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On the Garsia Lie Idempotent

Frédéric Patras, Christophe Reutenauer and Manfred Schocker

Abstract. The orthogonal projection of the free associative algebra onto the free Lie algebra is afforded by an idempotent in the rational group algebra of the symmetric group S_n, in each homogenous degree n. We give various characterizations of this Lie idempotent and show that it is uniquely determined by a certain unit in the group algebra of S_{n-1}. The inverse of this unit, or, equivalently, the Gram matrix of the orthogonal projection, is described explicitly. We also show that the Garsia Lie idempotent is not constant on descent classes (in fact, not even on coplactic classes) in S_n.

1 Introduction

The celebrated criterion of Friedrichs characterizes Lie polynomials in the algebra of noncommutative polynomials (equivalently, Lie elements in the tensor algebra). An equivalent version of this criterion states that a (noncommutative) polynomial is a Lie polynomial if and only if it is orthogonal to each proper shuffle (see [11, Problem 5.3.4], [14, 3.5.1]). This result motivated consideration of the orthogonal projection onto the free Lie algebra, as was pointed out by Garsia in [7]; its kernel is the space spanned by proper shuffles. This projection commutes with each homogeneous endomorphism of the algebra, since the algebra of Lie polynomials and the space of proper shuffles are both closed under these endomorphisms. Hence, in given degree n, the projection is afforded by an element of the group algebra $\mathbb{Q}S_n$ acting naturally, by permuting positions, on the homogeneous polynomials of degree n (Schur–Weyl duality). This element of the group algebra will necessarily be an idempotent: the *orthogonal Lie idempotent*, or *Garsia idempotent* (see [6], [14, 8.6.4]). We denote it by g_n.

2 Orthogonality

We denote by $\langle u, v\rangle$ the scalar product in $\mathbb{Q}S_n$ for which S_n is an orthonormal basis. Let $x \mapsto x^-$ be the linear endomorphism of $\mathbb{Q}S_n$ that extends the mapping $w \mapsto w^{-1}$ of S_n into itself.

Denote by $\mathcal{L}_n$ the space of multilinear Lie polynomials, naturally embedded in $\mathbb{Q}S_n$. Recall that $\mathcal{L}_n$ is a left ideal in $\mathbb{Q}S_n$ and that a *Lie idempotent* is an idempotent e in $\mathbb{Q}S_n$ such that $\mathcal{L}_n = \mathbb{Q}S_n\, e$, see [14, 8.4].

Note that the set of Lie idempotents is an affine subspace of $\mathbb{Q}S_n$, as is well-known; more precisely, if e is a Lie idempotent in $\mathbb{Q}S_n$, then the set of all Lie idempotents in $\mathbb{Q}S_n$ is $e + (1-e)\mathbb{Q}S_n e$. Indeed, idempotents e and f generate the same left ideal if

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and only if $ef = e$ and $fe = f$. If this condition holds, then $(e - f)e = e - f = (1 - e)(e - f)$, hence $f = e - (e - f)e = e - (1 - e)(e - f)e \in e + (1 - e)\,\mathbb{Q}S_n e$. Conversely, if $f = e + (1 - e)x\,e$, then $ef = e$ (since $e(1 - e) = 0$) and $fe = f$. Furthermore, f is idempotent since $f^2 = fef = fe = f$.

We obtain the following characterization of the Garsia idempotent g_n.

Theorem 1 *The following conditions are equivalent, for a Lie idempotent e in* $\mathbb{Q}S_n$*:*

(i) $e = g_n$;
(ii) $e = e^-$;
(iii) *e is the orthogonal projection of* 0 *onto the affine subspace of Lie idempotents;*
(iv) *e is of minimum norm among all Lie idempotents;*
(v) *the square of the norm of e is* $\frac{1}{n}$*;*
(vi) $\mathcal{L}_n$ *is orthogonal to* $\mathbb{Q}S_n(1 - e)$.

