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Abstract

Let G be the simple algebraic group $\mathrm{Sp}(2, 2)$, to be defined over $\mathbb{Q}$. It is a non-quasi-split, $\mathbb{Q}$-rank-two inner form of the split symplectic group Sp_8 of rank four. The cohomology of the space of automorphic forms on G has a natural subspace, which is spanned by classes represented by residues and derivatives of cuspidal Eisenstein series. It is called Eisenstein cohomology. In this paper we give a detailed description of the Eisenstein cohomology $H^q_{\mathrm{Eis}}(G, E)$ of G in the case of regular coefficients E. It is spanned only by holomorphic Eisenstein series. For non-regular coefficients E we really have to detect the poles of our Eisenstein series. Since G is not quasi-split, we are out of the scope of the so-called 'Langlands–Shahidi method' (cf. F. Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), 297–355; F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math. (2) **127** (1988), 547–584). We apply recent results of Grbac in order to find the double poles of Eisenstein series attached to the minimal parabolic P_0 of G. Having collected this information, we determine the square-integrable Eisenstein cohomology supported by P_0 with respect to arbitrary coefficients and prove a vanishing result. This will exemplify a general theorem we prove in this paper on the distribution of maximally residual Eisenstein cohomology classes.

Introduction

Let G be a connected, semisimple algebraic group defined over $\mathbb{Q}$ of $\mathbb{Q}$-rank $rk_{\mathbb{Q}}(G) \geqslant 1$, let E be a finite-dimensional, irreducible complex representation of the Lie group $G(\mathbb{R})$ of real points of G, and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic congruence subgroup. The study of the cohomology spaces $H^*(\Gamma, E)$ has been carried out over the last 40 years from various points of view and motivations, using and comparing several techniques. Among these techniques, the cohomology of arithmetic groups has major applications within the Langlands program, which itself originated in the attempt to solve classical problems of algebraic and analytic number theory, such as giving a satisfactory non-abelian class field theory. This approach to cohomology of an arithmetically defined group indicates a close connection to the theory of automorphic forms, in particular to cusp forms and Eisenstein series.

The link between $H^*(\Gamma, E)$ and automorphic forms was first provided in a conceptual way by Harder in the case of groups of $\mathbb{Q}$-rank one [Har75b, Har75a]. His method is of a differential

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geometric nature and uses the fact that the cohomology of Γ is isomorphic to the cohomology of a certain compact space $\Gamma\backslash\overline{X}$, which is a manifold with boundary $\partial(\Gamma\backslash\overline{X})$. In fact, $X = G(\mathbb{R})/K$ is the Riemannian symmetric space associated with the Lie group $G(\mathbb{R})$ and a maximal compact subgroup K, and $\Gamma\backslash\overline{X}$ is the Borel–Serre compactification of the locally symmetric quotient $\Gamma\backslash X$. With this framework at place, Harder showed that one can construct the 'cohomology at infinity', i.e. (up to isomorphy) the image of the natural restriction map $H^*(\Gamma\backslash\overline{X}, E) \to H^*(\partial(\Gamma\backslash\overline{X}), E)$ by means of Eisenstein series. This image is complementary within $H^*(\Gamma, E)$ to the cohomology of a space of square-integrable automorphic forms, which contains the cusp forms.

In the early 1990s, Franke finally proved in [Fra98] that such a decomposition can also be given in the general framework of an arbitrary connected, reductive algebraic group G. More precisely, Franke particularly showed that the cohomology of an arithmetic congruence subgroup $\Gamma \subset G(\mathbb{Q})$ decomposes as

$$H^*(\Gamma, E) = H_{\text{cusp}}(\Gamma, E) \oplus H_{\text{Eis}}(\Gamma, E)$$

into the cohomology space of classes represented by cuspidal automorphic forms and a natural complement called the Eisenstein cohomology of Γ. This is due to the adelic interpretation of $H^*(\Gamma, E)$ as a subspace of the space of K_Γ fixed vectors in

$$H^*(G, E) := H^*(\mathfrak{g}, K, \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E)$$

and an analogous decomposition of this cohomology as $H^*(G, E) = H_{\text{cusp}}(G, E) \oplus H_{\text{Eis}}(G, E)$. (Here K_Γ is an appropriate open, compact subgroup of the group of finite adelic points $G(\mathbb{A}_f)$ and $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is the usual space of (adelic) automorphic forms on G.)

In the case of regular coefficients E, i.e. the highest weight λ of E lies inside the open, positive Weyl chamber, the space of Eisenstein cohomology was investigated by Schwermer in [Sch94] together with Li in [LS04]. It was shown that under this assumption on E, each class in $H^*_{\text{Eis}}(G, E)$ can be represented by a bunch of Eisenstein series evaluated at a certain point in the region of holomorphy [Sch94, §§ 2 and 6]. This lead to a vanishing result in lower degrees of cohomology (cf. [LS04, Theorem 5.5]).

Still, for non-regular coefficients E, little is known in general: in contrast to the regular case, residues of Eisenstein series really enter the game (i.e. can contribute non-trivially to cohomology), when regarding non-regular coefficient modules E. The results gained so far suggest that the poles of Eisenstein series are encoded by poles and zeros (i.e. so-called special values) of automorphic L-functions. However, even for square-integrable residues the situation is not fully understood, since, e.g., a satisfactory theory of describing the residual spectrum of a non-quasi-split algebraic group G is not yet available.

On the other hand, finding the poles of Eisenstein series is not the only difficulty one encounters in this case. One also has to understand *whether* residual Eisenstein series contribute non-trivially to cohomology and, if they do contribute, *in which degrees* of cohomology. Again, all of these problems are entirely linked to deep questions of local and global representation theory and number theory, in particular the Langlands program.

In the present work we consider the above questions and approaches to Eisenstein cohomology regarding the connected, simple algebraic group $\mathrm{Sp}(2, 2)$ defined over $\mathbb{Q}$. It is a non-quasi-split $\mathbb{Q}$-rank-two form of the split symplectic group $\mathrm{Sp}_8/\mathbb{Q}$, the classical group of Cartan type C_4.

In § 1 the necessary facts about the automorphic cohomology $H^*(G, E)$ for a connected, semisimple algebraic group $G/\mathbb{Q}$ are reviewed. We recall the decomposition of Eisenstein cohomology $H^*_{\text{Eis}}(G, E)$ along the cuspidal support of the Eisenstein series in question, see

Theorem 1.1 (respectively, the original sources [FS98] or [MW95]): this is a decomposition along associate classes $\{P\}$ of a proper, parabolic $\mathbb{Q}$-subgroup P of G and certain (collections φ of) cohomological, cuspidal automorphic representations $\widetilde{\pi}$ of the corresponding Levi subgroups L of P.

In § 2, still for an arbitrary connected, semisimple algebraic group $G/\mathbb{Q}$, we deal with the question of how to breed the space of Eisenstein cohomology out of cohomological cuspidal automorphic representations $\widetilde{\pi}$ of the Levi subgroups L of parabolic $\mathbb{Q}$-subgroups $P \subsetneqq G$. Recall that P has a Levi decomposition $P = LN$ and a Langlands decomposition $P = MAN$, where N is a unipotent radical of P and A a maximal central $\mathbb{Q}$-torus of L. As in [FS98] we use the Eisenstein intertwining operator to obtain a map on the level of $(\mathfrak{g}, K)$-cohomology

$$H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \xrightarrow{E^q_\pi} H^*_{\mathrm{Eis}}(G, E). \tag{1}$$

Here, $W_{P,\widetilde{\pi}}$ is essentially the representation induced parabolically from $\widetilde{\pi}$. Further, $S_\chi(\mathfrak{a}^*)$ denotes the symmetric tensor algebra of the linear dual of $\mathfrak{a} = \mathrm{Lie}(A(\mathbb{R}))$. (The symbol '$\chi$' indicates an action of $\mathfrak{a}$ onto $S_\chi(\mathfrak{a}^*)$ by means of a character χ of $A(\mathbb{R})^\circ$. See § 2.2 for details.) This construction procedure of Eisenstein cohomology is explained in detail. In particular, we recall the notion of a class of type (π, w) (where w is a so-called Kostant representative with respect to the right action of the Weyl group of $L(\mathbb{C})$ on the Weyl group of $G(\mathbb{C})$): this is a non-trivial class in the right-hand side of (1). It follows from general results on $(\mathfrak{g}, K)$-cohomology that the derivative of χ must satisfy $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}}$. We may also suppose that it lies inside the closed, positive Weyl chamber C defined by P and A.

In § 2.3 we explain how the behavior of holomorphy of an Eisenstein series $E_P(f, \Lambda)$, $f \in W_{P,\widetilde{\pi}}$, $\Lambda \in \mathfrak{a}_\mathbb{C}$, interacts with the degree(s) of cohomology in which the image of E^q_π lies. The case of holomorphic Eisenstein series was already solved by Schwermer in [Sch83] and is summarized briefly in § 2.3.1.

Again, the residual case is the most difficult and remains unsolved in its full generality. We know by Langlands [Lan76] that the poles of the Eisenstein series $E_P(f, \Lambda)$ are those of its constant terms along parabolic subgroups. Assume that P is a self-associate, standard parabolic, then it suffices to consider the constant term along P itself. Putting $W(A) = N_{G(\mathbb{Q})}(A(\mathbb{Q}))/L(\mathbb{Q})$ (which is a subgroup of the Weyl group attached to the $\mathbb{Q}$-roots of G) we arrive at a decomposition of this constant term as a finite sum

$$E_P(f, \Lambda)_P = \sum_{w \in W(A)} M(\Lambda, \widetilde{\pi}, w)(f e^{\langle \Lambda + \rho_P, H_P(\cdot)\rangle}), \tag{2}$$

where $M(\Lambda, \widetilde{\pi}, w)$ are certain well-known meromorphic functions associated with Λ, $\widetilde{\pi}$ and $w \in W(A)$ (cf. § 2.3.2 for their precise definition and for the other symbols not explain here). So the behavior of holomorphy of $E_P(f, \Lambda)$ is given by the interplay of the poles and zeros of the finitely many functions $M(\Lambda, \widetilde{\pi}, w)$. If $M(\Lambda, \widetilde{\pi}, w)$ is residual at $\Lambda = \Lambda_0$, then we assume that we have normalized it to a holomorphic and non-vanishing function $N(\Lambda_0, \widetilde{\pi}, w)$. Put

$$W(A)_{\mathrm{res}} = \{w \in W(A) | M(\Lambda, \widetilde{\pi}, w) \text{ has a pole of order } \ell = \dim \mathfrak{a}_\mathbb{C} \text{ at } \Lambda = d\chi\}.$$

This means that the order of the pole is maximal and implies that the longest element w_0 of $W(A)$ (as a reduced word in the simple reflections generating $W(A)$) will be inside $W(A)_{\mathrm{res}}$. We prove the following new theorem in § 2.3.2 (cf. Theorem 2.1) on the degree of residual Eisenstein cohomology classes.

