits a proper monomorphism to an injective pair only if the kernel of the map  $\alpha$  is a proper kernel (cf. § 1.2). Hence, a pair  $\alpha$  whose kernel is not proper provides the desired example.

(d) *A proper subcategory  $P^*\mathfrak{R}'$  of  $P\mathfrak{R}$  relative to  $P^*\mathfrak{s}$ .* Let  $\mathfrak{R}$  be an additive category with abelian structure  $\mathfrak{s}$  and a proper subcategory  $\mathfrak{R}'$  of  $\mathfrak{R}$  relative to  $\mathfrak{s}$ . Then a “sufficiently large” proper subcategory of  $P\mathfrak{R}$  relative to  $P^*\mathfrak{s}$ , denoted by  $P^*\mathfrak{R}'$ , is given as follows. The class  $\mathcal{S}$  of proper objects is the class of all pairs  $[\alpha; A_1, A_2]$  such that the map  $\alpha \in (A_1, A_2)$  is a proper map of  $\mathfrak{R}'$ .



Since the objects of  $P^*\mathfrak{R}'$  are given by proper maps  $\alpha \in (A_1, A_2)$  of  $\mathfrak{R}'$ , there is a kernel  $A_0$  of  $\alpha$  and a cokernel  $A_3$  of  $\alpha$ . Thus, for every map  $f = \{\phi_1, \phi_2\}$  from  $\alpha$  to  $\alpha'$  of  $P^*\mathfrak{R}'$ , there are unique maps  $\phi_0 \in (A_0, A_0')$  and  $\phi_3 \in (A_3, A_3')$  of  $\mathfrak{R}'$  such that the diagram

$$\begin{array}{ccccccc} A_0 = \text{Ker } \alpha & \rightarrow & A_1 & \xrightarrow{\alpha} & A_2 & \rightarrow & \text{Coker } \alpha = A_3 \\ & & \phi_0 \downarrow & & \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow \\ A'_0 = \text{Ker } \alpha' & \rightarrow & A'_1 & \xrightarrow{\alpha'} & A'_2 & \rightarrow & \text{Coker } \alpha' = A'_3 \end{array}$$

is commutative. The maps  $\phi_0$  and  $\phi_3$  are referred to as 0th and 3rd *component* of  $f$ . An s.e.s.  $0 \rightarrow \alpha' \rightarrow \alpha \rightarrow \alpha'' \rightarrow 0$  in  $P^*\mathfrak{R}'$  is defined to be proper if the short sequences  $0 \rightarrow A'_i \rightarrow A_i \rightarrow A''_i \rightarrow 0$  ( $i = 0, 1, 2, 3$ ), given by the components, are proper s.e.s. in  $\mathfrak{R}'$ . The proof that this class of proper s.e.s. forms an abelian structure consists in a direct verification and is left to the reader. Furthermore, one can easily show that (i) a proper s.e.s. of  $P^*\mathfrak{R}'$  is a proper s.e.s. of  $P\mathfrak{R}$ ; (ii) all injectives of  $P^*\mathfrak{R}'$  are in  $\mathcal{Q}$ , and all projectives of  $P^*\mathfrak{R}'$  are in  $\mathcal{P}$ . In other words, the axioms (H), (H1), and (H1\*) hold. It remains to show that (H2) is satisfied; (H2\*) then follows by duality.

For a pair  $[\alpha; A_1, A_2]$  of  $P^*\mathfrak{R}'$ , an injective pair  $[\xi; X_1, X_2]$  and a proper monomorphism  $i = \{\kappa_1, \kappa_2\}$  from  $\alpha$  to  $\xi$  are constructed as follows. If  $A_1$  and  $A_2$  are proper, then  $A_3 = \text{Coker } \alpha$  is proper too. Thus, there are proper monomorphisms  $\iota_{A_\kappa} \in (A_\kappa, A_\kappa^i)$  of  $\mathfrak{R}'$  with  $A_\kappa^i$   $\mathfrak{s}$ -injective ( $\kappa = 1, 2, 3$ ). Put  $X_1 = A_1^i + A_2^i$  and  $X_2 = A_2^i + A_3^i$  with direct sum decompositions  $(A_1^i, A_2^i, \iota_1, \pi_1, \iota'_1, \pi'_1)$  and  $(A_2^i, A_3^i, \iota_2, \pi_2, \iota'_2, \pi'_2)$  respectively, and set

$$\xi = \iota_2 \pi'_1, \quad \kappa_1 = \iota_1 \iota_{A_1} + \iota'_1 \iota_{A_2} \alpha, \quad \kappa_2 = \iota_2 \iota_{A_2} + \iota'_2 \iota_{A_3} \pi_\alpha,$$

$\pi_\alpha$  being the canonical epimorphism from  $A_2$  to the cokernel  $A_3$ . Clearly,  $\xi$  is injective. Since  $\xi \kappa_1 = \kappa_2 \alpha$  and  $\pi'_2 \kappa_2 = \iota_{A_3} \pi_\alpha$ ,  $i = \{\kappa_1, \kappa_2\}$  is a map from  $\alpha$  to  $\xi$  with 3rd component  $\kappa_3 = \iota_{A_3}$ . Moreover,  $\pi_1 \kappa_1 = 1_{A_1}$  and  $\pi_2 \kappa_2 = 1_{A_2}$ , so that  $\kappa_1, \kappa_2$ , and  $\kappa_3$  are proper monomorphisms. Thus, by applying the axioms (P1-4) of  $\mathfrak{S}'$ , the map  $i = \{\kappa_1, \kappa_2\}$  is seen to be a proper monomorphism in  $P^*\mathfrak{R}'$ . Hence, (H2) is valid in  $P^*\mathfrak{R}'$ .

In the special case where  $\mathfrak{R}$  is a proper exact category (cf. § 1.3),  $P\mathfrak{R}$  with the structure  $P^*\mathfrak{S}$  is a proper abelian category.

SECTION 2. INJECTIVE AND PROJECTIVE HOMOTOPY GROUPS

**5. i- and p-homotopy.** The two dual concepts of homotopy defined by Eckmann and Hilton (3) for the category  $\mathfrak{M}$  of  $R$ -modules (using the analogy with fundamental properties of homotopy in topology) can be carried over to the following general case: We consider an additive category  $\mathfrak{R}$  with abelian structure  $\mathfrak{s}$  together with a proper subcategory  $\mathfrak{R}'$  relative to  $\mathfrak{s}$  (for the terminology, cf. § 1.2, § 1.3).

DEFINITION. Two maps  $\alpha, \beta \in (A, B)$  are *i-homotopic* ( $\alpha \underset{i}{\sim} \beta$ ) if for every proper monomorphism  $\mu \in (A, A')$  there is a map  $\gamma \in (A', B)$  such that

$\alpha - \beta = \gamma\mu$ ;  $\alpha, \beta \in (B, A)$  are  $p$ -homotopic ( $\alpha \sim_p \beta$ ) if for every proper epimorphism  $\epsilon \in (A', A)$  there is a map  $\delta \in (B, A')$  such that  $\alpha - \beta = \epsilon\delta$ .

The relations  $\alpha \sim_i \beta$  and  $\alpha \sim_p \beta$  both define an equivalence relation for all  $\alpha, \beta \in (A, B)$  which is compatible with the addition in  $(A, B)$ . Thus, the classes of  $i$ -homotopic maps in  $(A, B)$  form an abelian group  $\Pi^i(A, B)$ ; likewise, the classes of  $p$ -homotopic maps in  $(B, A)$  define an abelian group  $\Pi^p(A, B)$ .

For a proper object  $A$  (cf. § 1.3), a map  $\alpha \in (A, B)$  is  $i$ -homotopic to  $o$  if and only if it can be factored through a proper monomorphism  $\iota \in (A, A^i)$  with  $A^i$  injective:  $\alpha = \gamma\iota$  for a  $\gamma \in (A^i, B)$ . Hence, for a proper  $A$ ,  $\Pi^i(A, B)$  can be written as factor group  $(A, B)/\iota^*(A^i, B)$ ,  $\iota^*(A^i, B)$  denoting the image of  $\iota^*: (A^i, B) \rightarrow (A, B)$ ; by duality, it follows that  $\Pi^p(B, A) = (B, A)/\pi^*(B, A^p)$  for a proper epimorphism  $\pi \in (A^p, A)$  with  $A^p$  projective.

PROPOSITION 5.1. Let  $\alpha \in (A, B)$  and  $\beta \in (B, C)$  be two maps.

(i) If  $\alpha \sim_i o$ , then  $\beta\alpha \sim_i o$ .

(ii) If  $\beta \sim_i o$  and  $B$  is a proper object, then  $\beta\alpha \sim_i o$ .

*Proof.* The assertion (i) is trivial. To prove (ii), assume that  $B$  is proper. Then there is a proper monomorphism  $\iota \in (B, B^i)$  with  $B^i$  injective. If  $\beta \sim_i o$ , there is a  $\gamma^i \in (B^i, C)$  such that  $\beta = \gamma^i\iota$ . Let  $\mu \in (A, A')$  be a proper monomorphism. Then there exists a  $\gamma' \in (A', B^i)$  such that  $\alpha\mu = \gamma'\mu$ . Hence  $\gamma = \gamma^i\gamma'$  is a map in  $(A', C)$  with  $\beta\alpha = \gamma\mu$ .

Some consequences of 5.1 will be examined. Let  $\beta \in (B, B')$  be a map of  $\mathfrak{R}$ . If  $\phi \in (A, B) \sim_i o$ , then  $\beta_*\phi \sim_i o$ . Thus, for all  $A$  of  $\mathfrak{R}$ ,  $\beta_*: (A, B) \rightarrow (A, B')$  induces a homomorphism  $\beta_*^i: \Pi^i(A, B) \rightarrow \Pi^i(A, B')$ . Let  $\alpha \in (A', A)$  be a map of  $\mathfrak{R}'$ . If  $\phi \in (A, B) \sim_i o$ , then  $\alpha^*\phi \sim_i o$  (since  $A'$  is proper). Thus, for all  $B$  of  $\mathfrak{R}$ ,  $\alpha^*: (A, B) \rightarrow (A', B)$  induces a homomorphism  $\alpha^{*i}: \Pi^i(A, B) \rightarrow \Pi^i(A', B)$ . The triple of functions  $\Pi^i(A, B)$ ,  $\alpha^{*i}$ ,  $\beta_*^i$  forms then a functor of two variables in  $\mathfrak{R}'$  and in  $\mathfrak{R}$  with values in the category  $\mathfrak{G}$  of abelian groups, that is, a functor  $\Pi^i: \mathfrak{R}' \times \mathfrak{R} \rightarrow \mathfrak{G}$ .  $\Pi^i$  is an additive functor in both variables. If  $\mathfrak{R}$  is a proper abelian category (cf. § 1.3),  $\Pi^i$  is a functor of the category  $\mathfrak{R} \times \mathfrak{R}$ . The abelian groups  $\Pi^i(A, B)$  are the *injective homotopy groups* ( $i$ -homotopy groups) from  $A$  to  $B$ , and the functor  $\Pi^i: \mathfrak{R}' \times \mathfrak{R} \rightarrow \mathfrak{G}$  is called the  *$i$ -homotopy functor*.

The relation  $\alpha \sim_i \beta$  is compatible with the composition of maps, provided that  $\alpha$  and  $\beta$  are maps of  $\mathfrak{R}'$ . Thus, we can obtain from  $\mathfrak{R}'$  a new category  $\mathfrak{R}'^i$  as follows: The objects of  $\mathfrak{R}'^i$  are those of  $\mathfrak{R}'$ , the maps of  $\mathfrak{R}'^i$  are  $i$ -homotopy classes of the maps of  $\mathfrak{R}'$ , composition and sum in  $\mathfrak{R}'^i$  are induced by the corresponding operations in  $\mathfrak{R}'$ . Then  $\mathfrak{R}'^i$  is again an additive category.

The functor  $\Pi^i: \mathfrak{R}' \times \mathfrak{R} \rightarrow \mathfrak{G}$  can be considered as functor  $\Pi^i: \mathfrak{R}'^i \times \mathfrak{R} \rightarrow \mathfrak{G}$ . If  $\mathfrak{R}$  is a proper abelian category,  $\Pi^i$  is a functor of the category  $\mathfrak{R}^i \times \mathfrak{R}^i$ .