Proof In fact, we may replace S_n by any finite group G, $\mathbb{Q}$ by any subfield K of $\mathbb{R}$, and $\mathcal{L}_n$ by any left ideal I of KG; then let $\langle u, v\rangle$ denote the scalar product on KG for which G is an orthonormal basis; next, let $I^\perp$ be the subspace of KG orthogonal to I ($I^\perp$ plays the role of the subspace of multilinear shuffles in $\mathbb{Q}S_n$). The projection $\pi\colon KG \to I$ with kernel $I^\perp$ is left KG-linear, since $I, I^\perp$ are left ideals and since $\langle xy, xz\rangle = \langle y, z\rangle$ for x in G and y, z in KG.

Hence $\pi(x) = xg$, for some idempotent g in KG, defined by $g = \pi(1)$. This g will replace the Garsia idempotent in our general setting. Note that $KGg = I$.

We now take any idempotent e in KG such that $KGe = I$, and prove the equivalence of the six conditions above, where in (v), $\frac{1}{n}$ is replaced by e_1, the coefficient of 1 in e; indeed, for any Lie idempotent e, $e_1 = \frac{1}{n}$, see [14, Th. 8.14].

It is a well-known fact that for any idempotent e, the dimension of KGe is equal to $|G|e_1$, since for any projection p of a vector space, $\dim(\operatorname{Im} p) = \operatorname{Tr}(p)$.

(i) $\Leftrightarrow$ (vi): The kernel of the mapping $x \mapsto xe, KG \to KG$, is $KG(1 - e)$, and its image is $KGe = I$. Hence this mapping is the orthogonal projection π onto I if and only if I is orthogonal to $KG(1 - e)$. We conclude since $KGg = I$.

(i) $\Rightarrow$ (ii): An orthogonal projection is self-adjoint. Since the adjoint of $x \mapsto xe$ is $x \mapsto xe^-$ (because $\langle xy, z\rangle = \langle x, zy^-\rangle$), we must have $e = e^-$.

(ii) $\Rightarrow$ (iii): If f, f' are two idempotents with $I = KGf = KGf'$, then $ef = e = ef'$, because $e \in KGf = KGf'$. It follows that $\langle e, f - f'\rangle = \langle e^-, f - f'\rangle = \langle 1, e(f - f')\rangle = 0$, since $e = e^-$. Hence e is the orthogonal projection of 0 onto the affine subspace of idempotent generators of the left ideal I.

(iii) $\Leftrightarrow$ (iv): is clear.

(iii) $\Rightarrow$ (i): What has been already proved shows that g is the orthogonal projection of 0 onto the affine subspace of all idempotent generators of the left ideal I.. By unicity of this projection, we deduce that $e = g$.

(i) $\Rightarrow$ (v): Since $g = g^-$, we have $\langle g, g\rangle = \langle g, g^-\rangle = \langle g^2, 1\rangle = \langle g, 1\rangle = g_1 = \frac{\dim I}{|G|} = e_1$.

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(v) $\Rightarrow$ (iv): We know already that g is of minimum norm among all Lie idempotents, and that the square of this norm is $\frac{\dim I}{|G|}$. Hence e is also of minimum norm. ∎

To conclude this section, we compute the dimension of the affine space of Lie idempotents. This computation is classical.

Let e be a Lie idempotent. Recall that the set of all Lie idempotents is $e + (1-e)\mathbb{Q}S_n e$, hence the dimension we seek is simply the dimension of $(1-e)\mathbb{Q}S_n e$. Since e and $1-e$ are both idempotents, the latter dimension is classically equal to the scalar product of the characters of S_n acting on $\mathbb{Q}S_n e$ and $\mathbb{Q}S_n(1-e)$ (see [5, Th. 43.18]). If we denote the first character by χ, the second will be $\rho - \chi$, where ρ is the character of the regular representation, since $\mathbb{Q}S_n = \mathbb{Q}S_n e \bigoplus \mathbb{Q}S_n(1-e)$, as S_n-modules. If we denote by f_λ (resp., n_λ) the multiplicity of the irreducible representation corresponding to partition λ in the regular representation (resp., in the Lie representation), this dimension is therefore equal to $\sum_\lambda n_\lambda(f_\lambda - n_\lambda)$.