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Theorem. *In the notation used above, let* $0 \neq [\omega] \in H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$. *If all Eisenstein series* $E_P(f, \Lambda)$, $f \otimes 1$ *in the image of* ω, *have a pole of maximal possible order* $\ell = \dim \mathfrak{a}_\mathbb{C}$ *at* $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}} \in C$ *and if* $\operatorname{Im} N(d\chi, \widetilde{\pi}, w_0)$ *is a direct summand of* $\sum_{w \in W(A)_{\mathrm{res}}} \operatorname{Im} N(d\chi, \widetilde{\pi}, w)$, *then* $E^q_\pi([\omega])$ *contributes at least in degree* $q' := q + \dim N(\mathbb{R}) - 2l(w)$, *where* $l(w)$ *is the length of* w.

From § 3 on, we concentrate on the case $G = \mathrm{Sp}(2, 2)/\mathbb{Q}$. As mentioned earlier, G is a simple, connected, simply connected algebraic group over $\mathbb{Q}$, which is an non-quasi-split, $\mathbb{Q}$-rank-two inner form of Sp_8, the classical split group over $\mathbb{Q}$ of Cartan type C_4. Hence, the classes of associate and conjugate parabolic $\mathbb{Q}$-subgroups of G coincide and can be represented by the choice of three standard parabolic subgroups P_0 (a minimal subgroup) and P_1 and P_2 (two maximal subgroups). In order to construct Eisenstein cohomology, we need to obtain some knowledge on cohomology classes of type (π, w) as remarked before: $\pi = \chi\widetilde{\pi}$ with χ a certain character of $A_i(\mathbb{R})^\circ$ and $\widetilde{\pi}$ a cohomological cuspidal automorphic representation of $L_i(\mathbb{A})$ $(i = 0, 1, 2)$; and w is a Kostant representative of a coset with respect to the right action of the Weyl group of $L(\mathbb{C})$ on the Weyl group of $G(\mathbb{C})$. In § 4 the possible archimedean components $\widetilde{\pi}_\infty$ of cohomological cuspidal automorphic representations $\widetilde{\pi}$ are classified (cf. Lemma 4.1 and Proposition 4.2). These are irreducible unitary cohomological representations of the semisimple part $M_i(\mathbb{R})$ of the reductive Lie groups $L_i(\mathbb{R})$. We use the well-known Vogan–Zuckerman classification of such representations, cf. [VZ84].

Having gained this knowledge, in § 5 we then give a complete description of the $G(\mathbb{A}_f)$-module structure of the Eisenstein cohomology spaces $H^q_{\mathrm{Eis}}(G, E)$, under the assumption that the coefficient module E is regular. The case of each parabolic $\mathbb{Q}$-subgroup P_i, $i = 0, 1, 2$ is treated separately in three subsections. The main theorems describing the internal nature of Eisenstein cohomology classes with respect to regular coefficients are Theorems 5.3, 5.4 and 5.5. The general phenomenon that each Eisenstein class can be represented by (a finite number of) regular values of Eisenstein series (see [Sch94]) and the vanishing of $H^q_{\mathrm{Eis}}(G, E)$ below the half of $\dim X = 16$ (see [LS04]) is verified concretely in this case.

The much more difficult case of a general (meaning not necessarily regular) coefficient system E is dealt with in § 6. We concentrate on the contribution of the minimal parabolic P_0. The analysis of (residual) Eisenstein cohomology supported in P_0 might be viewed as a case study, which sources its interest in the absence of a good general theory from the following questions, which have been stated already above.

(a) How do we find the poles of an Eisenstein series $E_{P_0}(f, \Lambda)$?

(Working on this question is particularly interesting in our concrete case, since $G = \mathrm{Sp}(2, 2)$ (and so L_0) is not quasi-split, whence we are out of scope of the 'Langlands–Shahidi method' [Sha81, Sha88].)

(b) How do we control the contribution of the various resulting residues to Eisenstein cohomology?

In order to answer question (a), i.e. calculate the poles of $E_{P_0}(f, \Lambda)$, we have to normalize the operators $M(\Lambda, \widetilde{\pi}, w)$ of (2), meaning we have to find a function $r(\Lambda, \widetilde{\pi}, w)$ 'whose behavior of holomorphy we understand' such that $N(\Lambda, \widetilde{\pi}, w) = r(\Lambda, \widetilde{\pi}, w)^{-1} M(\Lambda, \widetilde{\pi}, w)$, to be called the *normalized* intertwining operator, is holomorphic and non-vanishing in the region we need it. For quasi-split groups a suggestion for such a normalization is provided by the Langlands–Shahidi method. However, as remarked before, our group is not quasi-split. We apply a little trick (cf. Proposition 6.2 or our original paper [Gro09, Proposition 3.1]), which allows us to

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obtain a good normalization of $M(\Lambda, \widetilde{\pi}, w)$ by only normalizing the local operators $M(\Lambda, \widetilde{\pi}_p, w)$, where p is a place where $G(\mathbb{Q}_p) = \mathrm{Sp}_8(\mathbb{Q}_p)$ (i.e. G splits). Then we use Proposition 6.1, which tells us that we can reduce the problem of normalizing $M(\Lambda, \widetilde{\pi}_p, w)$ at such places to the $\mathbb{Q}$-rank-one case. Still, we need some extra information, since we also have to normalize cuspidal representations of $L_0(\mathbb{A})$ which are locally not generic. At this point, we use the recent work of Grbac [Grb07, Grb09], which solves the question of how to normalize our operators for such non-generic local representations. The candidates for double poles of Eisenstein series are finally listed in our Propositions 6.6 and 6.8.

Question (b) is the most subtle matter. Here we confine ourselves to considering the space of square-integrable Eisenstein cohomology, (supported by P_0), denoted $H^q(\mathfrak{g}, K, L_{E,P_0} \otimes E)$. Its coefficient system L_{E,P_0} is a subspace of the residual spectrum of G and hence decomposes as a direct Hilbert sum over unitary residual automorphic representations of $G(\mathbb{A})$, each of which is generated by twice-iterated residues of Eisenstein series. By our Propositions 6.6 and 6.8 we can therefore determine the internal nature of a representative of a square-integrable Eisenstein cohomology class. This is contained in Theorem 6.9.

THEOREM. *Let P_0 be the minimal standard parabolic $\mathbb{Q}$-subgroup of $G = Sp(2,2)$ and E any irreducible, finite-dimensional complex-rational representation of $G(\mathbb{R})$. Then the square-integrable Eisenstein cohomology supported by P_0, $H^*(\mathfrak{g}, K, L_{E,P_0} \otimes E)$, is spanned by cohomology classes which are Eisenstein lifts of a class of type (π, w), $\pi = \chi\widetilde{\pi} \in \varphi_{P_0} \in \varphi \in \Psi_{E,P_0}$, $\widetilde{\pi} = \theta \mathbin{\hat{\otimes}} \tau$ and $w \in W^{P_0}$, such that necessarily one of the following conditions holds.*

If $d\chi$ is inside the open, positive Weyl chamber defined by P_0 and A_0:

(A) *if* $\dim\theta > 1$ *and* $\dim\tau > 1$,
$\widetilde{\pi} = \tau \mathbin{\hat{\otimes}} \tau$, $\chi_\tau = \mathbf{1}$, $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$;

(B) *if* $\dim\theta = 1$, $\dim\tau > 1$,
$\widetilde{\pi} = \mathbf{1} \otimes \tau$, $\chi_\tau = \mathbf{1}$, $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$;

(C) *if* $\dim\theta = \dim\tau = 1$, *then:*
 (1) $\widetilde{\pi} = \mathbf{1} \mathbin{\hat{\otimes}} \tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p\ \forall p \in S(B)$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$;
 (2) $\widetilde{\pi} = \tau \mathbin{\hat{\otimes}} \tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p\ \forall p \in S(B)$ *and* $d\chi = (\frac{5}{2}, \frac{1}{2})$;
 (3) $\widetilde{\pi} = \mathbf{1} \otimes \mathbf{1}$ *and* $d\chi = (\frac{7}{2}, \frac{3}{2}) = \rho_{P_0}$.

If $d\chi$ is on the boundary of the closed, positive Weyl chamber defined by P_0 and A_0:

(A) *if* $\dim\theta > 1$ *and* $\dim\tau > 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0)$ *or* $(1, 0)$;

(B) *if* $\dim\theta = 1$, $\dim\tau > 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$ *or* $(\frac{3}{2}, 0)$;

(B′) *if* $\dim\theta > 1$, $\dim\tau = 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$ *or* $(\frac{1}{2}, 0)$;

(C) *if* $\dim\theta = \dim\tau = 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(1, 1)$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$ *or* $(2, 0)$.

Our general Theorem 2.1 on the other hand gives a partial answer on how square-integrable Eisenstein cohomology classes are distributed in the various degrees. In addition, the classification of cohomological, irreducible, unitary representations of $G(\mathbb{R})$ given by [VZ84], essentially implies the following vanishing result (see Theorem 6.10).