Dually, there is a *p-homotopy functor*  $\Pi^p: \mathfrak{R} \times \mathfrak{R}' \rightarrow \mathfrak{G}$  given by the *projective homotopy groups*  $\Pi^p(A, B)$  and the corresponding induced homomorphisms  $\alpha^{*p}$  and  $\beta_{*p}$ . If we denote by  $\mathfrak{R}'^p$  the category where the maps are *p-homotopy classes* of the maps of  $\mathfrak{R}'$ ,  $\Pi^p$  can be considered as functor of  $\mathfrak{R} \times \mathfrak{R}'^p$ . The functors  $\Pi^i$  and  $\Pi^p$  are dual in the sense of § 1.1.

A map  $\phi \in (A', A)$  is an *i-equivalence* between  $A'$  and  $A$  if there exists a  $\psi \in (A, A')$  such that  $\phi\psi \underset{i}{\sim} 1_A$  and  $\psi\phi \underset{i}{\sim} 1_{A'}$ . Then  $A'$  and  $A$  are said to be of the same *i-type*. The notions of *p-equivalence* and *p-type* are defined dually.

**PROPOSITION 5.2.** *Let  $F: \mathfrak{R} \rightarrow \mathfrak{G}$  be an additive functor. If  $\alpha \underset{i}{\sim} o$  in  $\mathfrak{R}$  implies  $F\alpha = o$  in  $\mathfrak{G}$ , then an *i-equivalence*  $\phi$  in  $\mathfrak{R}$  induces an isomorphism  $F\phi$  in  $\mathfrak{G}$ .*

*Proof.* The assertion is a direct consequence of the additivity of  $F$  and of the definition of an *i-equivalence*. Note that if we take an arbitrary additive category  $\mathfrak{X}$  instead of  $\mathfrak{G}$ , then the conclusion would read:  $F\phi$  is an equivalence in  $\mathfrak{X}$ .

Let us examine the following consequences of 5.1 and 5.2. By 5.1, the assumption of 5.2 holds for both variables of the functor  $\Pi^i: \mathfrak{R}' \times \mathfrak{R} \rightarrow \mathfrak{G}$ . Thus, for a proper object  $A$ , the groups  $\Pi^i(A, B)$  are determined by the *i-type* of  $A$  and  $B$ , up to isomorphisms.

Clearly, an *i-equivalence* in  $\mathfrak{R}'$  induces an equivalence in  $\mathfrak{R}'^i$ , and vice versa. Let us consider, in particular, the situation where two functors  $F$  and  $F': \mathfrak{R}'^i \rightarrow \mathfrak{R}'^i$  and a natural equivalence  $h: F \rightarrow F'$  are given. The family of homomorphisms  $h(A)^{*i}: \Pi^i(F'A, B) \rightarrow \Pi^i(FA, B)$  defines then a natural isomorphism:  $\Pi^i(F' \times 1) \rightarrow \Pi^i(F \times 1)$  ( $1$  being the identity functor of  $\mathfrak{R}$ ).

The dual arguments are left to the reader.

It should be noted that the notions introduced in this section depend on the abelian structure  $\mathfrak{s}$  chosen in  $\mathfrak{R}$ . In an additive category  $\mathfrak{R}$ , there are in general several abelian structures; the one used to define *i-* and *p-homotopy* will be called *homotopy-structure*. In the special case where  $\mathfrak{R}$  has enough injectives and projectives relative to  $\mathfrak{s}$ , the proper “subcategory”  $\mathfrak{R}'$  will be  $\mathfrak{R}$  itself. Otherwise, a proper subcategory  $\mathfrak{R}'$  must be given explicitly.

**6. Suspensions.** A *suspension* of an object  $A$  is an object  $\Sigma A$  which can be embedded in a proper s.e.s. of  $\mathfrak{R}$

$$(6.1) \quad 0 \rightarrow A \xrightarrow{\iota} A^i \xrightarrow{\pi} \Sigma A \rightarrow 0$$

with  $A^i$  injective. For a proper object  $A$ , there always exists a suspension  $\Sigma A$ . Let  $\phi \in (A', A)$  be a map of  $\mathfrak{R}'$ . Then there are maps  $\phi^i \in (A'^i, A^i)$  and  $\Sigma\phi \in (\Sigma A', \Sigma A)$  such that the diagram

$$(6.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & A' & \xrightarrow{\iota'} & A'^i & \xrightarrow{\pi'} & \Sigma A \rightarrow 0 \\ & & \phi \downarrow & & \phi^i \downarrow & & \Sigma\phi \downarrow \\ 0 & \rightarrow & A & \xrightarrow{\iota} & A^i & \xrightarrow{\pi} & \Sigma A \rightarrow 0 \end{array}$$

is commutative.

PROPOSITION 6.1. *Let  $\phi \in (A', A)$  be a map of  $\mathbb{R}'$ , and  $\Sigma A'$  and  $\Sigma A$  arbitrary suspensions of  $A'$  and  $A$ . Then the map  $\Sigma\phi$  in diagram (6.2) has the following properties:*

- (i) *The  $i$ -homotopy class of  $\Sigma\phi$  is independent of the choice of  $\phi^i$ .*
- (ii) *If  $\phi \underset{i}{\sim} o$ , then  $\Sigma\phi \underset{i}{\sim} o$ .*
- (iii) *If  $\phi$  is an  $i$ -equivalence, then so is  $\Sigma\phi$ .*

*Proof.* (i) Let  $\{\phi_1^i, \Sigma\phi_1\}$  and  $\{\phi_2^i, \Sigma\phi_2\}$  be two pairs of maps which can be embedded in a diagram of the form (6.2). Put  $\phi_2^i - \phi_1^i = \alpha^i$ ,  $\Sigma\phi_2 - \Sigma\phi_1 = \alpha$ ; then diagram (6.2) is again commutative for  $\phi = o$ ,  $\phi^i = \alpha i$ ,  $\Sigma\phi = \alpha$ . Since  $\alpha^i i' = o$ , there is a  $\delta \in (\Sigma A', A^i)$  such that  $\alpha^i = \delta\pi'$ .  $\alpha\pi' = \pi\delta\pi'$  and  $\pi'$  epimorphic imply  $\alpha = \pi\delta$ . Clearly,  $\delta \underset{i}{\sim} o$ ; hence,  $\alpha \underset{i}{\sim} o$ , that is,  $\Sigma\phi_1 \underset{i}{\sim} \Sigma\phi_2$ .

(ii) If  $\phi \underset{i}{\sim} o$ , there is a  $\gamma \in (A'^i, A)$  such that  $\phi = \gamma i'$ . Thus,  $\phi^i$  can be chosen as  $\phi^i = \iota\gamma$ . Then  $\pi\phi^i = \Sigma\phi\pi' = o$ , so that  $\Sigma\phi = o$ . By (i), we have  $\Sigma\phi \underset{i}{\sim} o$  for any choice of  $\phi^i$ .

(iii) If  $\phi \in (A', A)$  is an  $i$ -equivalence, there is a  $\psi \in (A, A')$  such that  $\phi\psi - 1_A \underset{i}{\sim} o$  and  $\psi\phi - 1_{A'} \underset{i}{\sim} o$ . The maps  $\Sigma(\phi\psi - 1_A)$  and  $\Sigma(\psi\phi - 1_{A'})$  can be chosen as  $\Sigma\psi\Sigma\phi - 1_{\Sigma A}$  and  $\Sigma\phi\Sigma\psi - 1_{\Sigma A'}$  respectively. By (ii), we have therefore  $\Sigma\psi\Sigma\phi - 1_{\Sigma A} \underset{i}{\sim} o$  and  $\Sigma\phi\Sigma\psi - 1_{\Sigma A'} \underset{i}{\sim} o$ .

Assigning to each proper  $A$  an s.e.s. of the form (6.1), by 6.1 the pair of functions  $\Sigma A, \Sigma\phi$  forms a covariant functor  $\Sigma: \mathbb{R}'^i \rightarrow \mathbb{R}'^i$ . Besides,  $\Sigma$  is an additive functor. For two different assignments, the functors  $\Sigma$  obtained coincide up to natural  $i$ -equivalences.

The definition of the functor  $\Sigma$  may be iterated by setting  $\Sigma^n = \Sigma\Sigma^{n-1}$  for  $n > 1$ . Thus, we obtain additive covariant functors  $\Sigma^n: \mathbb{R}'^i \rightarrow \mathbb{R}'^i$ ,  $n \geq 1$ , again determined up to natural  $i$ -equivalences, whatever the choice of the s.e.s. (6.1).  $\Sigma^n$  is called  $n$ -fold suspension functor,  $\Sigma^n A$   $n$ -fold suspension of  $A$ .

Let us consider, first for a fixed assignment of s.e.s. (6.1), the composite additive functors  $\Pi^i(\Sigma^n \times 1): \mathbb{R}'^i \times \mathbb{R} \rightarrow \mathfrak{U}$  (1 being the identity functor of  $\mathbb{R}$ ). By the observation made at the end of § 2.5,  $\Pi^i(\Sigma^n \times 1)$  is determined up to natural isomorphisms, whatever the choice of the s.e.s. (6.1). The groups  $\Pi^i(\Sigma^n A, B)$  are the  $n$ th injective homotopy groups  $\Pi_n^i(A, B)$ , and the functor  $\Pi^i(\Sigma^n \times 1): \mathbb{R}'^i \times \mathbb{R} \rightarrow \mathfrak{U}$  is called the  $n$ th  $i$ -homotopy functor  $\Pi_n^i$ ,  $n \geq 0$ , ( $\Pi_0^i = \Pi^i$ )\*

A dual suspension of an object  $A$  is an object  $\Omega A$  which can be embedded in a proper s.e.s. of  $\mathbb{R}$

$$(6.1^*) \quad O \rightarrow \Omega A \rightarrow A \rightarrow A \rightarrow 0$$

\*It should be noted that the suspension  $\Sigma A$  plays the same rôle as the suspension of a topological space.

with  $A^p$  projective. Dually, we define  $n$ -fold dual suspension functors  $\Omega^n: \mathfrak{R}'^p \rightarrow \mathfrak{R}'^p$  and the composite functors  $\Pi^p(1 \times \Omega^n): \mathfrak{R} \times \mathfrak{R}'^p \rightarrow \mathfrak{G}$ . The groups  $\Pi^p(B, \Omega^n A)$  are the  $n$ th projective homotopy groups  $\Pi_n^p(A, B)$ ; the functor  $\Pi^p(1 \times \Omega^n)$  is denoted by  $\Pi_n^{p,*}$ .

It is easily seen that, for each proper object  $A$ , there is an injective resolution  $X$  in  $\mathfrak{R}'$ , that is, a proper exact sequence

$$0 \rightarrow A \xrightarrow{\partial_1} X_1 \xrightarrow{\partial_2} X_2 \rightarrow \dots$$

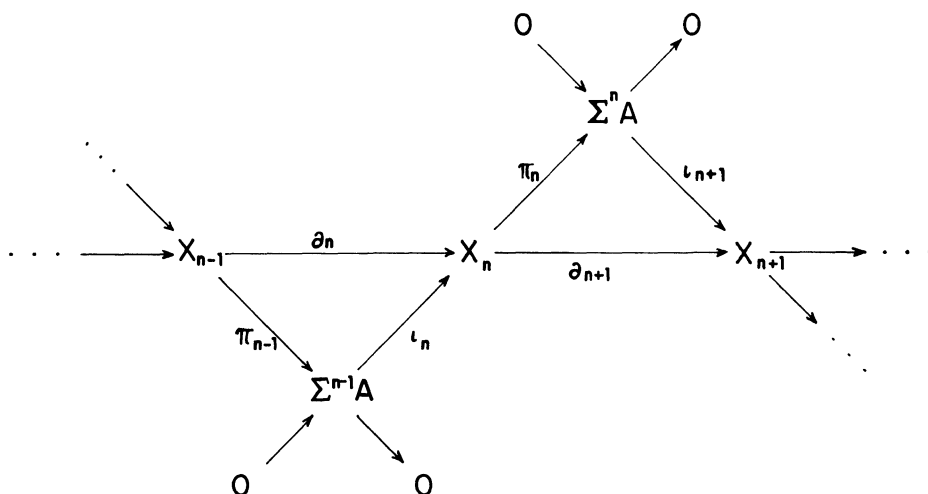
in  $\mathfrak{R}'$  where all  $X_i$  are injective. For any object  $B$ , let us denote the homology groups of the induced complex

$$(X, B): \dots \rightarrow (X_2, B) \xrightarrow{\partial_2^*} (X_1, B) \xrightarrow{\partial_1^*} (A, B) \rightarrow 0$$

as follows:  $H_n^i(A, B) = \text{Ker } \partial_n^* / \text{Im } \partial_{n+1}^*$  ( $n \geq 1$ ),  $H_0^i(A, B) = (A, B) / \text{Im } \partial_1^*$ . Defining in the usual way the induced homomorphisms of the homology groups, we obtain for a fixed resolution of  $A$  a sequence of functors  $H_n^i: \mathfrak{R}' \times \mathfrak{R} \rightarrow \mathfrak{G}$  ( $n = 0, 1, 2, \dots$ ). It will be shown that  $\Pi_n^i = H_n^i$  up to natural isomorphisms.