Note that f_λ is the number of Young tableaux of shape λ (see [12, Chapter 1]) and n_λ is the number of such tableaux whose major index is $\equiv 1$ modulo n (theorem of Kraskiewicz–Weyman, see [14, Cor. 8.10]). We may pursue a little further this calculation. Since f_λ is also the dimension of the irreducible representation corresponding to λ, $\sum_\lambda n_\lambda f_\lambda$ is the dimension of the Lie representation, that is, $(n-1)!$. And $\sum_\lambda n_\lambda^2$ is the scalar product of this representation by itself. Taking the formalism of symmetric functions, its character is $l_n = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d}$ (see [14, Chapter 8]) and thus $\langle l_n, l_n\rangle = \frac{1}{n^2}\sum_{d|n}\mu(d)^2\left(\frac{n}{d}\right)!\, d^{n/d}$, since $\langle p_d^{n/d}, p_d^{n/d}\rangle = \left(\frac{n}{d}\right)!\, d^{n/d}$ (see [12, Chapter 1]), and since the p_μ are mutually orthogonal. Thus the dimension of the affine subspace of Lie idempotents is $(n-1)! - \frac{1}{n^2}\sum_{d|n}\mu(d)^2\left(\frac{n}{d}\right)!\, d^{n/d}$.

For $n = 1$ to 8, it is $0, 0, 1, 4, 19, 98, 617, 4404$. In his thesis, the third author gives the following approximation for the numbers $\langle l_n, l_n\rangle$:

$$\frac{1}{n}(n-1)! \leq \langle l_n, l_n\rangle \leq \frac{1}{n-1}(n-1)!.$$

Furthermore, if n is prime, $\langle l_n, l_n\rangle$ is the smallest integer $\geq \frac{1}{n}(n-1)!$, that is $\frac{(n-1)!+1}{n}$ by Wilson's theorem, see [17, Korollar 5.3].

3 On Duchamp's Computation of the Garsia Idempotent

In order to compute the Garsia idempotent g_n, we follow the method of Gérard Duchamp [6]. We take the alphabet $\{1, 2, 3, \dots\}$ and consider each permutation as a word in this alphabet. Then the set of multilinear Lie polynomials of degree n is a left ideal $\mathcal{L}_n$ in $\mathbb{Q}S_n$; similarly, the space $\mathcal{S}_n$ spanned by proper multilinear shuffles of degree n is a left ideal in $\mathbb{Q}S_n$ (for example, $123 - 213 - 312 + 321 = [[1,2],3]$ is a multilinear Lie polynomial, and $1 \sqcup\!\sqcup\, 23 = 123 + 213 + 231$ is a proper shuffle; the reader may verify that they are indeed orthogonal with respect to the scalar product which has S_n as orthonormal basis).

Recall that the *shuffle* $u \sqcup\!\sqcup\, v$ of two words u, v is defined as the sum of all words w having u as subword, and v as complementary subword, with multiplicities. We

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say that word w *is a shuffle* of words u and v if w appears in $u \sqcup\!\sqcup v$; furthermore, we say that a word u is *increasing*, if for each i, its letter in i-th position is $\leq$ to its letter in $(i+1)$-th position; we shall consider in the sequel words which are shuffle of two increasing words, and we consider the number of such shuffles, where we shall distinguish between $u \sqcup\!\sqcup v$ and $v \sqcup\!\sqcup u$. For example, 3142 has two decompositions as shuffle of two increasing words, namely $34 \sqcup\!\sqcup 12$, and $12 \sqcup\!\sqcup 34$.