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Theorem. *If $E \neq \mathbb{C}$, then square-integrable Eisenstein cohomology supported by P_0 vanishes below degree three*

$$H^q(\mathfrak{g}, K, L_{E,P_0} \otimes E) = 0 \quad \text{for } q \leqslant 3.$$

If $E = \mathbb{C}$, then there is an epimorphism

$$H^0(\mathfrak{g}, K, L_{\mathbb{C},P_0}) \twoheadrightarrow H^0(G, \mathbb{C}) = \mathbb{C}$$

and

$$H^q(\mathfrak{g}, K, L_{\mathbb{C},P_0}) = 0 \quad \text{for } 1 \leqslant q \leqslant 3.$$

In fact, $q = 3$ is a sharp bound for the vanishing of $(\mathfrak{g}, K)$-cohomology in low degrees, so $H^4(\mathfrak{g}, K, L_{E,P_0} \otimes E)$ should not vanish. However, this should also follow from our Theorem 2.1, as we point out in § 6.4.

Finally, we give all necessary computational data (e.g. the sets of Kostant representatives w) in Tables A1–A8 that we have placed in a small appendix.

Notation and conventions

Throughout this paper, G will be a connected, simply connected, semisimple algebraic group over $\mathbb{Q}$ of rank $rk_{\mathbb{Q}}(G) \geqslant 1$ with finite center. Lie algebras of groups of real points of algebraic groups will be denoted by the same but fractional letter, e.g., $\mathrm{Lie}(G(\mathbb{R})) = \mathfrak{g}$. The complexification of a Lie algebra will be denoted by the subscript '$\mathbb{C}$', e.g., $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$. If $U(\mathfrak{g})$ is the universal enveloping algebra of the complex algebra $\mathfrak{g}_{\mathbb{C}}$, $Z(\mathfrak{g})$ denotes its center.

We use the standard terminology and hypotheses concerning algebraic groups and their subgroups to be found in [MW95, §§ I.1.4–I.1.12]. In particular, we assume that a minimal parabolic subgroup P_0 has been fixed and that $K_{\mathbb{A}} = K_{\mathbb{R}} \times K_{\mathbb{A}_f}$ is a maximal compact subgroup of the group $G(\mathbb{A})$ of adelic points of G which is in a *good position* with respect to P_0 (see [MW95, § I.1.4]). Then $K = K_{\mathbb{R}}$ is maximal compact in $G(\mathbb{R})$, hence comes with a Cartan involution ϑ. If H is a subgroup of G, we let $K_H = K \cap H(\mathbb{R})$.

Assume that L_0 is a Levi subgroup of P_0 which is invariant under ϑ and N_0 is the unipotent radical of P_0. Then we have the Levi decomposition $P_0 = L_0 N_0$ and if we additionally denote by A_0 a maximal, central $\mathbb{Q}$-split torus in L_0, then we also obtain the Langlands decomposition $P_0 = M_0 A_0 N_0$. As usual, $M_0 = \bigcap_\chi \ker \chi$, χ ranging over the group $X(L_0)$ of all $\mathbb{Q}$-characters on L_0. Let P be a standard parabolic $\mathbb{Q}$-subgroup of G. It has a unique Levi decomposition $P = L_P N_P$, with $L_P \supseteq L_0$ and also a unique Langlands decomposition $P = M_P A_P N_P$ with unique ϑ-stable split component $A_P \subseteq A_0$. If it is clear from the context we also omit the subscript 'P'. We write $\Delta(P, A)$ for the set of weights of the adjoint action of P with respect to A_P. Here ρ_P denotes the half-sum of these weights. In particular, $\rho = \rho_{P_0}$ is the half-sum of positive $\mathbb{Q}$-roots of G with respect to A_0.

Extend the Lie algebra $\mathfrak{a}$ of $A(\mathbb{R})$ to a ϑ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by adding a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$. The absolute root system of $\mathfrak{g}$ is denoted by $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, a simple subsystem (given by the obstruction that positivity on the system of absolute roots shall be compatible with the positivity on the set $\Delta_{\mathbb{Q}}$ of $\mathbb{Q}$-roots implied by the choice of the minimal pair (P_0, A_0)) is denoted Δ°. We also write Δ_M° for the set of absolute simple roots of $\mathfrak{m}$ with respect to $\mathfrak{b}$ (so $\Delta^\circ = \Delta_G^\circ$). The Weyl groups associated with Δ and $\Delta_{\mathbb{Q}}$ are denoted by $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $W_{\mathbb{Q}}$. We let $W^P = \{w \in W \mid w^{-1}(\alpha) > 0 \; \forall \, \alpha \in \Delta_M^\circ\}$. The elements of W^P are called *Kostant representatives*, cf. [BW80].

Using the fact that $K_{\mathbb{A}}$ is in a good position, we can extend the standard *Harish-Chandra height function* $H_P : P(\mathbb{A}) \to \mathfrak{a}^*$ given by $\prod_p |\chi(p)|_p = e^{\langle \chi, H_P(p) \rangle}$, with $\chi \in X(L)$ viewed as an element of $\mathfrak{a}^*_{\mathbb{C}}$, to a function on all of $G(\mathbb{A})$ by setting $H_P(g) := H_P(p)$, $g = kp$.

Let G be a connected, *reductive* group over $\mathbb{Q}$ and $\widetilde{\chi}$ a central character. As usual $L^2_{\mathrm{dis}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (respectively, $L^2_{\mathrm{dis}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$) denotes the discrete spectrum of G (respectively, the part of it consisting of functions with central character $\widetilde{\chi}$). It can be written as the direct sum of the cuspidal spectrum $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (respectively, $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$) and the residual spectrum $L^2_{\mathrm{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (respectively, $L^2_{\mathrm{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$). By [GGP69] the space $L^2_{\mathrm{dis}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$, decomposes as a direct Hilbert sum over all irreducible, admissible representations π of $G(\mathbb{A})$ with central character $\widetilde{\chi}$, each of which occurring with finite multiplicity $m_{\mathrm{dis}}(\pi)$. The same is therefore true for the cuspidal (respectively, residual) spectrum, if we replace the multiplicity by $m(\pi)$ (respectively, $m_{\mathrm{res}}(\pi)$). Every π can be written as a restricted tensor product $\pi = \otimes'_p \pi_p$, where p is a place of $\mathbb{Q}$, i.e. either a prime or ∞ and π_p is a local irreducible, admissible representation π_p of $G(\mathbb{Q}_p)$ (see [Fla79]). Further, π (and so all π_p) is unitary if and only if $\widetilde{\chi}$ is. Then π is the completed restricted tensor product $\pi = \hat{\otimes}'_p \pi_p$.

For any $G(\mathbb{A})$ representation σ, we write σ^∞ for the space of its smooth vectors and $\sigma_{(K)}$ for the space of K-finite vectors. Clearly, if σ is unitary, then $\sigma^\infty_{(K)}$ is a unitary $(\mathfrak{g}, K, G(\mathbb{A}_f))$ module.

1. Automorphic cohomology

1.1 Let E be a finite-dimensional, irreducible, complex-rational representation of $G(\mathbb{R})$ characterized by its highest weight λ. A starting point of our interest is the $G(\mathbb{A}_f)$-module structure of the $(\mathfrak{g}, K)$-cohomology of the space of (E-valued) adelic automorphic forms $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E$:

$$H^*(G, E) := H^*(\mathfrak{g}, K, \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E).$$

As is well known, and as we shall also see again later, in order to understand this cohomology space, one should understand the cohomological contribution of those automorphic representations $\pi = \pi_\infty \otimes \pi_f$ of $G(\mathbb{A})$ which have a cohomological infinite component π_∞.

By [Lan79, Proposition 2], a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module π is automorphic if and only if it is isomorphic to an irreducible subquotient of a parabolically induced representation $\pi' = \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{l},K_L)}[\sigma_{(K_L)}]$, σ being a cuspidal automorphic representation of a Levi subgroup L of a parabolic $\mathbb{Q}$-subgroup of G. This can be proved by use of the so-called Eisenstein intertwining operator (cf. § 2.2), which assigns, very roughly, to each function $f \in \pi'$ a regular value, a residue or a derivative of an Eisenstein series at a certain point (cf. [Fra98, Corollary 1, p. 236] for this more subtle approach).

Clearly, if π is cuspidal itself, then we can take $P = G$ and this Eisenstein summation process degenerates essentially to the identity function. Therefore, as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module, $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ decomposes as the space of cuspidal automorphic forms $\mathcal{A}_{\mathrm{cusp}}$ and the subrepresentation $\mathcal{A}_{\mathrm{Eis}}$, which is spanned as a representation by all subquotients of parabolically induced representations $\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{l},K_L)}[\sigma]$, with $P \neq G$. By the very definition of the Eisenstein intertwining operator, this subspace is spanned by Eisenstein series, residues and derivatives of such. We obtain the decomposition as $G(\mathbb{A}_f)$-modules

$$H^*(G, E) = H^*(\mathfrak{g}, K, \mathcal{A}_{\mathrm{cusp}} \otimes E) \oplus H^*(\mathfrak{g}, K, \mathcal{A}_{\mathrm{Eis}} \otimes E).$$

The first space is called *cuspidal cohomology* and is denoted by $H^*_{\mathrm{cusp}}(G,E)$, the second *Eisenstein cohomology*, to be denoted by $H^*_{\mathrm{Eis}}(G,E)$. Now, what we are interested in is the space $H^*_{\mathrm{Eis}}(G,E)$ of Eisenstein cohomology, on which we focus in this paper. Since $(\mathfrak{g},K)$-cohomology only takes into account representations which have a certain infinitesimal character, see [BW80], one can replace the space of all automorphic forms $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ by the space $\mathcal{A}_E$ consisting of those automorphic forms which are annihilated by a power of the ideal $\mathcal{Z}$ of $Z(\mathfrak{g})$, which annihilates the dual representation of E: $\mathcal{Z}\cdot\check{E}=0$,

$$\mathcal{A}_E=\{f\in\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\mid \mathcal{Z}^n f=0\ \text{ for some } n\}$$

and

$$H^*(G,E)=H^*(\mathfrak{g},K,\mathcal{A}_E\otimes E).$$

1.2 The spaces $\mathcal{A}_{E,P}$

In [FS98], Franke and Schwermer (and also Mœglin and Waldspurger in [MW95]) were able to give a much more detailed decomposition of the space $\mathcal{A}_E$, taking into account the cuspidal support along Levi subgroups of the Eisenstein series involved.