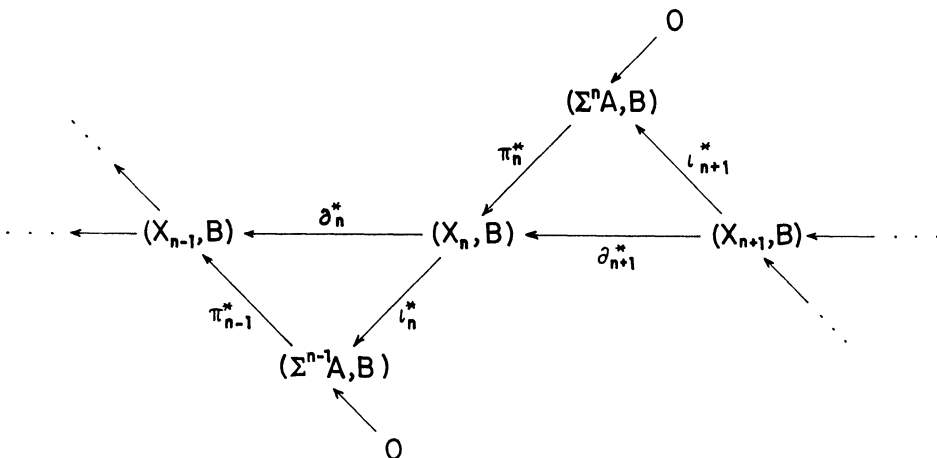
**PROPOSITION 6.2.** *Let  $\partial_{n+1} = \iota_{n+1}\pi_n$  be a canonical factorization of  $\partial_{n+1} \in (X_n, X_{n+1})$ . Then the  $\pi_n^*$  induce a family of isomorphisms  $p_n(A, B): \Pi_n^i(A, B) \rightarrow H_n^i(A, B)$  ( $n = 0, 1, 2, \dots$ ).*

*Proof.* Obviously, in an injective resolution  $X$  of  $A$ , the objects  $\text{Im } \partial_{n+1}^*$  are  $n$ -fold suspensions  $\Sigma^n A$  of  $A$ . Thus, the canonical factorizations  $\partial_{n+1} = \iota_{n+1}\pi_n$  give a commutative diagram



\*Again, it should be noted that the dual suspension  $\Omega A$  plays the same rôle as the loop space of a topological space.

where the diagonals are proper s.e.s. For any object  $B$ , the following induced diagram in  $\mathfrak{G}$



is given, where the diagonals are exact. By a standard argument in  $\mathfrak{G}$ , we obtain  $\text{Ker } \partial_n^* = \text{Ker } \iota_n^* = \pi_n^*(\Sigma^n A, B)$  and  $\text{Im } \partial_{n+1}^* = \pi_n^*(\iota_{n+1}^*(X_{n+1}, B))$ . In addition, the  $\pi_n^*$  are monomorphisms, and consequently isomorphisms between  $(\Sigma^n A, B)$  and  $\text{Ker } \partial_n^*$  as well as between  $\iota_{n+1}^*(X_{n+1}, B)$  and  $\text{Im } \partial_{n+1}^*$ . Thus, the  $\pi_n^*$  induce isomorphisms  $p_n(A, B)$  between the factor groups  $\Pi_n^i(A, B) = (\Sigma^n A, B) / \iota_{n+1}^*(X_{n+1}, B)$  and  $H_n^i(A, B) = \text{Ker } \partial_n^* / \text{Im } \partial_{n+1}^*$ .

By a straightforward verification, it is seen that the family  $p_n(A, B)$  defines a natural isomorphism  $p_n: \Pi_n^i \rightarrow H_n^i$ . We have thus shown in a precise way that  $\Pi_n^i = H_n^i$  up to natural isomorphisms. This subsequently justifies the notation  $H_n^i(A, B)$ .

There is a dual story in which the injective resolution is replaced by a projective one. The homology functors  $H_n^p$  obtained coincide then with the homotopy functors  $\Pi_n^p$ .

*Remark.* Proposition 6.2 says, among other things, that the functors  $\Pi_n^i$  can be given in two different ways: first, by means of the  $n$ -fold suspension, and second, as the homology functor of a complex constructed by means of the injective resolution. In the framework of (2), the first way of introducing  $\Pi_n^i$  corresponds to the construction of the *left-satellite functors* of  $\Pi^i$  with respect to the contravariant variable (2, III.1); the second way is (up to an extension of the complexes  $(X, B)$  by a term to the right) the construction of the *partially left-derived functors* of  $\text{Hom}$  with respect to the contravariant variable (2, V. 8). A dual interpretation can be given for the functors  $\Pi_n^p$ .

**7. Pairs.** The concept of “relative” homotopy groups in topology refers in general to a pair of spaces  $A_1 \subset A_2$ . The corresponding notion in an additive category is that of a monomorphism  $\alpha: A_1 \rightarrow A_2$ . Since the subse-

quent arguments can be made as well for arbitrary maps, we shall understand by a pair  $[\alpha; A_1, A_2]$  in  $\mathfrak{R}$  (often denoted by  $\alpha$  only) any map  $\alpha \in (A_1, A_2)$ .

Let us consider the additive category  $P\mathfrak{R}$  of pairs in  $\mathfrak{R}$  (cf. § 1.4). As homotopy structure in  $P\mathfrak{R}$  we choose the abelian structure  $P^*\mathfrak{R}$  introduced in Example (c) of § 1.4, and as proper subcategory of  $P\mathfrak{R}$ , we take the subcategory  $P^*\mathfrak{R}'$  introduced in Example (d) of § 1.4. Then it is clear what one is to understand by the functors  $\Pi_n^i: (P^*\mathfrak{R}')^i \times P\mathfrak{R} \rightarrow \mathfrak{U}$  and  $\Pi_n^p: P\mathfrak{R} \times (P^*\mathfrak{R}')^p \rightarrow \mathfrak{U}$ .

For a proper object  $A$ ,  $\iota A$  is a pair  $[\iota; A, A^i]$  given by a proper monomorphism  $\iota$  of  $\mathfrak{R}'$  from  $A$  to an injective  $A^i$ . Clearly,  $\iota A$  is an object of  $P^*\mathfrak{R}'$ . For a map  $\phi \in (A', A)$  of  $\mathfrak{R}'$ , there always exists a map  $\phi^i \in (A'^i, A^i)$  such that  $\iota\phi = \{\phi, \phi^i\}$  is a map from  $\iota A'$  to  $\iota A$ .

**PROPOSITION 7.1.** *Let  $\phi \in (A', A)$  be a map of  $\mathfrak{R}'$ . Then the map  $\iota\phi$  from  $\iota A$  to  $\iota A'$  has the following properties:*

- (i) *The  $i$ -homotopy class of  $\iota\phi$  in  $P\mathfrak{R}$  is independent of the choice of  $\phi^i$ .*
- (ii) *If  $\phi \underset{i}{\sim} o$ , then  $\iota\phi \underset{i}{\sim} o$ .*
- (iii) *If  $\phi$  is an  $i$ -equivalence, then so is  $\iota\phi$ .*

*Proof.* (i) Let  $\phi_1^i$  and  $\phi_2^i$  be two maps such that  $\{\phi, \phi_{1,2}^i\}$  are maps from  $\iota A'$  to  $\iota A$ . Now,  $\{\phi, \phi_2^i\} - \{\phi, \phi_1^i\} = \{o, \phi_2^i - \phi_1^i\}$  is easily seen to be  $i$ -homotopic to  $o$ .

(ii) If  $\phi \underset{i}{\sim} o$ , there is a  $\gamma \in (A'^i, A)$  such that  $\phi = \gamma\iota'$ . Thus,  $\phi^i$  can be chosen as  $\phi^i = \iota\gamma$ . Now,  $\iota\phi$  can be factored through a conveniently chosen proper monomorphism  $i$  from  $\iota A'$  to an injective pair.

(iii) The proof is analogous to that of proposition 6.1 (iii).

Assigning to each proper object  $A$  an s.e.s. of the form (6.1), that is, a determinate pair  $\iota A$ , by 7.1 the pair of functions  $\iota A, \iota\phi$  forms a covariant functor  $\iota: \mathfrak{R}'^i \rightarrow (P^*\mathfrak{R}')^i$ . This functor  $\iota$  is additive and determined up to natural  $i$ -equivalences, whatever the choice of  $A^i$  and  $\iota \in (A, A^i)$ .

With the functors  $\Sigma^n: \mathfrak{R}'^i \rightarrow \mathfrak{R}'^i, \iota: \mathfrak{R}'^i \rightarrow (P^*\mathfrak{R}')^i$ , determined up to natural  $i$ -equivalences, whatever the choice of the s.e.s. (6.1), and with the functor  $\Pi^i: (P^*\mathfrak{R}')^i \times P\mathfrak{R} \rightarrow \mathfrak{U}$  we can for each  $n \geq 1$  construct the composite functor  $\Pi^i(\iota\Sigma^{n-1} \times 1): \mathfrak{R}'^i \times P\mathfrak{R} \rightarrow \mathfrak{U}$  (1 being the identity functor of  $P\mathfrak{R}$ ). This functor is again additive and determined up to natural isomorphisms. It assigns to each proper object  $A$  and each pair  $\beta$  the "mixed" homotopy group  $\Pi^i(\iota\Sigma^{n-1}A, \beta)$ , called  $n$ th "relative" injective homotopy group  $P_n^i(A, \beta)$  from  $A$  to the pair  $\beta$ ; the functor  $\Pi^i(\iota\Sigma^{n-1} \times 1)$  is denoted by  $P_n^i, n \geq 1$ .\*

For a proper object  $A, \epsilon A$  is a pair  $[\epsilon; A^p, A]$  given by a proper epimorphism  $\epsilon$  of  $\mathfrak{R}'$  from a projective  $A^p$  to  $A$ . A dual construction to that of the functor  $\iota$

\*Note that the pair  $\iota\Sigma^{n-1}A$  is the analogue of the pair denoted in (4) by  $\iota_n A$  (cf. (4, p. 274)).

leads to the additive covariant functor  $\epsilon$  and for each  $n \geq 1$  to the definition of the composite functors  $\Pi^p(1 \times \epsilon\Omega^{n-1})$ . The group  $\Pi^p(\beta, \epsilon\Omega^{n-1}A)$ , is called the  $n$ th “relative” projective homotopy group  $P_n^p(\beta, A)$ ; the functor  $\Pi^p(1 \times \epsilon\Omega^{n-1})$  is denoted by  $P_n^p$ .†

The choice of  $P^*\mathfrak{B}$  as homotopy structure in  $P\mathfrak{K}$  enables us to identify (as in the topological case) the following “relative” homotopy groups with “absolute” homotopy groups:

$$(7.1) \quad P_n^i(A, \omega B) = \Pi_{n-1}^i(A, B) \text{ for } \omega B = [\omega; B, O],$$

$$(7.2) \quad P_n^i(A, \tilde{\omega}B) = \Pi_n^i(A, B) \text{ for } \tilde{\omega}B = [\tilde{\omega}; O, B];$$

dually,

$$(7.1^*) \quad P_n^p(\tilde{\omega}B, A) = \Pi_{n-1}^p(B, A) \text{ for } \tilde{\omega}B = [\tilde{\omega}, O, B],$$

$$(7.2^*) \quad P_n^p(\omega B, A) = \Pi_n^p(B, A) \text{ for } \omega B = [\omega, B, O].$$

We shall show in a precise way how these identifications are possible. Assigning to each map  $\phi \in (B, B')$  of  $\mathfrak{K}$  the maps  $\omega\phi = \{\phi, o\}$  from  $\omega B$  to  $\omega B'$  and  $\tilde{\omega}\phi = \{o, \phi\}$  from  $\tilde{\omega}B$  to  $\tilde{\omega}B'$ , we obtain additive covariant functors  $\omega$  and  $\tilde{\omega}: \mathfrak{K} \rightarrow P\mathfrak{K}$ .