Furthermore, $21534 \in S_5$ is the shuffle of two increasing words, but $321 \in S_3$ is not. It is a well-known fact, proved by Curtis Greene, that a permutation w is the shuffle of two increasing words if and only if its shape, in the Robinson–Schensted correspondence, is a partition with at most two parts; equivalently, w has no decreasing subsequence of length 3 (*i.e.*, for no subset I of cardinality 3, the mapping $w \mid I$ is decreasing); in other words, it is 321-avoiding. See [8, 15].

The idempotent g_n is equal to $\pi(12\dots n)$, where π is the projection onto $\mathcal{L}_n$ with kernel $\mathcal{S}_n$. Indeed, since $\mathcal{L}_n$ and $\mathcal{S}_n$ are left ideals in $\mathbb{Q}S_n$, π is left $\mathbb{Q}S_n$-linear, so that for any element P in $\mathbb{Q}S_n : \pi(P) = \pi(P(12\cdots n)) = P\pi(12\cdots n) = Pg_n$ (product in $\mathbb{Q}S_n$).

Now, it is a well-known fact that the $(n-1)!$ Lie polynomials $r(\sigma), \sigma \in S_n, \sigma(n) = n$, form a basis of $\mathcal{L}_n$, where r is the *right-to-left bracketing mapping*, that is,

$$r(i_1 \cdots i_n) = [i_1, \dots, [i_{n-1}, i_n] \cdots]$$

(see *e.g.*, [14, 5.6.2]). Hence, we may write $g_n = \sum_\sigma x_\sigma r(\sigma)$. Recall the notation $\langle , \rangle$ for the scalar product for which S_n is an orthonormal basis. Then $12\cdots n - g_n$ is orthogonal to $\mathcal{L}_n$, hence to each $r(\alpha)$, and we have:

$$\langle g_n, r(\alpha)\rangle = \langle 12\cdots n, r(\alpha)\rangle$$

for each α in S_n. This gives the system of $(n-1)!$ equations in the unknowns x_σ:

$$\sum_\sigma x_\sigma \langle r(\sigma), r(\alpha)\rangle = \delta_{12\cdots n,\alpha}$$

with the conditions: $\alpha, \sigma \in S_n, \alpha(n) = \sigma(n) = n$, because $\langle 12\cdots n, r(\alpha)\rangle = \delta_{12\cdots n,\alpha}$. This shows that (x_σ) is the first row of the inverse of the matrix

$$(\langle r(\sigma), r(\alpha)\rangle)_{\sigma(n)=\alpha(n)=n}$$

of size $(n-1)!$ by $(n-1)!$.

Duchamp [6, p. 238] gives a recursive procedure to compute the coefficients of this matrix; as a byproduct, he obtains that each nonzero entry is a power of 2, *cf.* Cor. 6.5.

Our aim here is to give an explicit combinatorial description of the coefficients of this matrix. For this we consider the *disjoint* union $S = \bigcup_{n\geq 0} S_n$ of all symmetric groups; it is a monoid, with product $S_n \times S_p \mapsto S_{n+p}, (u, v) \mapsto w$, where $w_i = u_i$ if $i \leq p, w_i = v_{i-p}$ if $i > p$. It is a well-known fact that S is a free monoid (see [4]). Its free generators are the connected permutations: a permutation w is *connected* if w, viewed as a mapping $[n] \to [n]$, does not stabilize any proper subinterval $[i]$ of $[n]$.

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Equivalently, w does not belong to any proper Young subgroup of S_n (a *Young subgroup* is a subgroup of the form $S_{i_1} \times \cdots \times S_{i_k}$, canonically embedded into S_n).

For later use, we note that u is connected if and only if:

$$(*) \qquad \forall j \in \{1, \ldots, n-1\}, \exists\, i, k \text{ such that } 1 \le i \le j < k \le n \text{ and } u_i > u_k.$$

The length of an element α of the free monoid S is denoted $L(\alpha)$; if $\alpha \in S_n$, it is the biggest k such that α belongs to a Young subgroup of the form $S_{i_1} \times \cdots \times S_{i_k}$ (note that there is always a unique minimal Young subgroup containing α).