First of all, the space $\mathcal{A}_E$ admits a certain decomposition as a direct sum with respect to the classes $\{P\}$ of associate parabolic $\mathbb{Q}$-subgroups $P\subseteq G$. This relies on such a decomposition of the space V_G of K-finite, left $G(\mathbb{Q})$-invariant, smooth functions $f:G(\mathbb{A})\to\mathbb{C}$ of uniform moderate growth, first proved by Langlands in a letter to Borel [Lan72]. See also [BLS96, Theorem 2.4]: $V_G=\bigoplus_{\{P\}}V_G(\{P\})$, where $V_G(\{P\})$ denotes the space of elements f in V_G which are negligible along Q for every parabolic $\mathbb{Q}$-subgroup $Q\subseteq G$, $Q\notin\{P\}$. Putting $\mathcal{A}_{E,P}=V_G(\{P\})\cap\mathcal{A}_E$ we obtain the desired decomposition of $\mathcal{A}_E$ as $(\mathfrak{g},K,G(\mathbb{A}_f))$-module

$$\mathcal{A}_E=\bigoplus_{\{P\}}\mathcal{A}_{E,P}.$$

Observe that $\mathcal{A}_{E,G}\subset V_G(\{G\})=L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))^\infty_{(K)}$. Hence,

$$H^q_{\mathrm{cusp}}(G,E)=H^q(\mathfrak{g},K,\mathcal{A}_{E,G}\otimes E),$$

and

$$H^q_{\mathrm{Eis}}(G,E)=\bigoplus_{\{P\},P\neq G}H^q(\mathfrak{g},K,\mathcal{A}_{E,P}\otimes E).$$

Since $V_G(\{G\})=L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))^\infty_{(K)}$ decomposes as a $(\mathfrak{g},K,G(\mathbb{A}_f))$-module as a direct sum over all cuspidal automorphic representations of $G(\mathbb{A})$, each occurring with finite multiplicity $m(\pi)$, by [BW80, XIII] we obtain a finite direct sum decomposition

$$H^*_{\mathrm{cusp}}(G,E)=\bigoplus_\pi H^*(\mathfrak{g},K,\pi\otimes E)^{m(\pi)}=\bigoplus_\pi(H^*(\mathfrak{g},K,(\pi_\infty)_{(K)}\otimes E)\otimes\pi_f^{\infty_f})^{m(\pi)},$$

the sum ranging over all cuspidal automorphic representations π of $G(\mathbb{A})$.

1.3 Eisenstein series

Also the summands $\mathcal{A}_{E,P}$ giving Eisenstein cohomology have a decomposition as $(\mathfrak{g},K,G(\mathbb{A}_f))$-module. We refer the reader to the original paper [FS98] for details.

Some technical assumptions and notation have to be fixed.

For $Q=LN=MAN$ associated with the standard parabolic P, φ_Q is a finite set of irreducible representations $\pi=\chi\widetilde{\pi}$ of $L(\mathbb{A})$, with $\chi:A(\mathbb{R})^\circ\to\mathbb{C}^*$ a continuous character and $\widetilde{\pi}$ an irreducible, unitary subrepresentation of $L^2_{\mathrm{cusp}}(L(\mathbb{Q})A(\mathbb{R})^\circ\backslash L(\mathbb{A}))$ of $L(\mathbb{A})$ whose central character induces a continuous morphism $A(\mathbb{Q})A(\mathbb{R})^\circ\backslash A(\mathbb{A}) \to U(1)$ and whose infinitesimal character matches the one of the dual of an irreducible subrepresentation of $H^*(\mathfrak{n}, E)$. This means that $\widetilde{\pi}$ is a unitary, cuspidal automorphic representation of $L(\mathbb{A})$ whose central and infinitesimal character satisfy the above conditions. Finally, three further 'compatibility conditions' have to be satisfied between these sets φ_Q, skipped here and written down in [FS98, § 1.2]. The family of all collections $\varphi = \{\varphi_Q\}$ of such finite sets is denoted by $\Psi_{E,P}$.

Now, let $W_{Q,\widetilde{\pi}}$ be the space of all smooth, K-finite functions

$$f : L(\mathbb{Q})N(\mathbb{A})A(\mathbb{R})^\circ\backslash G(\mathbb{A}) \to \mathbb{C},$$

such that for every $g \in G(\mathbb{A})$ the function $l \mapsto f(lg)$ on $L(\mathbb{A})$ is contained in the $\widetilde{\pi}$-isotypic component $\widetilde{\pi}^{m(\widetilde{\pi})}$ of $L^2_{\mathrm{cusp}}(L(\mathbb{Q})A(\mathbb{R})^\circ\backslash L(\mathbb{A}))$. For a function $f \in W_{Q,\widetilde{\pi}}$, $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $g \in G(\mathbb{A})$ an *Eisenstein series* is formally defined as

$$E_Q(f, \Lambda)(g) := \sum_{\gamma \in Q(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g) e^{\langle \Lambda + \rho_Q, H_Q(\gamma g)\rangle}.$$

If we set $(\mathfrak{a}^*)^+ := \{\Lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \Re e(\Lambda) \in \rho_Q + C\}$, where C equals the open, positive Weyl chamber with respect to $\Delta(Q, A)$, the series converges absolutely and uniformly on compact subsets of $G(\mathbb{A}) \times (\mathfrak{a}^*)^+$. It is known that $E_Q(f, \Lambda)$ is an automorphic form there and that the map $\Lambda \mapsto E_Q(f, \Lambda)(g)$ can be analytically continued to a meromorphic function on all of $\mathfrak{a}^*_{\mathbb{C}}$, cf. [MW95] or [Lan76, § 7]. It is known that the singularities Λ_0 (i.e. poles) of $E_Q(f, \Lambda)$ lie along certain affine hyperplanes of the form $R_{\alpha,t} := \{\xi \in \mathfrak{a}^*_{\mathbb{C}} \mid \langle \xi, \alpha\rangle = t\}$ for some constant t and some root $\alpha \in \Delta(Q, A)$, called 'root hyperplanes' ([MW95, Proposition IV.1.11(a)] or [Lan76, p. 131]). Choose a normalized vector $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ orthogonal to $R_{\alpha,t}$ and assume that Λ_0 is on no other singular hyperplane of $E_Q(f, \Lambda)$. Then define $\Lambda_0(u) := \Lambda_0 + u\nu$ for $u \in \mathbb{C}$. If c is a positively oriented circle in the complex plane around zero which is so small that $E_Q(f, \Lambda_0(\cdot))(g)$ has as no singularities on the interior of the circle with double radius, then

$$\mathrm{Res}_{\Lambda_0}(E_Q(f, \Lambda)(g)) := \frac{1}{2\pi i} \int_c E_Q(f, \Lambda_0(u))(g)\, du$$

is a meromorphic function on $R_{\alpha,t}$, called the *residue* of $E_Q(f, \Lambda)$ at Λ_0. Its poles lie on the intersections of $R_{\alpha,t}$ with the other singular hyperplanes of $E_Q(f, \Lambda)$. So one obtains a function holomorphic at Λ_0 in finitely many steps by taking successive residues as explained above.

1.4 The spaces $\mathcal{A}_{E,P,\varphi}$

Now we are able to turn to the desired decomposition of $\mathcal{A}_{E,P}$: for $\pi = \chi\widetilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$ let $\mathcal{A}_{E,P,\varphi}$ be the space of functions, spanned by all possible residues and derivatives of Eisenstein series defined via all $f \in W_{P,\widetilde{\pi}}$, at the value $d\chi$ inside the closed, positive Weyl chamber defined by $\Delta(P, A)$. It is a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module. Thanks to the functional equations (see [MW95, IV.1.10]) satisfied by the Eisenstein series considered, this is well defined, i.e. independent of the choice of a representative for the class of P (whence we took P itself) and the choice of a representation $\pi \in \varphi_P$. Finally, we obtain the following result.

Theorem 1.1 (Franke and Schwermer [FS98, Theorems 1.4 and 2.3]; see also Mœglin and Waldspurger [MW95, III, Theorem 2.6]). *There is direct sum decomposition as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module*

$$\mathcal{A}_{E,P} = \bigoplus_{\varphi \in \Psi_{E,P}} \mathcal{A}_{E,P,\varphi}$$

giving rise to

$$H^*_{\mathrm{Eis}}(G,E)=\bigoplus_{\{P\},P\neq G}\ \bigoplus_{\varphi\in\Psi_{E,P}} H^*(\mathfrak{g},K,\mathcal{A}_{E,P,\varphi}\otimes E).$$

2. Construction of Eisenstein cohomology

We now review a method to construct Eisenstein cohomology, using the notion of so-called '(π,w)-types'.