PROPOSITION 7.2. *The homomorphisms  $\sigma: (\Sigma^{n-1}A, B) \rightarrow (\iota\Sigma^{n-1}A, \omega B)$ , defined by  $\sigma\phi = \{\phi, o\}$  for all  $\phi \in (\Sigma^{n-1}A, B)$ , induce a family of isomorphisms  $s(A, B): \Pi_{n-1}^i(A, B) \rightarrow P_n^i(A, \omega B)$  ( $n = 1, 2, \dots$ ).*

*Proof.* It is easy to see that  $\sigma$  induces a homomorphism of the respective homotopy classes. The inverse homomorphism of  $\sigma$  is defined by  $\sigma^{-1}\{\phi_1, \phi_2\} = \phi_1$  for all  $\{\phi_1, \phi_2\} \in (\iota\Sigma^{n-1}A, \omega B)$ , and induces again a homomorphism of the homotopy classes.

PROPOSITION 7.3. *The homomorphisms  $\tilde{\sigma}: (\iota\Sigma^{n-1}A, \tilde{\omega}B) \rightarrow (\Sigma^n A, B)$ , defined by  $\tilde{\sigma}\{\phi_1, \phi_2\} = \phi_3$  (3rd component of  $\{\phi_1, \phi_2\}$ ) for all  $\{\phi_1, \phi_2\} \in (\iota\Sigma^{n-1}A, \tilde{\omega}B)$ , induce a family of isomorphisms  $\tilde{s}(A, B): P_n^i(A, \tilde{\omega}B) \rightarrow \Pi_n^i(A, B)$  ( $n = 1, 2, \dots$ ).*

*Proof.* It is easy to see that, with  $P^*\mathfrak{B}$  as homotopy structure in  $P\mathfrak{K}$ ,  $\tilde{\sigma}$  induces a homomorphism of the respective homotopy classes. Assume that  $\iota\Sigma^{n-1}A$  and  $\pi \in (X, \Sigma^n A)$  are given by the proper s.e.s.

$$O \rightarrow \Sigma^{n-1}A \xrightarrow{\iota} X \xrightarrow{\pi} \Sigma^n A \rightarrow O.$$

Then the inverse homomorphism of  $\tilde{\sigma}$  is defined by  $\tilde{\sigma}^{-1}\phi = \{o, \phi\pi\}$  for all  $\phi \in (\Sigma^n A, B)$ , and induces again a homomorphism of the homotopy classes.

By a straightforward verification, it is seen that the families  $s(A, B)$  and  $\tilde{s}(A, B)$  define natural isomorphisms  $s: \Pi_{n-1}^i \rightarrow P_n^i(1 \times \omega)$  and  $\tilde{s}: P_n^i(1 \times \tilde{\omega}) \rightarrow \Pi_n^i$  for all  $n \geq 1$ . The identifications (7.1) and (7.2) will be made under these natural isomorphisms. By duality, one obtains the identification (7.1\*) and (7.2\*).

†Note that the pair  $\epsilon\Omega^{n-1}A$  is the analogue of the pair denoted in (4) by  $\rho_n A$  (cf. (4, p. 274)).



With the functors  $\Sigma^n: (P^*\mathfrak{R}')^i \rightarrow (P^*\mathfrak{R}')^i$ ,  $\iota: \mathfrak{R}'^i \rightarrow (P^*\mathfrak{R}')^i$  and  $\Pi^i: (P^*\mathfrak{R}')^i \times P\mathfrak{R} \rightarrow \mathfrak{G}$ , we can construct the composite functor  $\Pi_n^i(\iota \times 1) = \Pi^i(\Sigma^n \iota \times 1): \mathfrak{R}'^i \times P\mathfrak{R} \rightarrow \mathfrak{G}$ . It will be shown that  $P_n^i = \Pi_{n-1}^i(\iota \times 1)$ , up to natural isomorphisms.

Consider the proper s.e.s. in  $\mathfrak{R}$ :

$$O \rightarrow A \xrightarrow{\iota} A^i \xrightarrow{\pi} \Sigma A \rightarrow O, \quad O \rightarrow \Sigma A' \xrightarrow{\iota} (\Sigma A)' \xrightarrow{\pi'} \Sigma^2 A \rightarrow O$$

with  $A^i, (\Sigma A)^i$  injective.

LEMMA 7.4. For the pair  $\iota A = [\iota; A, A^i]$ , a standard injective pair  $\xi A$  is given by  $\xi A = [\xi; A^i, A^i + (\Sigma A)^i]$  where  $\xi$  is a splitting monomorphism, and a (standard) suspension  $\Sigma \iota A$  of  $\iota A$  is given by  $\iota \Sigma A$ .

Proof. The pair  $\xi A = [\xi; X_1, X_2]$  and a map  $i = \{\kappa_1, \kappa_2\}$  from  $\iota A$  to  $\xi A$  are constructed as follows. Put  $X_1 = A^i, X_2 = A^i + (\Sigma A)^i$  with direct sum decomposition  $(A^i, (\Sigma A)^i, \iota_2, \pi_2, \iota_2', \pi_2')$ , and set  $\xi = \iota_2, \kappa_1 = \iota$  and  $\kappa_2 = \iota_2 \iota + \iota_2' \pi$ . Clearly,  $\xi$  is injective, and  $i$  is a monomorphism with 3rd component  $\kappa_3 = \iota'$ . In order to see that  $i$  is a proper monomorphism, and in order to examine the suspension  $\Sigma \iota A$  determined by  $i$ , let us consider the commutative diagram

$$(7.3) \quad \begin{array}{ccccc} & O & O & O & \\ & \downarrow & \downarrow & \downarrow & \\ O \rightarrow A & \xrightarrow{\iota} & A^i & \xrightarrow{\pi} & \Sigma A \rightarrow O \\ \kappa_1 = \iota \downarrow & & \downarrow \kappa_2 & & \downarrow \kappa_3 = \iota' \\ O \rightarrow A^i & \xrightarrow{\iota_2} & A^i + (\Sigma A)^i & \xrightarrow{\pi_2'} & (\Sigma A)^i \rightarrow O \\ & \pi \downarrow & \downarrow & & \downarrow \pi' \\ O \rightarrow \Sigma A & \rightarrow & \text{Coker } \kappa_2 & \rightarrow & \Sigma^2 A \rightarrow O \\ & \downarrow & \downarrow & & \downarrow \\ & O & O & & O \end{array}$$

where the columns are proper s.e.s. The maps in the last row are uniquely determined. Since the first two rows are proper s.e.s., by (P4\*), the last row is a proper s.e.s., too. Hence,  $i$  is a proper monomorphism. Furthermore, the second column splits, so that  $\text{Coker } \kappa_2$  is injective. Hence,  $\Sigma \iota A = \iota \Sigma A$ .

By induction with respect to  $n$ , it follows from 7.4 that  $\iota \Sigma^n A$  is an  $n$ -fold suspension of  $\iota A$ . Thus,  $\iota \Sigma^n = \Sigma^n \iota$  up to natural  $i$ -equivalences, and therefore  $P^i = \Pi_{n-1}^i(\iota \times 1)$  up to natural isomorphisms.

Finally, we state a simple consequence of the result just given and of Proposition 6.2. Let  $X$  be an injective resolution in  $P^*\mathfrak{R}'$  of the proper pair  $\iota A$ :

$$o \rightarrow \iota A \xrightarrow{d_1} \xi_1 \xrightarrow{d_2} \xi_2 \rightarrow \dots$$

For any pair  $\beta$ , let  $H_n^i(\iota A, \beta)$  denote the homology groups of the induced complex  $(X, \beta)$ . Then we have

PROPOSITION 7.5. *Let  $d_{n+1} = i_{n+1}p_n$  be a canonical factorization of  $d_{n+1} \in (\xi_n, \xi_{n+1})$ . Then the  $p_n^*$  induce a family of isomorphisms  $p_n(A, \beta): P_{n+1}^i(A, \beta) \rightarrow H_n^i(\iota A, \beta)$  ( $n = 0, 1, 2, \dots$ ).*

For a fixed resolution of  $\iota A$ , a homology functor  $H_n^i: \mathfrak{R}' \times P\mathfrak{R} \rightarrow \mathfrak{G}$  can be defined by means of the groups  $H_n^i(\iota A, \beta)$ . The family  $p_n(A, \beta)$  defines then a natural isomorphism  $p_n: P_{n+1}^i \rightarrow H_n^i$ .

REMARK. It should be noted that, in choosing the homotopy structure in  $P\mathfrak{R}$ , we did not take the abelian structure  $P\mathfrak{s}$  (cf. § 1.4, Example (c)) which would have been the most obvious one to choose. The reason for this is: if the homotopy structure is  $P\mathfrak{s}$ , the identifications  $P_n^i(A, \omega B) = \Pi_n^i(A, B)$  and  $P_n^p(\omega B, A) = \Pi_n^p(B, A)$  cannot be obtained. If we insist on these identifications (which are essential in the topological analogue), the most natural choice to define homotopy in  $P\mathfrak{R}$  is the class  $P^*\mathfrak{s}$ . This choice has been made, although  $P^*\mathfrak{s}$  has an unpleasant feature: it is not necessarily an exact structure, even if  $\mathfrak{s}$  is exact. Hence, in order to avoid having to develop the whole theory again in the category of pairs, we were compelled to carry out the procedure of §§ 1.5 and 1.6 in additive categories with abelian structure (instead of in exact categories). But now an additional feature of  $P^*\mathfrak{s}$  appears: If  $\mathfrak{s}$  is not an exact structure,  $P\mathfrak{R}$  has not enough injectives and projectives (relative to  $P\mathfrak{s}$ ), even if this is the case for  $\mathfrak{R}$  (relative to  $\mathfrak{s}$ ). To face this new difficulty, we introduced the concept of a proper subcategory  $\mathfrak{R}'$  of  $\mathfrak{R}$  relative to  $\mathfrak{s}$ .

**8. Double pairs.** The introduction of pairs and “relative” homotopy groups can be done in the category  $P\mathfrak{R}$  of pairs in  $\mathfrak{R}$  with homotopy structure  $P^*\mathfrak{s}$  in the very same way as in the category  $\mathfrak{R}$  with homotopy structure  $\mathfrak{s}$ . To do this, one considers the category  $P^2\mathfrak{R} = P(P\mathfrak{R})$  of pairs in  $P\mathfrak{R}$  with homotopy structure  $P^*(P^*\mathfrak{s})$  and the proper subcategory  $P^*(P^*\mathfrak{R}')$  of  $P^2\mathfrak{R}$ . Then it is clear what one is to understand by the functors  $\Pi_n^i: (P^*(P^*\mathfrak{R}'))^i \times P^2\mathfrak{R} \rightarrow \mathfrak{G}$ .

The objects of  $P^2\mathfrak{R}$  are pairs  $[a; \alpha_1, \alpha_2]$  of pairs  $\alpha_1, \alpha_2$  in  $\mathfrak{R}$  (often denoted by  $a$  only). They will also be called *double pairs in  $\mathfrak{R}$* , and  $P^2\mathfrak{R}$  the category of double pairs in  $\mathfrak{R}$ .

Let  $i$  denote the functor from  $(P^*\mathfrak{R}')^i$  to  $(P^*(P^*\mathfrak{R}'))^i$  which corresponds to the functor  $\iota$  from  $\mathfrak{R}'^i$  to  $(P^*\mathfrak{R}')^i$  (cf. § 2.7). By means of  $i$ , the functors  $P_n^i: (P^*\mathfrak{R}')^i \times P^2\mathfrak{R} \rightarrow \mathfrak{G}$  are constructed. Moreover, we have now a composite functor  $k = i\iota: \mathfrak{R}'^i \rightarrow (P^*(P^*\mathfrak{R}'))^i$ , determined up to natural  $i$ -equivalences. This functor suggests for each  $n \geq 2$  the definition of an additive functor  $\Pi^i(k\Sigma^{n-2} \times 1): \mathfrak{R}'^i \times P^2\mathfrak{R} \rightarrow \mathfrak{G}$  ( $1$  being the identity functor of  $P^2\mathfrak{R}$ ), determined up to natural isomorphisms. It assigns to each proper object  $A$  and each double pair  $b$  the “mixed” homotopy groups  $\Pi^i(k\Sigma^{n-2}A, b)$ , called  *$n$ th “relative” injective homotopy groups  $T_n^i(A, b)$*  from  $A$  to the double pair  $b$ ; the functor  $\Pi^i(k\Sigma^{n-2} \times 1)$  is denoted by  $T_n^i$ ,  $n \geq 2$ .