Coming back to our initial problem, note that the mapping r is left $\mathbb{Q}S_n$-linear; that is, for permutations α, σ in S_n, one has $r(\alpha \circ \sigma) = \alpha \circ r(\sigma)$.

Now, the following theorem yields an explicit description of the coefficients of the matrix $(\langle r(\alpha), r(\beta)\rangle)_{\alpha,\beta \in S_n, \alpha(n)=\beta(n)=n}$.

Theorem 2

(1) *Let $w \in S_n$ with $w(n) = n$,* i.e., *$w = un$ as word. Then $\langle r(12\cdots n), r(w)\rangle$ is the number of ways of writing u as the shuffle of two increasing words.*

(2) *Let $u \in S_p$ be the shuffle of two increasing words ($p \ge 1$). Then there are 2^k ways of writing u as shuffle of two increasing words, where k is the length of u in the free monoid S.*

Proof (1) The following formula is well-known:

$$r(12\cdots n) = \sum (-1)^{|y|} xny,$$

where the sum is taken over all words x, y such that xny is a permutation in S_n, that x is increasing, y is decreasing, and where $|y|$ is the length of the word y.

For example, $r(1234) = [1, [2, [3, 4]]] = 1234 - 1243 - 1342 + 1432 - 2341 + 2431 + 3421 - 4321$.

Likewise, if we take $w = un$ as in the statement, and write $u = u_1 \cdots u_{n-1}$ as a product of letters, then we have:

$$r(un) = \sum (-1)^q u_{i_1} \cdots u_{i_p} n u_{j_q} \cdots u_{j_1},$$

where the conditions of summation are: $i_1 < \cdots < i_p, j_1 < \cdots < j_q, p + q = n - 1, \{i_1, \ldots, i_p, j_1, \ldots, j_q\} = \{1, \ldots, n-1\}$.

If we compare the two formulas, we find that the scalar product $\langle r(12 \ldots n), r(un)\rangle$ is equal to the number of $i_1, \ldots, i_p, j_1, \ldots, j_q$ satisfying the previous conditions, and moreover $u_{i_1} < \cdots < u_{i_p}, u_{j_1} < \cdots < u_{j_q}$: in other words, it is the number of ways of writing u as the shuffle of two increasing words.

(2) We call *factorization* of u a pair (r, s) of increasing words such that u is a shuffle of r and s. We assume first that u is connected, that is, $k = 1$. We show, by induction on the length (as word, not element of S!), that for each prefix x of u, proper and nonempty, each factorization (r, s) of x has at most one extension to a factorization (r', s') of a longer prefix x' of u (meaning that r is a prefix of r', and s a prefix of s').

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This will suffice since the prefix u_1 of length 1 of u has exactly two factorizations (u_1, ε) and (ε, u_1), with $\varepsilon =$ empty word.

So, let (r, s) be a factorization of x. Let j be the length of x. In $(*)$, take the smallest possible k. Let (r', s') be a factorization of the prefix x' of length k of u (if it exists). Then u_i appears in r for example; it implies that u_k cannot appear in r', since $u_i > u_k$.

Hence u_k appears in s'. For $k' = j+1, \dots, k-1$, we have $u_{k'} > u_k$ (otherwise $u_{k'} < u_k < u_i$, which contradicts the minimality of k). Hence $s' = su_k$, $r' = ru_{j+1} \cdots u_{k-1}$, which proves unicity of extension.

Suppose now that u is of length k in the free monoid $S : u = v_1 v_2 \cdots v_k, v_i \in S$ connected, and the product is taken in the monoid S. Each factorization of u determines a factorization of each v_i, and conversely, a collection of factorizations of the $v_i's$ determines a factorization of u. This implies the theorem. ∎

It is well-known that the number of 321-avoiding permutations in S_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. This shows that the number of nonzero elements in each row, or column, of the matrix $(\langle r(\sigma), r(\alpha)\rangle)$ considered by Duchamp, and above, is the Catalan number.