2.1 Classes of type (π,w)

Take $\pi=\chi\widetilde{\pi}\in\varphi_P$ and consider the symmetric tensor algebra

$$S_\chi(\mathfrak{a}^*)=\bigoplus_{n\geqslant 0}\bigodot^n \mathfrak{a}^*_{\mathbb{C}},$$

where $\bigodot^n\mathfrak{a}^*_{\mathbb{C}}$ is the symmetric tensor product of n copies of $\mathfrak{a}^*_{\mathbb{C}}$, as a module under $\mathfrak{a}_{\mathbb{C}}$: via the natural identification $\mathfrak{a}_{\mathbb{C}}\xrightarrow{\sim}\mathfrak{a}^*_{\mathbb{C}}$ it is an $\mathfrak{a}_{\mathbb{C}}$-module acted upon by $\xi\in\mathfrak{a}_{\mathbb{C}}\cong\mathfrak{a}^*_{\mathbb{C}}$ via multiplication with $\langle\xi,\rho_P+d\chi\rangle+\xi$ (within the symmetric tensor algebra). This explains the subscript 'χ'. We extend this action trivially on $\mathfrak{l}_{\mathbb{C}}$ and $\mathfrak{n}_{\mathbb{C}}$ to obtain an action of the Lie algebra $\mathfrak{p}_{\mathbb{C}}$ on the Banach space $S_\chi(\mathfrak{a}^*)$. Observe that one can equivalently regard $S_\chi(\mathfrak{a}^*)$ as the space of polynomials p on $\mathfrak{a}_{\mathbb{C}}$. Thus, we can define a $P(\mathbb{A}_f)$-module structure via the rule

$$(q\cdot p)(\xi)=e^{\langle\xi+d\chi+\rho_P,H_P(q)\rangle}p(\xi),$$

for $q\in P(\mathbb{A}_f)$, $\xi\in\mathfrak{a}_{\mathbb{C}}$ and $p\in S_\chi(\mathfrak{a}^*)$. There is a continuous linear isomorphism

$$\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}\mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{l},K_L)}[\widetilde{\pi}^\infty_{(K_L)}\otimes S_\chi(\mathfrak{a}^*)]^{m(\widetilde{\pi})}\xrightarrow{\sim}W_{P,\widetilde{\pi}}\otimes S_\chi(\mathfrak{a}^*),$$

induced by the tensor map $\otimes$ and the evaluation of functions $f\in C^\infty(G(\mathbb{A}),(\widetilde{\pi}^\infty)^{m(\widetilde{\pi})})$ at the identity, $f\mapsto ev_{id}(f):g\mapsto f(g)(id)$, so in particular one can view the right-hand side as a $(\mathfrak{g},K,G(\mathbb{A}_f))$-module by transport of structure. Doing this, Franke [Fra98, pp. 256–257] shows that

$$\begin{aligned}&H^q(\mathfrak{g},K,W_{P,\widetilde{\pi}}\otimes S_\chi(\mathfrak{a}^*)\otimes E)\\ &\quad\cong\bigoplus_{\substack{w\in W^P\\ -w(\lambda+\rho)|_{\mathfrak{a}_{\mathbb{C}}}=d\chi}}\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[H^{q-l(w)}(\mathfrak{m},K_M,(\widetilde{\pi}_\infty)_{(K_M)}\otimes{}^\circ F_w)\otimes\mathbb{C}_{d\chi+\rho_P}\otimes\widetilde{\pi}_f^{\infty_f}]^{m(\widetilde{\pi})}.\end{aligned}\tag{3}$$

Here ${}^\circ F_w$ is the finite-dimensional representation of $M(\mathbb{C})$ with highest weight $w(\lambda+\rho)-\rho|_{\mathfrak{b}_{\mathbb{C}}}$ and $\mathbb{C}_{d\chi+\rho_P}$ the one-dimensional, complex $P(\mathbb{A}_f)$-module on which $q\in P(\mathbb{A}_f)$ acts by multiplication by $e^{\langle d\chi+\rho_P,H_P(q)\rangle}$. A non-trivial class in a summand of the right-hand side is called a cohomology class *of type* (π,w), $\pi\in\varphi_P$, $w\in W^P$. (This notion was first introduced in [Sch83].)

Further, as $L(\mathbb{R})\cong M(\mathbb{R})\times A(\mathbb{R})^\circ$, $\widetilde{\pi}_\infty$ can be regarded as an irreducible, unitary representation of $M(\mathbb{R})$. Therefore, a (π,w) type consists of an irreducible representation $\pi=\chi\widetilde{\pi}$ whose unitary part $\widetilde{\pi}=\widetilde{\pi}_\infty\hat{\otimes}\widetilde{\pi}_f$ has at the infinite place an irreducible, unitary representation $\widetilde{\pi}_\infty$ of the semisimple group $M(\mathbb{R})$ with non-trivial $(\mathfrak{m},K_M)$-cohomology with respect to ${}^\circ F_w$.

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2.2 The Eisenstein map

In order to construct Eisenstein cohomology classes, we start from a class of type (π, w). Since we are interested in cohomology, we can by (3) assume without loss of generality that $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_{\mathbb{C}}}$ lies inside the closed, positive Weyl chamber defined by $\Delta(P, A)$.

We reinterpret $S_\chi(\mathfrak{a}^*)$ as the (Banach) space of formal, finite $\mathbb{C}$-linear combinations of differential operators $\partial^\alpha/\partial\Lambda^\alpha$ on the complex, l-dimensional vector space $\mathfrak{a}_{\mathbb{C}}$. It is understood that some choice of Cartesian coordinates $z_1(\Lambda), \ldots, z_l(\Lambda)$ on $\mathfrak{a}_{\mathbb{C}}$ has been fixed and $\alpha = (n_1, \ldots, n_l) \in \mathbb{N}_0^l$ denotes a multi-index with respect to these. As a consequence of [MW95, Proposition IV.1.11], there exists a polynomial $0 \neq q(\Lambda)$ on $\mathfrak{a}_{\mathbb{C}}$ such that for every $f \in W_{P,\widetilde{\pi}}$ the function

$$\Lambda \mapsto q(\Lambda) E_P(f, \Lambda)$$

is holomorphic at $d\chi$. Since $\mathcal{A}_{E,P,\varphi}$ can be written as the space which is generated by the coefficient functions in the Taylor series expansion of $q(\Lambda)E_P(f, \Lambda)$ at $d\chi$, f running through $W_{P,\widetilde{\pi}}$, we are able to define a surjective homomorphism of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules $E_{P,\pi}$

$$W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \xrightarrow{E_{P,\pi}} \mathcal{A}_{E,P,\varphi}$$

$$f \otimes \frac{\partial^\alpha}{\partial\Lambda^\alpha} \mapsto \frac{\partial^\alpha}{\partial\Lambda^\alpha}(q(\Lambda)E_P(f, \Lambda))|_{d\chi}$$

and obtain a well-defined map in cohomology

$$H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \xrightarrow{E_\pi^q} H^*(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E). \tag{4}$$

2.3 Degrees of Eisenstein cohomology classes

2.3.1 *Regular Eisenstein series.* Suppose that $[\omega] \in H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ is a class of type (π, w), represented by a morphism ω, such that for all elements $f \otimes (\partial^\alpha/\partial\Lambda^\alpha)$ in its image, $E_{P,\pi}(f \otimes (\partial^\alpha/\partial\Lambda^\alpha)) = (\partial^\alpha/\partial\Lambda^\alpha)(q(\Lambda)E_P(f, \Lambda))|_{d\chi}$ is just the regular value $E_P(f, d\chi)$ of the Eisenstein series $E_P(f, \Lambda)$, which is assumed to be holomorphic at the point $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_{\mathbb{C}}}$ inside the closed, positive Weyl chamber defined by $\Delta(P, A)$. Then $E_\pi^q([\omega])$ is a non-trivial Eisenstein cohomology class

$$E_\pi^q([\omega]) \in H^q(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E).$$

This is a consequence of [Sch83, Theorem 4.11].

2.3.2 *Residual Eisenstein series.* In the residual case, there might no longer be an unique degree, in which the image of E_π^q contributes. However, we can still single out a certain degree in which residual Eisenstein series contribute, if they have a pole of *maximal possible order* at $\Lambda = d\chi$ and satisfy some extra condition to be introduced below. (Observe that this maximum is precisely the dimension of $\mathfrak{a}_{\mathbb{C}}$.)

Let us explain this. As a matter of fact, the poles of the Eisenstein series $E_P(f, \Lambda)$ are those of its constant terms [Lan76]. Furthermore, it is enough to consider the constant term along associate parabolic subgroups. Indeed, due to the functional equation satisfied by Eisenstein series (cf. [MW95, § IV.1.10]) it suffices to consider the constant term along the standard parabolic subgroup $P' \in \{P\}$, which is conjugate to $\overline{P}$, the parabolic opposite to P. For sake of simplicity we assume that P is a self-associate, i.e. $P = P'$. Put

$$\mathrm{I}_{P,\widetilde{\pi},\Lambda} := \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \mathrm{Ind}_{(\mathfrak{l},K_M)}^{(\mathfrak{g},K)} [\widetilde{\pi}_{(K_M)}^\infty \otimes \mathbb{C}_{\Lambda+\rho_P}]^{m(\widetilde{\pi})},$$

where we assume that $q \in P(\mathbb{A}_f)$ acts on $\mathbb{C}_{\Lambda+\rho_P}$ by multiplication with $e^{\langle \Lambda+\rho_P, H_P(q)\rangle}$. Then the constant term along P can be written as a finite sum over certain Weyl group elements $w \in W(A) := N_{G(\mathbb{Q})}(A(\mathbb{Q}))/L(\mathbb{Q})$

$$E_P(f,\Lambda)_P = \sum_{w \in W(A)} M(\Lambda, \widetilde{\pi}, w)(f e^{\langle \Lambda+\rho_P, H_P(\cdot)\rangle}), \tag{5}$$

see, e.g., [MW95, Propsition II.1.7], and the poles of the Eisenstein series are determined by the mutual influence of the poles of the $(\mathfrak{g}, K, G(\mathbb{A}_f))$-intertwining operators

$$M(\Lambda, \widetilde{\pi}, w) : \mathrm{I}_{P,\widetilde{\pi},\Lambda} \to \mathrm{I}_{P,w(\widetilde{\pi}),w(\Lambda)}$$
$$M(\Lambda, \widetilde{\pi}, w)\psi(g) = \int_{N(\mathbb{Q})\cap wN(\mathbb{Q})w^{-1}\backslash N(\mathbb{A})} \psi(w^{-1}ng)\, dn.$$