By application of the results of § 1.7 to  $P\mathfrak{R}$  and  $P^2\mathfrak{R}$ , one shows that:

$P_n^i = T_n^i(1 \times \bar{o})$  for the functor  $\bar{o}: P\mathfrak{R} \rightarrow P^2\mathfrak{R}$  corresponding to  $\bar{\omega}: \mathfrak{R} \rightarrow P\mathfrak{R}$ ;  
 $T_{n+1}^i = P_n^i(i \times 1)$ ,  $n \geq 1$ .

Let  $[\alpha_1; A_{11}, A_{12}]$  and  $[\alpha_2; A_{21}, A_{22}]$  be pairs in  $\mathfrak{R}$ , and  $a = \{\phi_1, \phi_2\}$  a map from  $\alpha_1$  to  $\alpha_2$  of  $P\mathfrak{R}$ . The double pair  $[a; \alpha_1, \alpha_2]$  in  $\mathfrak{R}$  is completely characterized by a “directed” commutative diagram in  $\mathfrak{R}$

$$(8.1) \quad \begin{array}{ccc} A_{11} & \xrightarrow{\alpha_1} & A_{12} \\ \phi_1 \downarrow & \downarrow \downarrow & \downarrow \phi_2 \\ A_{21} & \xrightarrow{\alpha_2} & A_{22} \end{array}$$

where the double arrow determines the pairs that are mapped. Let  $a$  and  $a'$  be two double pairs, and  $F = \{f_1, f_2\}$  a map from  $a$  to  $a'$  of  $P^2\mathfrak{R}$ . Then  $F$  is characterized by the matrix

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

where  $\{\phi_{i1}, \phi_{i2}\} = f_i$  ( $i = 1, 2$ ). The maps  $\phi_{ij}$  are called the *components* of  $F$  in  $\mathfrak{R}$ .

For a double pair  $f = [\{\phi_1, \phi_2\}; \alpha_1, \alpha_2]$ , given by a diagram of the form (8.1), the transposed “directed” diagram defines again a double pair  $[\{\alpha_1, \alpha_2\}; \phi_1, \phi_2]$ , called the *transposed  $f^T$  of  $f$* . For double pairs  $a$  and  $b$ , and for a map  $F \in (a, b)$  given by a matrix of the above form, the transposed matrix is a map in  $(a^T, b^T)$ , called the *transposed map  $F^T$  of  $F$* . The pair of functions  $a^T, F^T$  forms an additive covariant functor  $T: P^2\mathfrak{R} \rightarrow P^2\mathfrak{R}$ , called *transposition*. Clearly,  $TT = 1$  (1 being the identity functor of  $P^2\mathfrak{R}$ ).

The transposition  $T$  gives an additional “structure” in the category  $P^2\mathfrak{R}$  which will be exploited in the proof of the exactness of the homotopy sequences of a triple (see § 3.10). There we shall need the result that  $T_n^i = T_n^i(1 \times T)$  up to natural isomorphism (see 8.4 below).\*

LEMMA 8.1. *If  $a$  is injective in  $P^*(P^*\mathfrak{R}')$ , then so is  $a^T$ .*

The verification is straightforward and is left to the reader.

LEMMA 8.2. *The transposition  $T: (a, b) \rightarrow (a^T, b^T)$  induces a natural isomorphism  $t: \Pi^i(k \times 1) \rightarrow \Pi^i(Tk \times T)$ .*

*Proof.* Clearly,  $T$  defines a family of isomorphisms  $T(a, b): (a, b) \rightarrow (a^T, b^T)$ , natural with respect to both variables. Let  $a = kA$  and  $F \in (a, b)$  be  $i$ -homotopic to  $o$ . Since  $a$  is proper, there is a proper monomorphism  $I \in (a, a^i)$  of  $P^*(P^*\mathfrak{R}')$  with  $a^i$  injective and a map  $G \in (a^i, b)$  such that  $F = GI$ . Then,  $F^T = G^T I^T$ , where  $I^T \in (a^T, a^{iT})$ . By 8.1,  $a^{iT}$  is injective; thus,  $F^T \in (a^T, b^T)$  is  $i$ -homotopic to  $o$ . Since  $TT = 1$  and  $a^T$  is proper too,  $F^T \underset{i}{\sim} o$  implies  $F \underset{i}{\sim} o$ . Hence,  $T(kA, b)$  induces a family of isomorphisms  $t(A, b): \Pi^i(kA, b) \rightarrow \Pi^i((kA)^T, b^T)$ , natural with respect to both variables.

\*The topological analogue of the following arguments can be found in (4, § 7).

LEMMA 8.3.  $Tk = k$  up to natural  $i$ -equivalence.

*Proof.* This follows from the fact that  $(kA)^T$  and  $kA$  are  $i$ -equivalent.

By 8.2 and 8.3 we obtain

PROPOSITION 8.4. *The transposition  $T: (a, b) \rightarrow (a^T, b^T)$  defines natural isomorphisms  $t: T_n^i \rightarrow T_n^i(1 \times T)$ .*

The formulation of the notions and results dual to those enunciated above is left to the reader. Note that the transposition is self-dual in  $P^2\mathfrak{A}$ .

SECTION 3. THE EXACT HOMOTOPY SEQUENCES

**9. The exact homology sequence.** We state here a well-known theorem of homological algebra which will be used to prove the exactness of the homotopy sequences of a pair (see section 3.10). The only category considered below is the category  $\mathfrak{G}$  of abelian groups.

The terminology used here is that of (2) with some changes that we are going to explain. Considering left-complexes  $X$  and their homology groups  $H(X)$ , we lower the indices to avoid writing negative numbers. We shall speak of an *exact sequence*

$$O \rightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \rightarrow O$$

of left-complexes if the short sequences

$$O \rightarrow X'_n \xrightarrow{i_n} X_n \xrightarrow{p_n} X''_n \rightarrow O$$

given by the components are exact for  $n \geq 1$ , but only left-exact for  $n = 0$ . Then the following theorem is *still* valid.

THEOREM 9. *Let  $O \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow O$  be an exact sequence of left-complexes. Then there is a sequence of (connecting) homomorphisms  $\Delta_n: H_n(X'') \rightarrow H_{n-1}(X')$  such that the sequence of homology groups (homology sequence)*

$$\dots \rightarrow H_n(X') \rightarrow H_n(X) \rightarrow H_n(X'') \xrightarrow{\Delta_n} H_{n-1}(X') \rightarrow \dots \rightarrow H_0(X'')$$

is exact.

The demonstration of Theorem 9 is contained in (2, chapter IV), where an explicit description of the homomorphisms  $\Delta_n$  is also given. Moreover, the connecting homomorphisms  $\Delta_n$  are natural homomorphisms in the sense that the homology functors  $H_n$  form a connected sequence of functors.

**10. The exact homotopy sequences of a pair.** Let  $A$  be a proper object and  $\beta \in (B_1, B_2)$  an arbitrary map of  $\mathfrak{A}$ . Then one associates to  $A$  and the pair  $\beta = [\beta; B_1, B_2]$ , as in (3), the sequence of homotopy groups

$$S^*(A, \beta): \dots \rightarrow \Pi_n^i(A, B_1) \xrightarrow{\beta_*^i} \Pi_n^i(A, B_2) \xrightarrow{j_*^i} P_n^i(A, \beta) \xrightarrow{\partial_*^i} \Pi_{n-1}^i(A, B_1) \rightarrow \dots \rightarrow \Pi_0^i(A, B_2),$$

called *injective homotopy sequence of the pair*  $\beta$  with respect to  $A$ . The homomorphisms in  $S_*(A, \beta)$  are defined as follows: (i)  $\beta_*^i$  is the homomorphism induced by  $\beta \in (B_1, B_2)$ ; (ii) identifying  $\Pi_n^i(A, B_2) = P_n^i(A, \tilde{\omega}B_2)$ ,  $j_*^i$  is the homomorphism induced by  $j = \{o, 1_{B_2}\}$  from  $\tilde{\omega}B_2$  to  $\beta$ ; (iii) identifying  $\Pi_{n-1}^i(A, B_1) = P_n^i(A, \omega B_1)$ ,  $\partial_*^i$  is the homomorphism induced by  $\partial = \{1_{B_1}, o\}$  from  $\beta$  to  $\omega B_2$ .

**THEOREM 10.** *The injective homotopy sequence  $S_*(A, \beta)$  is exact.*

*Proof.* Let  $X$  be an injective resolution of  $\iota A$ :

$$o \rightarrow \iota A \xrightarrow{d_1} \xi_1 \xrightarrow{d_2} \xi_2 \rightarrow \dots$$

such that the  $\xi_n$  are standard injective pairs in the sense of Lemma 7.4, and let  $(X, \beta)$  denote the induced left-complex  $\dots \rightarrow (\xi_2, \beta) \rightarrow (\xi_1, \beta) \rightarrow (\iota A, \beta) \rightarrow o$ . Since

$$o \rightarrow \tilde{\omega}B_2 \xrightarrow{j} \beta \xrightarrow{\partial} \omega B_1 \rightarrow o$$

is obviously an s.e.s. in  $P\mathfrak{R}$ , the sequences

$$O \rightarrow (\xi_n, \tilde{\omega}B_2) \xrightarrow{j_*} (\xi_n, \beta) \xrightarrow{\partial_*} (\xi_n, \omega B_1)$$

are exact for all  $n \geq 0$  ( $\xi_0 = \iota A$ ). Since, for each  $n \geq 1$ ,  $\xi_n$  is a standard injective pair  $\{\xi_n; X_1, X_2\}$ , there are maps  $\xi_n' \in (X_2, X_1)$  such that  $\xi_n' \xi_n = 1_{X_1}$ . From this it is easily seen that the homomorphisms  $\partial_*$  are epimorphisms for all  $n \geq 1$ . Consequently, the sequence of left-complexes  $O \rightarrow (X, \tilde{\omega}B_2) \rightarrow (X, \beta) \rightarrow (X, \omega B_1) \rightarrow O$  is exact in the sense of § 3.9. The homology groups  $H_n((X, \beta))$  are exactly the groups  $H_n^i(\iota A, \beta)$  of § 2.7. Hence, by 7.5 and 9, the sequence

$$S_*(A, \beta) : \dots \rightarrow P_{n+1}^i(A, \tilde{\omega}B_2) \xrightarrow{j_*^i} P_{n+1}^i(A, \beta) \xrightarrow{\partial_*^i} P_{n+1}^i(A, \omega B_1) \\ \xrightarrow{p_{n-1}^{-1} \Delta_n p_n} P_n^i(A, \tilde{\omega}B_2) \rightarrow \dots \rightarrow P_1^i(A, \omega B_1)$$

is exact. Identifying  $P_{n+1}^i(A, \tilde{\omega}B_2) = \Pi_{n+1}^i(A, \beta_2)$  and  $\Omega_{n+1}^i(A, \omega B_1) = \Pi_n^i(A, B_1)$  by the natural isomorphisms  $\tilde{s}$  and  $s$  respectively, one obtains the injective homotopy sequence  $S_*(A, \beta)$  up to the following deviations: (1) in  $S_*'(A, \beta)$ , one has the composite homomorphisms  $\Delta_n^i = \tilde{s} p_{n-1}^{-1} \Delta_n p_n s$  instead of the homomorphisms  $\beta_*^i$  induced by  $\beta$ ; (2) the sequence  $S_*'(A, \beta)$  ends with the term  $\Pi_0^i(A, B_1)$ . It is easy to see that, by the assumption  $\Delta_n^i = \beta_*^i$  for  $n \geq 1$ , the sequence  $S_*'(A, \beta)$  can be extended by the term

$$\Pi_0^i(A, B_1) \xrightarrow{\beta_*^i} \Pi_0^i(A, B_2)$$

such that it remains exact.