We show that the previous computation has an equivalent formulation in the symmetric group algebra $\mathbb{Q}S_{n-1}$; we consider S_{n-1} as the subgroup of S_n of permutations fixing n, and consequently $\mathbb{Q}S_{n-1}$ as a subalgebra of $\mathbb{Q}S_n$. We denote by $x \mapsto \tilde{x}$ the linear mapping $\mathbb{Q}S_n \to \mathbb{Q}S_{n-1}$ fixing each permutation in S_{n-1}, and sending the others to 0.

Recall that we had written $g_n = \sum_\sigma x_\sigma r(\sigma)$, where the sum is over all σ in S_{n-1}, and $x_\sigma \in \mathbb{Q}$; note that, as is well-known (it follows from the first formula in the proof of Th. 2), $\widetilde{r(\sigma)} = \sigma$, that is, the only permutation fixing n and appearing in $r(\sigma)$ is σ itself. Hence $\widetilde{g_n} = \sum_\sigma x_\sigma \sigma$, and if we know $\widetilde{g_n}$, we know the coefficients x_σ, hence g_n. In fact, one has $g_n = \widetilde{g_n}\omega_n$ in $\mathbb{Q}S_n$ where $\omega_n = r(12\cdots n)$, by left equivariance of $r : g_n = \sum_{\sigma\in S_{n-1}} x_\sigma r(\sigma) = \sum x_\sigma \sigma r(12\cdots n) = \widetilde{g_n}\omega_n$.

Theorem 3

(1) *In $\mathbb{Q}S_{n-1}$, $\widetilde{g_n}$ is invertible and $\widetilde{g_n}^{-1} = \sum_{\sigma\in S_{n-1}} \langle r(12\cdots n), r(\sigma)\rangle\sigma$.*
(2) *In the algebra $\mathbb{Q}S_{n-1}$, one has the equality $\widetilde{g_n}^{-1} = \widetilde{\omega_n \overline{\omega_n}}$.*
(3) *The matrix $(\langle r(\alpha), r(\beta)\rangle)_{\alpha,\beta\in S_{n-1}}$ is the image of $\widetilde{g_n}^{-1}$ under the right regular representation of S_{n-1}.*

Proof

(1): We have to show that in $\mathbb{Q}S_{n-1}$ the product of $\widetilde{g_n}$ by

$$\sum_{\sigma\in S_{n-1}} \langle r(12\cdots n), r(\sigma)\rangle\sigma$$

is equal to $12\cdots n$. This product is equal to $\sum_{\alpha,\sigma\in S_{n-1}} x_\alpha \langle r(12\cdots n), r(\sigma)\rangle \alpha\sigma$. The coefficient of β in this product is

$$\sum_{\alpha\sigma=\beta} x_\alpha \langle r(12\cdots n), r(\sigma)\rangle = \sum_{\alpha\sigma=\beta} x_\alpha \langle r(\alpha), r(\alpha\sigma)\rangle,$$

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by left equivariance of r; this is equal to

$$\sum_{\alpha} x_\alpha \langle r(\alpha), r(\beta)\rangle = \delta_{12\cdots n,\beta},$$

by the system of equations satisfied by the x_α. This proves that the product is equal to $12\cdots n$.

(2): Note that for $x, y \in \mathbb{Q}S_n$, $\tilde{x}\tilde{y} = \widetilde{xy}$, as is verified when x, y are permutations. Hence we have $\widetilde{g}_n\widetilde{\omega_n\omega_n^-} = \widetilde{g_n\omega_n\omega_n^-}$. Now $\widetilde{g}_n\omega_n = g_n$, as is proved before the theorem. Hence $\widetilde{g}_n\widetilde{\omega_n\omega_n^-} = \widetilde{g_n\omega_n^-} = \widetilde{(\omega_n g_n)^-} = 12\cdots n$, because $\omega_n g_n = \omega_n$, since $\omega_n \in \mathbb{Q}S_n g_n$, and because the only permutation appearing in ω_n and fixing n is $12\cdots n$.