Let us assume that $E_P(f,\Lambda)$ has got a pole of order ℓ at the point $d\chi$ for an $f \in W_{P,\widetilde{\pi}}$. Then the residue of the Eisenstein series $\mathrm{Res}_{d\chi} E_P(f,\Lambda)$ will be obtained via the constant term map in the sum of the images $J(d\chi, \widetilde{\pi}, w)$ of those normalized intertwining operators $N(d\chi, \widetilde{\pi}, w)$ for which $M(\Lambda, \widetilde{\pi}, w)$ has a pole of at least order ℓ at $\Lambda = d\chi$. (By a normalization we mean a function which results out of $M(\Lambda, \widetilde{\pi}, w)$ when dividing out the poles, i.e. more precisely, we assume that we have found a meromorphic function $r(\Lambda, \widetilde{\pi}, w)$ such that $N(\Lambda, \widetilde{\pi}, w) = r(\Lambda, \widetilde{\pi}, w)^{-1} M(\Lambda, \widetilde{\pi}, w)$ is holomorphic and non-vanishing in a region containing $d\chi$.) This set of operators therefore defines a subset $W(A)_{\mathrm{res}} \subseteq W(A)$, given by

$$W(A)_{\mathrm{res}} = \{w \in W(A) \mid M(\Lambda, \widetilde{\pi}, w) \text{ has a pole of at least order } \ell \text{ at } \Lambda = d\chi\}.$$

In particular, if we assume that ℓ is maximal, then $M(\Lambda, \widetilde{\pi}, w_0)$, with w_0 the longest element of $W(A)$, will be among these operators, i.e. $w_0 \in W(A)_{\mathrm{res}}$. Now, let $[\omega] \in H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ be a class represented by a morphism ω having only functions $f \otimes 1$ in its image whose associated Eisenstein series $E_P(f,\Lambda)$ have a pole of maximal possible order at the uniquely determined point $\Lambda = d\chi$. We recall that the class $[E_{P,\pi}(\omega)_P]$ which is represented by the constant term of the residues $\mathrm{Res}_{d\chi} E_P(f,\Lambda)$ along P equals the natural restriction $\mathrm{res}^q_P(E^q_\pi([\omega]))$ of the class $E^q_\pi([\omega])$ to the face $e'(P)_{\mathbb{A}} := P(\mathbb{Q})\backslash P(\mathbb{A})/K_P A(\mathbb{R})^\circ$ of the adelic Borel–Serre-compactification of $S := G(\mathbb{Q})\backslash G(\mathbb{A})/K$. As this will not play a big role here, we refer the reader to [Sch83, Satz 1.10] and [Roh96] for details. Having observed this, we see that

$$\mathrm{res}^q_P(E^q_\pi([\omega])) \in H^q\biggl(\mathfrak{g}, K, \sum_{w \in W(A)_{\mathrm{res}}} J(d\chi, \widetilde{\pi}, w) \otimes E\biggr).$$

The reader should observe that the sum $\sum_{w \in W(A)_{\mathrm{res}}} J(d\chi, \widetilde{\pi}, w)$ will not be direct in general. This is the point where we introduce the extra condition mentioned already at the beginning of this subsection: from now on we assume that $J(d\chi, \widetilde{\pi}, w_0)$ is a *direct* summand of our coefficient space, i.e. there is a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module N such that

$$J(d\chi, \widetilde{\pi}, w_0) \oplus N = \sum_{w \in W(A)_{\mathrm{res}}} J(d\chi, \widetilde{\pi}, w). \tag{6}$$

This assumption is not too strong. However, it enables us to write $\mathrm{res}^q_P(E^q_\pi([\omega]))$ as $\mathrm{res}^q_P(E^q_\pi([\omega])) = [\Omega_{w_0}] \oplus [\Omega_N]$, where clearly $[\Omega_{w_0}] \in H^q(\mathfrak{g}, K, J(d\chi, \widetilde{\pi}, w_0) \otimes E)$ and $[\Omega_N] \in H^q(\mathfrak{g}, K, N \otimes E)$. We now show that $[\Omega_{w_0}]$ might be viewed as a cohomology class in a certain degree q'.

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As P is self-associate, we have $L(\mathbb{A}) = w_0 L(\mathbb{A}) w_0^{-1} = \overline{L}(\mathbb{A})$ and $N(\mathbb{A}) = w_0 \overline{N}(\mathbb{A}) w_0^{-1}$. This implies that we can rewrite the intertwining operator $M(\Lambda, \widetilde{\pi}, w_0)$ as

$$M(\Lambda, \widetilde{\pi}, w_0)\psi(g) = \int_{\overline{N}(\mathbb{A})} \psi(n w_0^{-1} g)\, dn$$

and hence $M(\Lambda, \widetilde{\pi}, w_0)\psi \in \mathrm{I}_{\overline{P},\widetilde{\pi},\Lambda}$, the representation induced from the *opposite* parabolic $\overline{P}$. Therefore, it is justified to look at the image $J(d\chi, \widetilde{\pi}, w_0)$ of $N(d\chi, \widetilde{\pi}, w_0)$ as a subspace of $\mathrm{I}_{\overline{P},\widetilde{\pi},d\chi}$. However, this implies further that $[\Omega_{w_0}]$ can be viewed as a cohomology class in $H^q(\mathfrak{g}, K, \mathrm{I}_{\overline{P},\widetilde{\pi},d\chi} \otimes E)$. Clearly, the degrees in which $J(d\chi, \widetilde{\pi}, w_0)$ has cohomology with respect to E are determined by its infinite component

$$J(d\chi, \widetilde{\pi}_\infty, w_0) \hookrightarrow \mathrm{Ind}_{(\mathfrak{l},K_M)}^{(\mathfrak{g},K)}[(\widetilde{\pi}_\infty)_{(K_M)} \otimes \mathbb{C}_{d\chi+\rho_{\overline{P}}}]^{m(\widetilde{\pi})}.$$

As a consequence of the first half of the proof of [BW80, V, Proposition 1.5], $[\Omega_{w_0}]$ defines in this case (i.e. if all Eisenstein series $E_P(f, \Lambda)$, $f \otimes 1$ in the image of ω, have a pole of maximal possible order ℓ at $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}}$ and if (6) holds) a cohomology class in degree $q' := q + \dim N(\mathbb{R}) - 2l(w)$.

By (3) $r = q - l(w)$ is a degree, in which $\widetilde{\pi}_\infty$ has $(\mathfrak{m}, K_M)$-cohomology. So we have proved the following theorem.

THEOREM 2.1. *Let $[\omega] \in H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ be a non-trivial class of type (π, w), $\pi = \chi\widetilde{\pi}$, $w \in W^P$ such that $\widetilde{\pi}_\infty$ has non-zero $(\mathfrak{m}, K_M)$-cohomology in degree $r = q - l(w)$ with respect to ${}^\circ F_w$. Suppose that all Eisenstein series $E_P(f, \Lambda)$, $f \otimes 1$ in the image of ω, have a pole of maximal possible order $\ell = \dim \mathfrak{a}_\mathbb{C}$ at the uniquely determined point $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}}$ and that $J(d\chi, \widetilde{\pi}, w_0)$ is a direct summand of $\sum_{w \in W(A)_{\mathrm{res}}} J(d\chi, \widetilde{\pi}, w)$. Then the restriction of $E_\pi^q([\omega])$ to the face $e'(P)_\mathbb{A}$ has a summand which defines an Eisenstein cohomology class in degree $r + \dim N(\mathbb{R}) - l(w)$.*

Remark (Maximal parabolic P). If P maximal, then P will automatically be a self-associate if G is not of type A_n $(n \geqslant 2)$, D_n (n odd) or E_6. Assume that P is a self-associate. Then only the longest (since it is the only non-trivial) Weyl group element $w \in W(A)$ can contribute a pole to an Eisenstein series and we are in the situation considered above. We recall further that if P is not self-associate, then $E_P(f, \Lambda)$ will be holomorphic for $\Re e(\Lambda) \geqslant 0$.

Clearly, if $rk_\mathbb{Q}(G) = 1$, then any proper parabolic P will be self-associate and hence the above statement always applies to these groups.

3. The group Sp(2, 2)

3.1 We now collect necessary, basic facts concerning the group $G = \mathrm{Sp}(2, 2)$. Therefore, let B be a quaternion algebra over $\mathbb{Q}$ with canonical involution $x \mapsto \overline{x}$, such that $B \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{H}$ where $\mathbb{H}$ equals the real Hamilton quaternions. We denote by $S(B)$ the finite set of places p where B does not split, i.e. $B \otimes_\mathbb{Q} \mathbb{Q}_p$ is a division algebra. Suppose that $f : B^n \times B^n \to B$ is a Hermitian form of signature (p, q), where $0 \leqslant q \leqslant p$ with $n = p + q$ and B^n is being regarded as a B-right module. We suppose that f is equivalent to $(x, y) \mapsto \sum_{i=1}^p x_i \overline{y}_i - \sum_{j=1}^q x_{j+p}\overline{y}_{j+p}$. Then we define $\mathrm{Sp}(p, q)$ to be the group of all B-linear automorphisms of B^n leaving f invariant:

$$\mathrm{Sp}(p, q) = \{g \in M_n(B) \mid g^* K_{p,q} g = K_{p,q}\}.$$

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Here, $g^* = (\overline{g}_{ji})_{i,j} = \overline{g}^t$ and

$$K_{p,q} := \begin{pmatrix} id_{p\times p} & 0 \\ 0 & -id_{q\times q} \end{pmatrix}.$$

Here $\mathrm{Sp}(p,q)$ is a connected, simply connected, simple algebraic group over $\mathbb{Q}$ of ranks $rk_{\mathbb{Q}}(G) = rk_{\mathbb{R}}(G) = \min(p,q)$. It is a non-quasi-split inner form of Sp_{2n}, the split group of type C_n. From now on let $G = \mathrm{Sp}(2,2)$. A maximal compact subgroup K of $G(\mathbb{R})$ is isomorphic to $K = \mathrm{Sp}(2) \times \mathrm{Sp}(2)$.