The proof that  $\Delta_n^i = \beta_*^i$  for  $n \geq 1$  can be given in two ways: either by a direct verification with the explicitly given homomorphisms  $s, \tilde{s}, p_n$  and  $\Delta_n$ ,

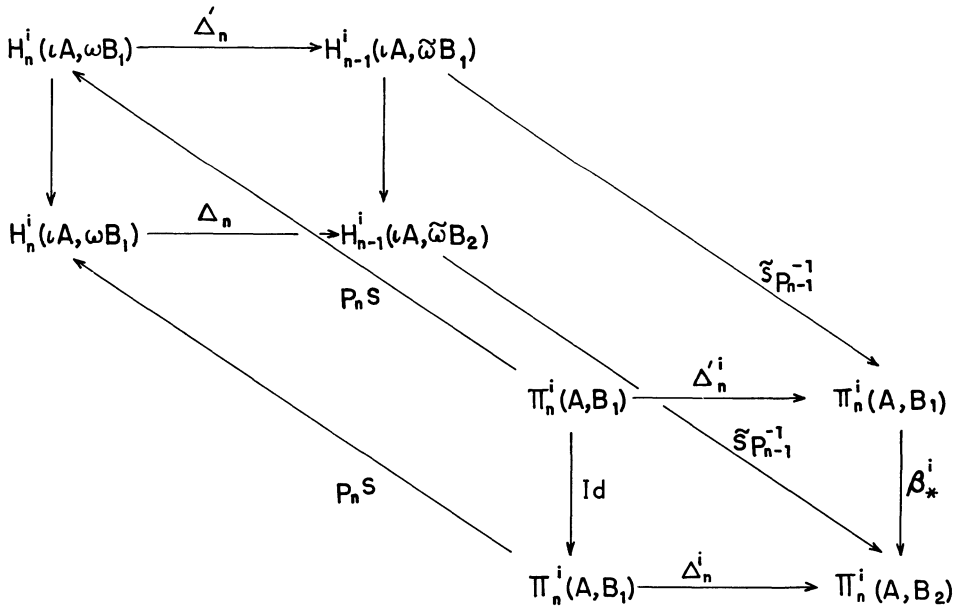
or by the following consideration of their naturality property. For an arbitrary object  $B$ , consider the s.e.s.

$$o \rightarrow \tilde{\omega}B \xrightarrow{j'} 1_B \xrightarrow{\partial'} \omega B \rightarrow o$$

where  $1_B = [1_B; B, B]$ ,  $j' = \{o, 1_B\}$ ,  $\partial' = \{1_B, o\}$ . The connecting homomorphisms  $\Delta_n': H_n^i(\iota A, \omega B) \rightarrow H_{n-1}^i(\iota A, \tilde{\omega}B)$  are then isomorphisms; hence, the  $\Delta_n'^i = \tilde{s}p_{n-1}^{-1}\Delta_n'p_n s$  define natural isomorphisms:  $\Pi_n^i \rightarrow \Pi_n^i$ ,  $n \geq 1$ . The commutative diagram

$$\begin{array}{ccccccc} o & \rightarrow & \tilde{\omega}B_1 & \xrightarrow{j'} & 1_{B_1} & \xrightarrow{\partial'} & \omega B_1 \rightarrow o \\ (o, \beta) \downarrow & & (1, \beta) \downarrow & & (1, o) \downarrow & & \\ o & \rightarrow & \tilde{\omega}B_2 & \xrightarrow{j} & \beta & \xrightarrow{\partial} & \omega B_1 \rightarrow o \end{array}$$

induces the following commutative diagrams



Hence,  $\Delta_n^i = \beta_*^i \Delta_n'^i$  where the  $\Delta_n'^i$  are natural isomorphisms. Thus,  $\Delta_n^i$  can be replaced by  $\beta_*^i$ .

In addition, we have shown that the homomorphisms  $\beta_*^i, j_*^i, \partial_*^i$  are natural in the following sense. If  $\beta'$  is another pair and  $f$  a map from  $\beta$  to  $\beta'$ , then  $f$  "induces a map" from  $S_*(A, \beta)$  to  $S_*(A, \beta')$ .

Dually, to  $A$  and the pair  $\beta = [\beta; B_2, B_1]$ , one associates the sequence

$$S^*(\beta, A) : \dots \rightarrow \Pi_n^p(B_1, A) \xrightarrow{\beta_*^p} \Pi_n^p(B_2, A) \xrightarrow{j_*^p} P_n^p(\beta, A) \xrightarrow{\partial_*^p} \Pi_{n-1}^p(B_1, A) \rightarrow \dots \rightarrow \Pi_0^p(B_2, A),$$

called *projective homotopy sequence of the pair*  $\beta$  with respect to  $A$ . The homomorphisms in  $S^*(\beta, A)$  are then defined as follows: (i)  $\beta^{*p}$  is the homomorphism induced by  $\beta \in (B_2, B_1)$ ; (ii) identifying  $\Pi_n^p(B_2, A) = P_n^p(\omega B_2, A)$ ,  $j^{*p}$  is the homomorphism induced by  $j = \{1_{B_2}, o\}$  from  $\beta$  to  $\omega B_2$ ; (iii) identifying  $\Pi_{n-1}^p(B_1, A) = P_n^p(\tilde{\omega} B_1, A)$ ,  $\partial^{*p}$  is the homomorphism induced by  $\partial = \{o, 1_{B_1}\}$  from  $\tilde{\omega} B_1$  to  $\beta$ .

**THEOREM 10\*.** *The projective homotopy sequence  $S^*(\beta, A)$  is exact.*

Replacing  $\mathfrak{R}$  by  $P\mathfrak{R}$ , we obtain for a proper pair  $\alpha$  and any double pair  $b = [b; \beta_1, \beta_2]$  the exact homotopy sequences  $S_*(\alpha, b)$  and  $S^*(b, \alpha)$ .\* Let us take for  $\alpha$  a special pair of the form  $\alpha = \iota A$  (cf. § 2.7). Then, using the identifications  $P_{n+1}^i = \Pi_n^i(\iota \times 1)$  and  $T_{n+1}^i = P_n^i(\iota \times 1)$  (1 being the identity functor in  $P\mathfrak{R}$  and  $P^2\mathfrak{R}$  respectively), we have an exact sequence of “relative” homotopy groups

$$S^*(A, b) : \dots \rightarrow P_{n+1}^i(A, \beta_1) \xrightarrow{b_*^i} P_{n+1}^i(A, \beta_2) \xrightarrow{j_*^i} T_{n+1}^i(A, b) \xrightarrow{\partial_*^i} P_n^i(A, \beta_1) \rightarrow \dots \rightarrow P_1^i(A, \beta_2),$$

called *injective homotopy sequence of the double pair*  $b$  with respect to  $A$ .

**11. The exact homotopy sequences of a triple.** A *triple*  $[\beta, \gamma; B, C, D]$  in  $\mathfrak{R}$  (often denoted by  $[\beta, \gamma]$  only) is a sequence of the form

$$B \xrightarrow{\beta} C \xrightarrow{\gamma} D.$$

To such a triple, a “directed” commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\gamma\beta} & D \\ \beta \downarrow & \downarrow \downarrow & \downarrow 1_D \\ C & \xrightarrow{\gamma} & D \end{array}$$

can be associated. This directed diagram determines a double pair  $b = [(\beta, 1_D); \gamma\beta, \gamma]$  again denoted by  $[\beta, \gamma]$ . When, subsequently, we speak of a triple, we always mean the double pair associated in the above way.

Let us consider the exact homotopy sequences of the triple  $b = [\beta, \gamma]$  and of its transposed  $b^T$ , both with respect to a proper object  $A$ :

$$S^*(A, b) : \dots \rightarrow P_{n+1}^i(A, \gamma\beta) \xrightarrow{b_*^i} P_{n+1}^i(A, \gamma) \xrightarrow{j_*^i} T_{n+1}^i(A, b) \xrightarrow{\partial_*^i} P_n^i(A, \gamma\beta) \rightarrow \dots;$$

$$S^*(A, b^T) : \dots \rightarrow P_{n+1}^i(A, \beta) \xrightarrow{b_*^T} P_{n+1}^i(A, 1_D) \xrightarrow{j_*^T} T_{n+1}^i(A, b^T) \xrightarrow{\partial_*^T} P_n^i(A, \beta) \rightarrow \dots$$

Obviously,  $P_{n+1}^i(A, 1_D) = 0$  for all  $n \geq 0$ . Thus, the homomorphisms  $\partial_*^T$  are (natural) isomorphisms. By 8.4, the transposition  $T$  defines natural isomor-

\*These sequences are the analogues of the exact sequences  $S_*(\alpha, b)$  and  $S^*(b, \alpha)$  of (4, § 4). However, their exactness does not need a separate proof in this context.

phisms  $t: T_{n+1}^i \rightarrow T_{n+1}^i(1 \times T)$ . Hence, putting  $c_*^i = \partial_*^i t^{-1}(\partial_*'^i)^{-1}$  and  $\Delta^i = \partial_*'^i t j_*^i$ , one obtains an exact sequence

$$S_*(A, [\beta, \gamma]): \dots \rightarrow P_n^i(A, \beta) \xrightarrow{c_*^i} P_n^i(A, \gamma\beta) \xrightarrow{b_*^i} P_n^i(A, \gamma) \rightarrow \xrightarrow{\Delta^i} P_{n-1}^i(A, \beta) \rightarrow \dots \rightarrow P_1^i(A, \gamma).$$

called *injective homotopy sequence of the triple*  $[\beta, \gamma]$  with respect to  $A$  (cf. (4), Corollary 7.8). The homomorphisms in  $S_*(A, [\beta, \gamma])$  can be given (more conveniently) as follows: (i)  $c_*^i$  is the homomorphism induced by  $c = \{1_B, \gamma\}$  from  $\beta$  to  $\gamma\beta$ ; (ii)  $b_*^i$  is the homomorphism induced by  $b = \{\beta, 1_D\}$  from  $\gamma\beta$  to  $\gamma$ ; (iii) identifying  $P_n^i(A, \gamma) = T_n^i(A, \bar{\sigma}\gamma)$  (cf. § 2.8),  $\Delta^i$  is the homomorphism induced by

$$\Delta \begin{bmatrix} o & o \\ \phi_1 & \phi_2 \end{bmatrix} = (o, \phi_1)$$

for all

$$\begin{bmatrix} o & o \\ \phi_1 & \phi_2 \end{bmatrix}$$

from  $k\Sigma^{n-2}A$  to  $\bar{\sigma}\gamma$ .

Proof of (i) and (iii).

(i)  $c_* = \partial_* T^{-1} \partial_*'^{-1}$  and  $T^{-1} = T$ . Given  $\{\phi_1, \phi_2\}$  from  $k\Sigma^{n-1}A$  to  $\beta$ , choose

$$F = \begin{bmatrix} \phi_1 & \phi_2 \\ \gamma\phi_2 & * \end{bmatrix}$$

from  $k\Sigma^{n-1}A$  to  $b^T$ . Then  $\partial_*'^i F = \{\phi_1, \phi_2\}$  and

$$\partial_* T(F) = \begin{bmatrix} \phi_1 & \gamma\phi_2 \\ \phi_2 & * \end{bmatrix} = \{\phi_1, \gamma\phi_2\}.$$

Hence,  $c = \{1_B, \gamma\}$ .

(iii)  $\Delta = \partial_*'^i T j_*$ . Given

$$G = \begin{bmatrix} o & o \\ \phi_1 & \phi_2 \end{bmatrix}$$

from  $k\Sigma^{n-2}A$  to  $\bar{\sigma}\gamma$ ,

$$\Delta G = \partial_*'^i T(G) = \partial_*'^i T \begin{bmatrix} o & o \\ \phi_1 & \phi_2 \end{bmatrix} = \partial_*'^i \begin{bmatrix} o & \phi_1 \\ o & \phi_2 \end{bmatrix} = \{o, \phi_1\}.$$

Dually, for a triple  $[\gamma, \beta] = [\gamma, \beta; D, C, B]$ , one obtains the exact sequence

$$S^*([\gamma, \beta], A): \dots \rightarrow P_n^p(\beta, A) \xrightarrow{c^{*p}} P_n^p(\beta\gamma, A) \xrightarrow{b^{*p}} P_n^p(\gamma, A) \rightarrow \xrightarrow{\Delta^p} P_{n-1}^p(\beta, A) \rightarrow \dots \rightarrow P_1^p(\gamma, A),$$



called *projective homotopy sequence of the triple*  $[\gamma, \beta]$  with respect to  $A$  (cf. **(4)**, Corollary 7.8\*). The homomorphisms in  $S^*([\gamma, \beta], A)$  are then given as follows: (i)  $c^*$  is the homomorphism induced by  $c = \{\gamma, 1_B\}$  from  $\beta\gamma$  to  $\beta$ ; (ii)  $b^*$  is the homomorphism induced by  $b = \{1_D, \beta\}$  from  $\gamma$  to  $\beta\gamma$ ; (iii) identifying  $P_n^p(\gamma, A) = T_n^p(o\gamma, A)$ ,  $\Delta^p$  is the homomorphism induced by

$$\Delta \begin{bmatrix} \phi_2 & \phi_1 \\ o & o \end{bmatrix} = (\phi_1, o).$$

**12. The exact homotopy sequences of a fibration.** Under certain assumptions for the pairs  $\beta$  in  $\mathfrak{R}$ , the “relative” sequences  $S_*(A, \beta)$  and  $S^*(\beta, A)$  become “absolute” sequences, that is, sequences without “relative” homotopy groups. In analogy to the situation in topology, these pairs are referred to as *fibrations* (cf. also **(3)**).