(3): Let $M(\sigma)$ denote the image of $\sigma \in S_{n-1}$ under the right regular representation of S_{n-1}. In other words, $M(\sigma)_{\alpha,\beta} = 1$ if $\alpha\sigma = \beta$, 0 otherwise.

The image of $\widetilde{g}_n^{-1}$ is therefore $\sum_{\sigma\in S_{n-1}} \langle r(12\cdots n), r(\sigma)\rangle M(\sigma)$. The (α,β)-entry of this matrix is $\langle r(12\cdots n), r(\sigma)\rangle$, where $\alpha\sigma = \beta$. This is, by left equivariance of r, equal to $\langle r(\alpha), r(\beta)\rangle$, what was to be shown. ∎

Remark Similar calculations show that $\widetilde{g}_n\omega_n\tilde{g}_n^{-1} = \omega_n^-$. In other words, $\widetilde{g}_n$ conjugates ω_n and ω_n^-, which illustrates the fact that ω_n and ω_n^- are idempotents that yield the same character.

In order to prove this formula, recall that $g_n = \widetilde{g}_n\omega_n$ and thus $\widetilde{g}_n^{-1} g_n = \omega_n$. Furthermore, $g_n = g_n^-$ implies $\widetilde{g}_n^{\,-} = \tilde{g}_n$ and $(\widetilde{g}_n^{-1})^- = \widetilde{g}_n^{-1}$. Hence $\tilde{g}_n\omega_n\widetilde{g}_n^{-1} = g_n\widetilde{g}_n^{-1} = g_n^-(\widetilde{g}_n^{-1})^- = (\widetilde{g}_n^{-1}g_n)^- = \omega_n^-$.

4 Garsia Idempotent and Descent Algebra

All of the classical Lie idempotents are elements of the Solomon descent algebra $\mathcal{D}_n$ (see [14, Chapter 8]). This algebra is the linear span in $\mathbb{Q}S_n$ of the elements

$$\Delta^D = \sum_{\mathrm{Des}(\pi)=D} \pi$$

indexed by subsets D of $\{1,\ldots,n-1\}$, where $\mathrm{Des}(\pi) = \{i \le n-1 \mid \pi(i) > \pi(i+1)\}$ is the *descent set* of $\pi \in S_n$.

However, we have the following negative result, which is well known for small values of n.

Theorem 4 *The Garsia Lie idempotent g_n does not lie in $\mathcal{D}_n$ for $n \ge 4$.*

Proof It follows from [1, Th. 1.2] that

$$g_n\Delta^D = (-1)^{|D|} g_n.$$

Comparing coefficients of the identity in S_n, gives

$$\sum_{\mathrm{Des}(\pi^{-1})=D} k_\pi = (-1)^{|D|}\frac{1}{n},$$

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where $g_n = \sum_{\pi \in S_n} k_\pi \pi$.

Assume for a contradiction that g_n is constant on descent classes, then $g_n = g_n^-$ is also constant on inverse descent classes, hence the above formula implies

$$k_\pi = \frac{1}{n} \frac{(-1)^{|D|}}{c_D}$$

for all $\pi \in S_n$ with $\mathrm{Des}(\pi^{-1}) = D$, where c_D denotes the number of permutations in S_n with descent set D. However, the number of descents of $\pi = 24135 \cdots n \in S_n$ is 1, while the number of descents of $\pi^{-1} = 31425 \cdots n$ is 2, hence $k_{\pi^{-1}} < 0 < k_\pi$, a contradiction. ■

Corollary The Garsia Lie idempotent g_n does not lie in the coplactic algebra, for $n \geq 4$.