3.2 Parabolic groups

We fix a minimal parabolic $P_0 = L_0 N_0 = M_0 A_0 N_0$ as in the introduction. We see that

$$L_0 \cong \mathrm{GL}_1(B) \times \mathrm{GL}_1(B)$$

and so

$$M_0 = \mathrm{SL}_1(B) \times \mathrm{SL}_1(B).$$

Further, A_0 can be chosen such that $\mathrm{Lie}(A_0(\mathbb{R})) = \mathfrak{a}_0$, with

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, a = \mathrm{diag}(a_1, a_2) \in M_2(\mathbb{R}) \right\}$$

and we can identify the set of $\mathbb{Q}$- and $\mathbb{R}$-roots of G with

$$\Delta_{\mathbb{Q}} = \Delta(\mathfrak{g}, \mathfrak{a}_0) = \{\pm\beta_i \pm \beta_j, 1 \leqslant i < j \leqslant 2\} \cup \{\pm 2\beta_1, \pm 2\beta_2\},$$

where β_i is the linear functional on $\mathfrak{a}_0$ extracting the value a_i. The simple $\mathbb{Q}$-roots are $\Delta_{\mathbb{Q}}^{\circ} = \{\beta_1 - \beta_2, 2\beta_2\}$. The unipotent radical N_0 of P_0 is of dimension 14.

There are two standard, maximal parabolic $\mathbb{Q}$-subgroups P_1, P_2 (the latter being the Siegel parabolic). Explicitly, we obtain

$$\begin{aligned} L_1 &\cong \mathrm{GL}_1(B) \times \mathrm{Sp}(1,1) \\ M_1 &= \mathrm{SL}_1(B) \times \mathrm{Sp}(1,1) \\ A_1 &= \{g \in A_0 | a_2 = 1\} \\ \dim N_1(\mathbb{R}) &= 11 \\ K_{M_1} &\cong \mathrm{SL}_1(\mathbb{H}) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1) \end{aligned}$$

and

$$\begin{aligned} L_2 &\cong \mathrm{GL}_2(B) \\ M_2 &= \mathrm{SL}_2(B) \\ A_2 &= \{g \in A_0 | a_1 = a_2\} \\ \dim N_2(\mathbb{R}) &= 10 \\ K_{M_2} &\cong \mathrm{Sp}(2). \end{aligned}$$

3.3 Root data

For $i = 0, 1, 2$, extend $\mathfrak{a}_i$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by adding a Cartan subalgebra $\mathfrak{b}_i$ of $\mathfrak{m}_i$. We may take

$$\mathfrak{b}_0 = \left\{ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \middle| \, b = \mathrm{diag}(b_1, b_2) \in iM_2(\mathbb{R}) \right\}.$$

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Then the absolute root system of G is given as

$$\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) = \{\pm\lambda_i \pm \lambda_j, 1 \leqslant i < j \leqslant 4\} \cup \{\pm 2\lambda_i, 1 \leqslant i \leqslant 4\}$$

where λ_i equals the functional sending $H \in \mathfrak{h}_\mathbb{C}$ to

$$\lambda_i(H) = \begin{cases} b_i + a_i & 1 \leqslant i \leqslant 2, \\ b_{i-2} - a_{i-2} & 3 \leqslant i \leqslant 4. \end{cases}$$

A simple subsystem which is compatible with the choice of positivity on $\mathfrak{a}_0^*$ is hence

$$\Delta^\circ = \{\underbrace{\lambda_1 + \lambda_3}_{=:\alpha_1}, \underbrace{-\lambda_2 - \lambda_3}_{=:\alpha_2}, \underbrace{\lambda_2 + \lambda_4}_{=:\alpha_3}, \underbrace{-2\lambda_4}_{=:\alpha_4}\}.$$

The highest weight λ of an irreducible, finite-dimensional representation E of $G(\mathbb{R})$ may be written as $\lambda = \sum_{i=1}^4 c_i \alpha_i$, where c_i are non-negative half-integers.

The corresponding systems of simple roots for the three standard parabolics are

$$\begin{aligned} \Delta^\circ_{M_0} &= \{\alpha_1, \alpha_3\} \\ \Delta^\circ_{M_1} &= \{\alpha_1, \alpha_3, \alpha_4\} \\ \Delta^\circ_{M_2} &= \{\alpha_1, \alpha_2, \alpha_3\}. \end{aligned}$$

Clearly, the restrictions of the roots $\alpha_j \in \Delta^\circ \backslash \Delta^\circ_{M_i}$ to $\mathfrak{a}_i$ gives the set of simple roots within $\Delta(P_i, A_i)$. For later purpose we also fix the following notation for the corresponding fundamental weights: ω_{ij}, $j = 1, 2, 3$, denotes the jth fundamental weight of $M_i(\mathbb{C})$, $i = 1, 2$. The fundamental weights of $M_0(\mathbb{C})$ are denoted by ω_{01} and ω_{02}.

We list the tables of values $w(\lambda + \rho) - \rho|_{\mathfrak{b}_{i_\mathbb{C}}}$, $w \in W^{P_i}$, and $\langle -w(\lambda + \rho)|_{\mathfrak{a}_{i_\mathbb{C}}}, \alpha_j\rangle$, $\alpha_j \in \Delta(P_i, A_i)$, $w \in W^{P_i}$ (and therefore also the sets W^{P_i}) in the appendix.

4. Cohomological representations for the three standard Levi subgroups

4.1 Recall the notion of (π, w)-types and the construction process of Eisenstein cohomology described in § 2. We need to find the cohomological, irreducible, unitary representations of $M_i(\mathbb{R})$. Denote the set of irreducible, unitary representations by $\widehat{M_i(\mathbb{R})}$, the cohomological representations among them by $\widehat{M_i(\mathbb{R})}_{\mathrm{coh}}$. Connected, semisimple Lie groups are of 'type I' (or 'tame' in the sense of Kirillow and Bernstein), so by the Künneth rule (cf. [BW80, § I.1.3])

$$\begin{aligned} \widehat{M_0(\mathbb{R})}_{\mathrm{coh}} &= \widehat{\mathrm{SL}_1(\mathbb{H})}_{\mathrm{coh}} \hat{\otimes} \widehat{\mathrm{SL}_1(\mathbb{H})}_{\mathrm{coh}}, \\ \widehat{M_1(\mathbb{R})}_{\mathrm{coh}} &= \widehat{\mathrm{SL}_1(\mathbb{H})}_{\mathrm{coh}} \hat{\otimes} \widehat{\mathrm{Sp}(1,1)}_{\mathrm{coh}}, \\ \widehat{M_2(\mathbb{R})}_{\mathrm{coh}} &= \widehat{\mathrm{SL}_2(\mathbb{H})}_{\mathrm{coh}}. \end{aligned}$$

4.2 Compact factors

The cohomological representations of a $\mathrm{SL}_1(\mathbb{H})$-factor of $M_i(\mathbb{R})$, $i = 0, 1$, are easily determined in the next lemma. For the sake of simplicity we identify ${}^\circ F_w$ with its restriction to this factor.

LEMMA 4.1. *Let $w \in W^P$ ($P = P_0$ or P_1) and V be an irreducible, unitary representation of $SL_1(\mathbb{H})$. Then*

$$H^q(\mathfrak{sl}_1(\mathbb{H}), SL_1(\mathbb{H}), V \otimes {}^\circ F_w) = \begin{cases} \mathbb{C} & \text{if } q = 0 \text{ and } V = {}^\circ F_w \\ 0 & \text{else.} \end{cases}$$

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Proof. Since $\mathrm{SL}_1(\mathbb{H})$ is compact, relative Lie algebra cohomology with respect to $V \otimes {}^\circ F_w$ is one-dimensional, if $V \cong {}^\circ \check{F}_w$ (the representation contragredient to ${}^\circ F_w$) and $q=0$ and vanishes otherwise. By [Sch94, Proposition 4.13], and our Tables A5 and A6 (see Appendix A), respectively, our Table A1 (see Appendix A) we see that ${}^\circ \check{F}_w \cong {}^\circ F_w$. □

4.3 Non-compact factors

The paper [VZ84] provides a full classification of irreducible, unitary, cohomological representations of a connected semisimple Lie group. In order to apply it to the simple Lie groups $\mathrm{Sp}(1,1)$ and $\mathrm{SL}_2(\mathbb{H})$, let us fix a maximal compact Cartan algebra $\mathfrak{t}_1 \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ of $\mathfrak{sp}(1,1)$ (respectively, $\mathfrak{t}_2 \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathbb{R}$ of $\mathfrak{sl}_2(\mathbb{H})$). We can arrange that with respect to this Cartan algebra the system of positive roots looks like $\Delta_1^+ = \{\mu_1 \pm \mu_2, 2\mu_1, 2\mu_2\}$, (respectively, $\Delta_2^+ = \{\mu_1 \pm \mu_2, \mu_1 \pm \mu_3, \mu_2 \pm \mu_3\}$). Take a finite-dimensional, irreducible, complex representation F of $\mathrm{Sp}(1,1)$ (respectively, $\mathrm{SL}_2(\mathbb{H})$) with highest weight μ with respect to Δ_1^+ (respectively, Δ_2^+). Skipping the details, we obtain the following proposition.