DEFINITION. A pair  $[\beta; B_1, B_2]$  is an *i-fibration relative to a proper object*  $A$  if it has the following properties: (i) the map  $\beta \in (B_1, B_2)$  has a kernel  $B_0'$ , called the fibre of  $\beta$ , (ii) there is an injective resolution  $X$  of  $A$  such that, for each map  $\phi_2 \in (X_n, B_2)$  where  $X_n$  is an injective object occurring in  $X$ , there exists a map  $\phi_1 \in (X_n, B_1)$  with  $\phi_2 = \beta\phi_1$ . If property (ii) holds for arbitrary injective objects  $X_n$  of  $\mathfrak{R}'$ , one speaks simply of an *i-fibration*.

A pair  $[\beta; B_2, B_1]$  is a *p-fibration relative to a proper object*  $A$  if the dual properties hold; the cokernel  $B_0$  of the map  $\beta \in (B_2, B_1)$  is then called the *cofibre of*  $\beta$ .

Let  $A$  be a proper object and  $[\beta; B_1, B_2]$  an *i-fibration relative to*  $A$ . The canonical monomorphism from the fibre  $B_0$  to  $B_1$  will be denoted by  $\kappa$ . For  $A$  and the double pair  $b$  given by the “directed” commutative diagram

$$(12.1) \quad \begin{array}{ccc} B_0 & \longrightarrow & O \\ \kappa \downarrow & \Downarrow & \downarrow \\ B_1 & \xrightarrow{\beta} & B_2 \end{array}$$

one proves the following lemma.

LEMMA 12.1  $T_n^i(A, b) = o$  for all  $n \geq 2$ .

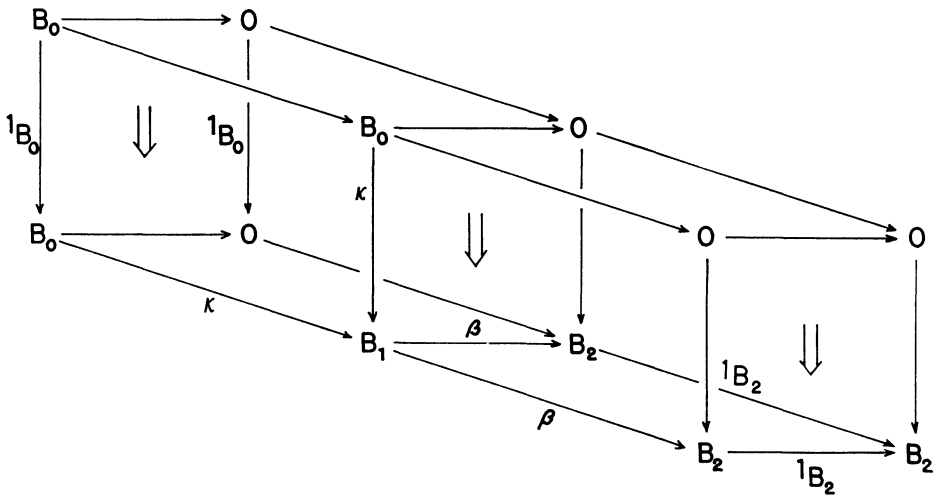
Proof. Let  $X$  be an injective resolution of the double pair  $a = kA$ :

$$o \rightarrow a \xrightarrow{D_1} x_1 \xrightarrow{D_2} x_2 \rightarrow \dots,$$

such that the injective double pairs  $x_n$  are given by diagrams of the form (8.1), where the term  $A_{21}$  coincides with an injective object  $X_n$  of  $\mathfrak{R}$  for which fibration-property (ii) of  $\beta$  holds. By Lemma 7.4, it is easily seen that such a resolution of  $kA$  exists. We denote by  $(X, b)$  the induced left-complex

$$\dots \rightarrow (x_2, b) \xrightarrow{D_2^*} (x_1, b) \xrightarrow{D_1^*} (a, b) \rightarrow O.$$

The diagram



clearly determines an exact sequence

$$o \rightarrow b' \xrightarrow{I} b \xrightarrow{P} b'' \text{ in } P^2\mathfrak{K}.$$

Thus, the sequences

$$O \rightarrow (x_n, b') \xrightarrow{I^*} (x_n, b) \xrightarrow{P^*} (x_n, b'')$$

are exact for all  $n \geq 0$  ( $x_0 = kA$ ). The pair  $\beta$  being an  $i$ -fibration, it will be shown that the  $P^*$  are epimorphisms for  $n \geq 1$ . Consequently, the sequence of left-complexes  $O \rightarrow (X, b') \rightarrow (X, b) \rightarrow (X, b'') \rightarrow O$  is exact in the sense of § 3.9.

Let

$$F = \begin{bmatrix} o & o \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

be a map from  $x_n$  to  $b''$ ,  $n \geq 1$ . The double pair  $x_n$  is given by a diagram of the form (8.1), where  $A_{21} = X_n$  has the property just mentioned. So there is a map  $\phi_{21}' \in (A_{21}, B_1)$  such that  $\phi_{21} = \beta\phi_{21}'$ . Since  $\beta\phi_{21}'\phi_1 = \phi_{21}\phi_1 = o$  (for the notation  $\phi_1$ , see diagram (8.1)), there is a  $\phi_{11}' \in (A_{11}, B_0)$  such that  $k\phi_{11}' = \phi_{21}'\phi_1$ . Hence, we have a map

$$G' = \begin{bmatrix} \phi_{11}' & o \\ \phi_{21}' & \phi_{22} \end{bmatrix}$$

from  $x_n$  to  $b$  such that  $P_*G' = F$ .

Now, by the same arguments as in the proof of Theorem 10, the sequence of homotopy groups

$$\dots \rightarrow T_{n+2}^i(A, b') \rightarrow T_{n+2}^i(A, b) \rightarrow T_{n+2}^i(A, b'') \rightarrow T_{n+1}^i(A, b') \rightarrow \dots \rightarrow T_2^i(A, b'')$$

is exact. Furthermore,

$$T_n^i(A, b'') = T_n^i(A, \partial 1_{B_2}) = P_n^i(A, 1_{B_2}) = o$$

and

$$T_n^i(A, b') = T_n^i(A, (o1_{B_0})^T) = T_n^i(A, o1_{B_0}) = P_{n-1}^i(A, 1_{B_0}) = o$$

for all  $n \geq 2$ . Hence,  $T_n^i(A, b) = 0$  for all  $n \geq 2$ .

Consider a diagram of the form (12.1), first without the assumption that  $\beta$  is an  $i$ -fibration. For the corresponding double pair  $b = [b; \omega B_0, \beta]$ , its transposed  $b^T = [b^T; \kappa, \tilde{\omega} B_2]$ , and for the pairs  $[\kappa; B_0, B_1]$ ,  $[\beta; B_1, B_2]$ , we have the exact sequences  $S_*(A, b)$ ,  $S_*(A, b^T)$ ,  $S_*(A, \kappa)$ ,  $S_*(A, \beta)$ . These can be embedded into the diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
 & & T_{n+1}^i(A, b^T) & & T_{n+1}^i(A, b) & & \\
 & & \downarrow & & \downarrow & & \\
 \dots \rightarrow \Pi_n^i(A, B_1) & \xrightarrow{j_*^i} & P_n^i(A, \kappa) & \xrightarrow{\partial_*^i} & \Pi_{n-1}^i(A, B_0) = P_n^i(A, \omega B_0) & \xrightarrow{\kappa_*^i} & \Pi_{n-1}^i(A, B_1) \rightarrow \dots \\
 (12.2) \text{ Id } \downarrow & \mathbf{1} & \downarrow & b_*^i & \mathbf{3} & \downarrow & b_*^i \mathbf{2} & \downarrow & Jd \\
 \dots \rightarrow \Pi_n^i(A, B_1) & \xrightarrow{\beta_*^i} & P_n^i(A, \tilde{\omega} B_2) = \Pi_n^i(A, B_2) & \longrightarrow & P_n^i(A, \beta) & \xrightarrow{\partial_*^i} & \Pi_{n-1}^i(A, B_1) \rightarrow \dots \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
 & & T_n^i(A, b^T) & & T_n^i(A, b) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \vdots & & \vdots & & & & 
 \end{array}$$

satisfying certain commutativity relations.

LEMMA 12.2. *The diagrams (1) and (2) are commutative; diagram (3) is anticommutative.*

*Proof.* **1.** Identify  $\Pi_n^i(A, B_1) = P_n^i(A, \tilde{\omega} B_1)$ , and let  $\{o, \phi\}$  be a map from  $\iota \Sigma^{n-1} A$  to  $\tilde{\omega} B_1$ . Then  $b_*^T j_*^i \{o, \phi\} = b_*^T \{o, \phi\} = \{o, \beta \phi\}$ . Thus,  $b_*^T j_*^i = \beta_*$ .

**2.** Given a map  $\{\phi, o\}$  from  $\iota \Sigma^{n-1} A$  to  $\omega B_0$ , then  $\partial_* b_* \{\phi, o\} = \partial_* \{\kappa \phi, o\} = \kappa \phi$ . Thus,  $\partial_* b_* = \kappa_*$ .

**3.** Let  $\{\phi_1, \phi_2\}$  be a map from  $\iota \Sigma^{n-1} A$  to  $\kappa$ . One has to show that

$$\begin{aligned}
 (b_* \partial_*^i + j_*^i b_*^T) \{\phi_1, \phi_2\} &= b_* \partial_*^i \{\phi_1, \phi_2\} + j_*^i b_*^T \{\phi_1, \phi_2\} = \{\kappa \phi_1, o\} + \{o, \beta \phi_2\} \\
 &= \{\kappa \phi_1, \beta \phi_2\} = (\kappa, \beta)_* \{\phi_1, \phi_2\} \sim_i o.
 \end{aligned}$$

This can be done by considering the homotopy sequence of the triple  $[\kappa, \beta]$  where  $\{\kappa, \beta\}_*^i = \{\kappa, 1\}_*^i \{1, \beta\}_*^i = b_*^i c_*^i = o$ . Thus,

$$b_* \partial_*^i \sim_i - j_*^i b_*^T.$$

Assuming now that  $\beta$  is an  $i$ -fibration relative to  $A$ , the following theorem is obtained (cf. (3, Theorem 5.2)).

**THEOREM 12.3.** *Let  $A$  be a proper object, and  $[\beta; B_1, B_2]$  an  $i$ -fibration relative to  $A$ ;  $[\kappa; B_0, B_1]$  is the pair given by the canonical monomorphism  $\kappa$  from the fibre  $B_0$  of  $\beta$  to  $B_1$ . Then there are natural isomorphisms*

$$u: P_n^i(A, \kappa) \rightarrow \Pi_n^i(A, B_2) \quad \text{for } n \geq 1,$$

and

$$v: \Pi_{n-1}^i(A, B_0) \rightarrow P_n^i(A, \beta) \quad \text{for } n \geq 1,$$

such that (using the notation of diagram (12.2)) the following identifications can be made: (1)  $uj_*^i = \beta_*^i$ , (2)  $\partial_*'^i v = \kappa_*^i$ , (3)  $v^{-1}j_*^i = -\partial_*'^i u^{-1}$ .

*Proof.* Put  $u = b_*^{T^i}$  and  $v = b_*^i$ . Lemma 12.1 implies, for  $n \geq 2$ ,  $u$  and  $v$  to be isomorphisms. For  $n = 1$ , it only follows that  $b_*^{T^i}$  and  $b_*^i$  are monomorphisms. But, by a separate verification, it is easily seen that they are epimorphisms, and therefore isomorphisms. The relations (1), (2), and (3) are exactly the commutativity relation of Lemma 12.2.

Applying 12.3 to the ‘‘relative’’ homotopy sequences  $S_*(A, \beta)$  and  $S_*(A, \kappa)$  we obtain two exact ‘‘absolute’’ homotopy sequences

$$T^*(A, \beta): \dots \rightarrow \Pi_n^i(A, B_1) \xrightarrow{\beta_*^i} \Pi_n^i(A, B_2) \xrightarrow{\Delta^i = v^{-1}j_*^i} \Pi_{n-1}^i(A, B_0) \xrightarrow{\kappa_*^i} \Pi_{n-1}^i(A, B_1) \rightarrow \dots$$

$$T^*(A, \kappa): \dots \rightarrow \Pi_n^i(A, B_0) \xrightarrow{\kappa_*^i} \Pi_n^i(A, B_1) \xrightarrow{\beta_*^i} \Pi_n^i(A, B_2) \xrightarrow{\Delta'^i = \partial_*'^i u^{-1}} \Pi_{n-1}^i(A, B_0) \rightarrow \dots$$

which are one and the same sequence up to a sign of the ‘‘connecting homomorphisms’’  $\Delta^i$  and  $\Delta'^i: v^{-1}j_*^i = -\partial_*'^i u^{-1}$ . This sequence is called *injective homotopy sequence of the  $i$ -fibration  $\beta$  relative to  $A$* .