Proof The third author has shown that a Lie element in $\mathbb{Q}S_n$, which is also in the *coplactic algebra* (see [3, 13]), is necessarily contained in $\mathcal{D}_n$, see [16]. Hence, the corollary follows from the theorem. ■

5 Special Lie Idempotents

Motivated by Theorem 3, we say that a Lie idempotent e in $\mathbb{Q}S_n$ is *special* if

$$\tilde{e} = \sum_{\substack{\sigma \in S_n \\ \sigma(n)=n}} e_\sigma \sigma$$

is invertible in $\mathbb{Q}S_{n-1}$. Of course, the Garsia idempotent g_n is special by Th. 3. Moreover, $\frac{1}{n}\omega_n = \frac{1}{n} r(12 \cdots n)$ (the idempotent of Dynkin–Specht–Wever) is special, since $\tilde{\omega}_n = 12 \cdots n$, as already mentioned before Th. 3.

Recall that the *canonical idempotent* is $\frac{1}{n}\rho_n$, with

$$\rho_n = \sum_{\pi \in S_n} \frac{(-1)^{\mathrm{des}(\pi)}}{\binom{n-1}{\mathrm{des}(\pi)}} \pi,$$

where $\mathrm{des}(\pi) = \#\,\mathrm{Des}(\pi)$, see [14, Th. 8.16].

Then ρ_4 is not special. Indeed, $\widetilde{\rho_4} = 123 - \frac{1}{3}(132 + 213 + 231 + 312) + \frac{1}{3}321$. The sum of its coefficients is 0, so $\widetilde{\rho_4}$ is in a proper ideal of $\mathbb{Q}S_3$, and not invertible.

Special Lie idempotents are characterized by the following result.

Theorem 5 *The following conditions are equivalent, for a Lie idempotent e:*

(i) *e is special;*

(ii) $\{\sigma e \mid \sigma \in S_n, \sigma(n) = n\}$ *is a basis of the space* $\mathcal{L}_n$ *of multilinear Lie polynomials.*

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Proof We know that (i) and (ii) hold for $\omega_n = r(12\cdots n)$. Hence $\{\sigma\omega_n \mid \sigma(n) = n\}$ is a basis of $\mathcal{L}_n$, which is of dimension $(n-1)!$. Since KS_{n-1} (considered as a subspace of KS_n) is also of dimension $(n-1)!$, we see that the mapping $KS_{n-1} \to KS_n, \alpha \mapsto \alpha\omega_n$ is injective.

Now, e is special if and only if $KS_{n-1}\tilde{e} = KS_{n-1}$, that is: $\dim(KS_{n-1}\tilde{e}) = (n-1)!$.

On the other hand, (ii) holds for e if and only if $\dim(KS_{n-1}e) = (n-1)!$; now, we have $e = \tilde{e}\omega_n$ (by the same argument as for g_n, before the proof of Th. 3), and since the mapping above is injective, the latter condition is equivalent to $\dim(KS_{n-1}\tilde{e}) = (n-1)!$, which ends the proof. ■

We may deduce a result already proved in [2].

Corollary The Klyachko Lie idempotent $\kappa_n = \frac{1}{n}\sum_{\pi\in S_n} \varepsilon^{\mathrm{maj}(\pi)}\pi$ is special.

Here ε is a primitive n-th root of 1 and $\mathrm{maj}(\pi)$ is the sum of the descents of π, see [9] or [14, Th. 8.17].

Proof It follows from Klyachko's work that if C is a set of coset representatives of S_n under right multiplication by the subgroup generated by the long cycle $(12\cdots n)$, then $\{\sigma\kappa_n \mid \sigma \in C\}$ is a basis of the space of multilinear Lie elements, see [14, Cor. 8.20]. We may take $C = \{\sigma \in S_n \mid \sigma(n) = n\}$, hence κ_n is special by the theorem. ■

Note that the results of this section extend some of those of [10, Section 9], where the authors use the supplementary hypothesis that e is in the descent algebra.

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Laboratoire J.-A. Dieudonné
Université de Nice
Parc Valrose
06108 Nice
cedex 02
France

Université du Québec à Montréal
Département de mathématiques
CP 8888, succ. Centre-Ville
Montréal, QC
H3C 3P8
e-mail: christo@math.uqam.ca

Department of Mathematics
University of Wales Swansea
Singleton Park
Swansea SA2 8PP
Wales
U.K.

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