Let  $[\beta; B_2, B_1]$  be a  $p$ -fibration relative to  $A$ . The canonical epimorphism from  $B_1$  to the cofibre  $B_0$  of  $\beta$  will be denoted by  $\gamma$ . The formulation of the theorem dual to 12.3 is left to the reader. We shall only write down the exact ‘‘absolute’’ homotopy sequences dual to  $T_*(A, \beta)$  and  $T_*(A, \kappa)$ :

$$T^*(\beta, A): \dots \rightarrow \Pi_n^p(B_1, A) \xrightarrow{\beta_*^p} \Pi_n^p(B_2, A) \xrightarrow{\Delta^p} \Pi_{n-1}^p(B_0, A) \xrightarrow{\gamma_*^p} \Pi_{n-1}^p(B_1, A) \rightarrow \dots$$

$$T^*(\gamma, A): \dots \rightarrow \Pi_n^p(B_0, A) \xrightarrow{\gamma_*^p} \Pi_n^p(B_1, A) \xrightarrow{\beta_*^p} \Pi_n^p(B_2, A) \xrightarrow{\Delta'^p} \Pi_{n-1}^p(B_0, A) \rightarrow \dots$$

These are again one and the same sequence (up to a sign of  $\Delta^p$  and  $\Delta'^p$ ) which is called *projective homotopy sequence of the  $p$ -fibration  $\beta$  relative to  $A$* .

SECTION 4. HOMOTOPY THEORY IN SOME SPECIAL CATEGORIES

**13. Homotopy for R-homomorphisms.** Let  $R$  be a ring with unit element. Consider the additive category  $\mathfrak{M}$  of  $R$ -modules and choose the

exact structure  $t$  of  $\mathfrak{M}$  as homotopy structure. Then one obtains the homotopy concepts for  $R$ -modules introduced in (3).

Since  $\mathfrak{M}$  is a proper abelian category, that is, there are enough  $t$ -injectives and  $t$ -projectives,  $\mathfrak{M}$  can be chosen as its own proper subcategory. Hence, the functors  $\Pi_n^i$  and  $\Pi_n^p$  are functors of the category  $\mathfrak{M} \times \mathfrak{M}$  (respectively  $\mathfrak{M}^i \times \mathfrak{M}$  and  $\mathfrak{M} \times \mathfrak{M}^p$ ).

The category  $P\mathfrak{M}$  of pairs in  $\mathfrak{M}$  with homotopy structure  $P^*t$  is still a proper category; namely, we have  $P^*\mathfrak{M} = P\mathfrak{M}$ . Hence, the functors  $P_n^i$  and  $P_n^p$  are functors of the categories  $\mathfrak{M} \times P\mathfrak{M}$  and  $P\mathfrak{M} \times \mathfrak{M}$  (respectively  $\mathfrak{M}^i \times P\mathfrak{M}$  and  $P\mathfrak{M} \times \mathfrak{M}^p$ ). However,  $P^*t$  is a non-exact abelian structure.

The category  $P^2\mathfrak{M}$  of double pairs in  $\mathfrak{M}$  with abelian structure  $P^*(P^*t)$  is no longer a proper category. Here we have to consider the proper subcategory  $P^*(P\mathfrak{M})$  relative to  $P^*t$ , which is different from  $P^2\mathfrak{M}$ .

Let  $R$  be an associative algebra over a commutative ring  $K$ . Then there is a natural ring-homomorphism  $\phi: K \rightarrow R$  (cf. (2, IX, 1)). By means of this ring-homomorphism, the  $\phi$ -relative structure  $\mathfrak{s}_\phi$  can be introduced in  $\mathfrak{M}$ . The  $\mathfrak{s}_\phi$ -injectives and  $\mathfrak{s}_\phi$ -projectives are known as *weakly injective* and *weakly projective* modules (cf. (2, X, 8)). Taking  $\mathfrak{s}_\phi$  as homotopy structure—we shall speak of the *weak homotopy structure* in  $\mathfrak{M}$ —the corresponding homotopy relations and homotopy groups will be called *weak homotopies* and *weak homotopy groups*.

The weak homotopy structure in  $\mathfrak{M}$  is very convenient in the following context. There is a simple covariant functor  $C: \mathfrak{M} \rightarrow \mathfrak{M}$  which assigns to each module a weakly injective module, namely  $CA = \text{Hom}_S(R, A)$ . A covariant functor  $E: \mathfrak{M} \rightarrow \mathfrak{M}$  with the dual property is given by  $EA = R \otimes_S A$ .

**14. Chain homotopy.** Let us consider the additive category  $D\mathfrak{M}$  of complexes in  $\mathfrak{M}$  (cf. § 1.4). As homotopy structure in  $D\mathfrak{M}$ , we choose the abelian structure  $\mathfrak{s}_1$  introduced in Example (b) of § 1.4. The notations there will be used freely in this part. We are going to show that the concepts of  $i$ - and  $p$ -homotopy both coincide with the concept of chain-homotopy.

Let  $A$  and  $A'$  be two complexes. A *chain map*  $\phi: A \rightarrow A'$  of degree  $p$  is a sequence  $(\phi_i)_{i \in \mathbb{Z}}$  of  $R$ -homomorphisms  $\phi_i \in (A_i, A'_{i+p})$  such that  $\phi_{i+1}\alpha_i = \alpha_{i+p}'\phi_i$ . Thus, the chain maps of degree 0 are exactly the maps of the additive category  $D\mathfrak{M}$ .

Two chain maps  $\phi, \psi: A \rightarrow A'$  of degree  $p$  are *homotopic* ( $\phi \simeq \psi$ ) if there is a sequence  $(\sigma_i)_{i \in \mathbb{Z}}$  of  $R$ -homomorphisms  $\sigma_i \in (A_i, A'_{i+p-1})$  such that  $\phi_i - \psi_i = \sigma_{i+1}\alpha_i + (-1)^p\alpha'_{i+p-1}\sigma_i$ . In particular, a map  $\phi \in (A, B)$  is homotopic to 0 if  $\phi_i = \sigma_{i+1}\alpha_i + \alpha'_{i-1}\sigma_i$  for a sequence  $(\sigma_i)_{i \in \mathbb{Z}}$  of  $R$ -homomorphisms  $\sigma_i \in (A_i, A'_{i-1})$ .

The chain maps  $\phi: A \rightarrow B$  of degree  $p$  form an abelian group  $\text{Hom}_p(A, B)$  with respect to the usual addition of maps. The relation  $\phi \simeq \psi$  defines an equivalence relation for all  $\phi, \psi \in \text{Hom}_p(A, B)$  which is compatible with the

addition in  $\text{Hom}_p(A, B)$ . Thus, the classes of homotopic chain maps in  $\text{Hom}_p(A, B)$  again form an abelian group  $\mathfrak{S}\text{om}_p(A, B)$ .

A map  $\phi \in (A, B)$  of  $D\mathfrak{M}$  is  $i$ -homotopic to  $o$  if and only if it can be factored by the proper monomorphism  $\mu$  from  $A$  to  $X(A)$  constructed in Example (b) of § 1.4.

LEMMA 14.1 *A chain map  $\phi \in \text{Hom}_o(A, B)$  is homotopic to  $o$  if and only if there is a  $\gamma \in \text{Hom}_o(X(A), B)$  such that  $\phi = \gamma\mu$ .*

*Proof.* Let  $\phi = (\phi_i)_{i \in \mathbb{Z}}$  be in  $\text{Hom}_o(A, B)$ , and assume that there is a  $\gamma = (\gamma_i)_{i \in \mathbb{Z}} \in \text{Hom}_o(X(A), B)$  such that  $\phi = \gamma\mu$ , that is (with the notation of Example (b)),  $\phi_i = \gamma_i(\iota_i'\alpha_i + \iota_i)$  for all  $i$ . Put  $\sigma_i = \gamma_{i-1}\iota'_{i-1}$ . Then  $(\sigma_i)_{i \in \mathbb{Z}}$  is a sequence of homomorphisms  $\sigma_i \in (A_i, B_{i-1})$  such that  $\sigma_{i+1}\alpha_i + \beta_{i-1}\sigma_i = \phi_i$ .

Conversely, assume that  $\phi \simeq o$ , that is, that there exists a sequence  $(\sigma_i)_{i \in \mathbb{Z}}$  of homomorphisms  $\sigma_i \in (A_i, B_{i-1})$  such that  $\sigma_{i+1}\alpha_i + \beta_{i-1}\sigma_i = \phi_i$ . Put  $\gamma_i = \beta_{i-1}\sigma_i\pi_i + \sigma_{i+1}\pi_i'$ . Then  $\gamma_{i+1}\xi_i = \beta_i\gamma_i$ , and  $\gamma_i\mu_i = \phi_i$ .

Dually, a map  $\phi \in (B, A)$  is  $p$ -homotopic to  $o$  if and only if it can be factored by the proper epimorphism  $\epsilon$  from  $Y(A)$  to  $A$  constructed in Example (b) of § 1.4. The proof of the following lemma is left to the reader.

LEMMA 14.2 *A chain map  $\phi \in \text{Hom}_o(B, A)$  is homotopic to  $o$  if and only if there is a  $\gamma \in \text{Hom}_o(B, Y(A))$  such that  $\phi = \epsilon\gamma$ .*

Now it follows immediately that a map  $\phi \in (A, B)$ , considered as chain map in  $\text{Hom}_o(A, B)$ , is homotopic to  $o$  if and only if  $\phi \simeq_i o$  and  $\phi \simeq_p o$ . Thus  $\Pi^i(A, B) = \Pi^p(A, B) = \mathfrak{S}\text{om}_o(A, B)$ .

For a complex  $A$ , the construction of a suspension  $\Sigma A$  will be given explicitly.  $X(A)$  and  $\mu \in (A, X(A))$  are to be constructed as in Example (b) of § 1.4. A complex  $A'$  and a map  $\epsilon \in (X(A), A)$  are given as follows:  $A'_i = A_{i+1}$ ,  $\alpha'_i = -\alpha_{i+1}$ ,  $\epsilon_i = \alpha_i\pi_i - \pi_i'$  (with the notation of Example (b)). Then,  $\alpha'_i\epsilon_i = \epsilon_{i+1}\xi_i$  and the sequences

$$O \rightarrow A_i \xrightarrow{\mu_i} X_i \xrightarrow{\epsilon_i} A'_i \rightarrow O$$

are splitting s.e.s. of  $\mathfrak{M}$ . Hence,  $A' = \Sigma A$ . Iterating this construction, one sees that an  $n$ -fold suspension  $\Sigma^n A$  of  $A$  is given by the complex  $A'$  where  $A'_i = A_{i+n}$ ,  $\alpha'_i = (-1)^n\alpha_{i+n}$ . Dually, an  $n$ -fold dual suspension  $\Omega^n A$  of  $A$  is given by the complex  $A'$  where  $A'_i = A_{i-n}$ ,  $\alpha'_i = (-1)^n\alpha_{i-n}$ .

Clearly, one obtains isomorphisms  $t_p: (\Sigma^p A, B) \rightarrow \text{Hom}_{-p}(A, B)$  by setting  $t_p(\phi_i)_{i \in \mathbb{Z}} = ((-1)^{p^i}\phi_{i-p})_{i \in \mathbb{Z}}$  for all  $\phi = (\phi_i)_{i \in \mathbb{Z}} \in (\Sigma^p A, B)$ . By 14.1,  $\phi \simeq_i o$  if and only if  $t_p\phi \simeq o$ . Hence,  $t_p$  induces an isomorphism from  $\Pi_p^i(A, B)$  to  $\mathfrak{S}\text{om}_{-p}(A, B)$ . Similarly, an isomorphism between  $\Pi_p^i(A, B)$  and  $\text{Hom}_{-p}(A, B)$  can be given. Thus, up to explicit isomorphisms,  $\Pi_p^i(A, B) = \Pi_p^p(A, B) = \mathfrak{S}\text{om}_{-p}(A, B)$  for all  $p \geq 0$ .

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