Proposition 4.2 (Vogan and Zuckerman [VZ84]). *For each μ there is an integer $j_1(\mu)$, $0 \leqslant j_1(\mu) \leqslant 2$ such that the irreducible, unitary $(\mathfrak{sp}(1,1), Sp(1) \times Sp(1))$-modules with non-trivial cohomology with respect to F are the uniquely determined irreducible, unitary representations $A_j(\mu)$, $j_1(\mu) \leqslant j \leqslant 1$ having the property*

$$H^q(\mathfrak{sp}(1,1), Sp(1) \times Sp(1), A_j(\mu) \otimes F) = \begin{cases} \mathbb{C} & \text{if } q=j \text{ or } q = 4-j \\ 0 & \text{otherwise} \end{cases}$$

together with the two irreducible, unitary $(\mathfrak{sp}(1,1), Sp(1) \times Sp(1))$-modules $A^+(\mu)$, $A^-(\mu)$ with

$$H^q(\mathfrak{sp}(1,1), Sp(1) \times Sp(1), A^\pm(\mu) \otimes F) = \begin{cases} \mathbb{C} & \text{if } q=2 \\ 0 & \text{otherwise.} \end{cases}$$

This integer is given as

$$j_1(\mu) = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu = k\mu_1,\ k=1,2,3,\ldots \\ 2 & \text{otherwise.} \end{cases}$$

Analogously, there is an integer $j_2(\mu)$, $0 \leqslant j_2(\mu) \leqslant 3$ such that the irreducible, unitary $(\mathfrak{sl}_2(\mathbb{H}), Sp(2))$-modules with non-trivial cohomology with respect to F are the uniquely determined irreducible, unitary representations $B_j(\mu)$, $j_2(\mu) \leqslant j \leqslant 2$ having the property

$$H^q(\mathfrak{sl}_2(\mathbb{H}), Sp(2), B_j(\mu) \otimes F) = \begin{cases} \mathbb{C} & \text{if } q=j \text{ or } q = 5-j \\ 0 & \text{otherwise.} \end{cases}$$

This integer is given as

$$j_2(\mu) = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu = k\mu_1,\ k=1,2,3,\ldots \\ 2 & \text{if } \mu \circ \vartheta = \mu \\ 3 & \text{otherwise.} \end{cases}$$

Remark. One can see this also by use of the isomorphisms $\mathrm{SO}(4,1)^\circ \cong \mathrm{PSp}(1,1)$ and $\mathrm{SO}(5,1)^\circ \cong \mathrm{PSL}_2(\mathbb{H})$ of real Lie groups and the classification of $\widehat{\mathrm{SO}(n,1)}_{\mathrm{coh}}$ as given essentially in [BW80] and, later on, completely in [RS87].

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The condition $j_2(\mu) = 3$ can be interpreted as $F \ncong \check{F}$, see [BC83, Corollary 1.6(a)].

4.4 We have to compare weights with respect to maximally non-compact Cartans to weights in $\mathfrak{t}^*_{i_\mathbb{C}}$. Therefore, let $\varpi_{ij} \in \mathfrak{t}^*_{i_\mathbb{C}}$, be the fundamental weights corresponding to the simple roots in Δ_i^+ and consider the linear maps given by

$$\varphi_1 : (\mathfrak{sp}(1,1) \cap \mathfrak{b}_1)^*_\mathbb{C} \to \mathfrak{t}^*_{1_\mathbb{C}}, \quad \omega_{12} \mapsto \varpi_{12}, \quad \omega_{13} \mapsto \varpi_{11}$$

and

$$\varphi_2 : \mathfrak{b}^*_{2_\mathbb{C}} \to \mathfrak{t}^*_{2_\mathbb{C}}, \quad \omega_{21} \mapsto \varpi_{22}, \quad \omega_{22} \mapsto \varpi_{21}, \quad \omega_{23} \mapsto \varpi_{23}.$$

These are isomorphisms respecting the choices of positivity on each side and transferring fundamental representations to fundamental representations.

In particular, we can compare highest weights of irreducible representations of $\mathrm{Sp}(1,1)$ and $\mathrm{SL}_2(\mathbb{H})$ with respect to the two Cartan subalgebras and their choices of positivity by applying the corresponding map φ_i.

5. Eisenstein cohomology of Sp(2, 2) with respect to regular coefficients

5.1 Having listed the sets W^{P_i}, $i = 0, 1, 2$, in our Appendix A, and the cohomological representations of the groups $M_i(\mathbb{R})$ in the last section, we are now ready to attack the problem of determining the Eisenstein cohomology of G. In view of our § 2, we need to construct the spaces $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P} \otimes E)$ for each class $\{P\}$ of proper, associate parabolic $\mathbb{Q}$-subgroups of G. We remark that for $G = \mathrm{Sp}(2,2)$ the associate classes and conjugacy classes of parabolic $\mathbb{Q}$-subgroups coincide, hence we can suppose that P is one of the groups P_0, P_1 or P_2.

This section deals with the case of *regular* coefficients E. That means the highest weight λ of E has strictly positive integer coefficients with respect to a decomposition according to the fundamental weights. Recall the following crucial result on Eisenstein cohomology with respect to regular coefficients E, which in our particular case reads as follows.

THEOREM 5.1 (Schwermer [Sch94]; see also Franke [Fra98, Theorem 19.II]). *Residual Eisenstein series do not contribute to the Eisenstein cohomology of G with respect to regular E. More precisely, if Π is a set of representatives of irreducible representations $\pi = \chi\widetilde{\pi}$ of the Levi components $L(\mathbb{A})$ of standard parabolic $\mathbb{Q}$-subgroups of G, which give rise to non-trivial maps E^*_π. Then E^q_π is an isomorphism and we obtain*

$$H^q_{\mathrm{Eis}}(G,E) \cong \bigoplus_{\pi \in \Pi} \bigoplus_{\substack{w \in W^P \\ -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}} = d\chi}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} [H^{q-l(w)}(\mathfrak{m}, K_M, (\widetilde{\pi}_\infty)_{(K_M)} \otimes {}^\circ F_w) \otimes \mathbb{C}_{d\chi+\rho_P} \otimes \widetilde{\pi}_f^{\infty_f}]^{m(\widetilde{\pi})}.$$

5.2 The minimal parabolic subgroup

In order to perform the construction via (π, w)-types, we need to know for which $w \in W^{P_0}$, $\Lambda_w = -w(\lambda+\rho)|_{\mathfrak{a}_{0_\mathbb{C}}}$ lies inside the closed, positive Weyl chamber. This is achieved explicitly in Tables A7 and A8 (see Appendix A) and we see that only very few elements in W^{P_0} can actually satisfy this condition. These are underlined in Table A8. Among them, only six elements satisfy it for sure, i.e. for all coefficient systems E (even non-regular systems). The others need some extra condition on the highest weight λ which might also not be satisfied by a regular representation E.

It is given in Table A8 (see Appendix A). We denote by $W^+(\lambda)$ the set of $w \in W^{P_0}$ giving rise to $\Lambda_w \in \overline{C}$.

Remark 5.2. General theory, as developed in [Sch94], tells us that Λ_w to make part of the closed, positive Weyl chamber must at least satisfy $l(w) \geqslant \frac{1}{2} \dim N_0(\mathbb{R}) = 7$. However, instead of looking at all $w \in W^{P_0}$ having $l(w) \geqslant 7$, it would have been enough to consider those $w \in W^{P_0}$ giving rise to the inequality

$$l(w^{P_i/P_0}) \geqslant \frac{\dim N_0(\mathbb{R})}{2 \dim N_i(\mathbb{R})}, \quad i = 1, 2. \tag{7}$$

This follows from [Sch94, Theorem 6.4]. Here, the Weyl group element w^{P_i/P_0} is defined as follows: let W^{P_i/P_0} be the set of representatives of minimal length for the right cosets of $W(\mathfrak{m}_{0_\mathbb{C}}, \mathfrak{b}_{0_\mathbb{C}})$ in $W(\mathfrak{m}_{i_\mathbb{C}}, \mathfrak{b}_{i_\mathbb{C}})$. Such representatives are unique by [Kos61, Proposition 5.13]. Now, for a given $w \in W^{P_0}$ there are uniquely determined elements $w^{P_i/P_0} \in W^{P_i/P_0}$, $w^{P_i} \in W^{P_i}$ satisfying $w = w^{P_i/P_0} \circ w^{P_i}$ and $l(w) = l(w^{P_i/P_0}) + l(w^{P_i})$, see [Sch94, Proposition 4.7].

In our cases, (7) reads as

$$l(w^{P_1/P_0}) \geqslant \tfrac{7}{11} \quad \text{and} \quad l(w^{P_2/P_0}) \geqslant \tfrac{7}{10},$$

meaning that we only have to consider those $w \in W^{P_0}$ which are neither in W^{P_1} nor in W^{P_2}. In fact, these elements can be excluded by direct means as Tables A7 and A8 in our Appendix A show.

Collecting this information we obtain the following theorem.

Theorem 5.3. *Let E be an irreducible, finite-dimensional, complex-rational representation of $G(\mathbb{R}) = Sp(2,2)$ with regular highest weight $\lambda = \sum_{i=1}^4 c_i\alpha_i$. The summand*

$$H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) = \bigoplus_{\varphi \in \Psi_{E,P_0}} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0,\varphi} \otimes E)$$

in the Eisenstein cohomology $H^q_{\mathrm{Eis}}(G, E)$ is given as a $G(\mathbb{A}_f)$-module by

$$H^q(\mathfrak{g}_0, K, \mathcal{A}_{E,P_0} \otimes E) = \bigoplus_{\substack{w \in W^+(\lambda)\\ l(w)=q}} \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\\ \widetilde{\pi}_\infty = {}^\circ F_w,\\ d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{0_\mathbb{C}}}\\ E^q_\pi \neq 0}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_0(\mathbb{A}_f)} [\mathbb{C}_{d\chi+\rho_{P_0}} \otimes \widetilde{\pi}_f^{\infty_f}]^{m(\widetilde{\pi})}$$

for $8 \leqslant q \leqslant 13$

$$H^{14}(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) = \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\\ \widetilde{\pi}_\infty = {}^\circ F_w,\\ l(w)=14\\ d\chi = \lambda+\rho|_{\mathfrak{a}_{0_\mathbb{C}}}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_0(\mathbb{A}_f)} [\mathbb{C}_{d\chi+\rho_{P_0}} \otimes \widetilde{\pi}_f^{\infty_f}]^{m(\widetilde{\pi})}$$

$$H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) = 0 \quad \textit{otherwise}.$$

All of these spaces are entirely built up by cohomology classes representable by regular values of Eisenstein series.

Proof. Recalling the construction process via (π, w)-types, and the result on cohomological, irreducible, unitary representations in Lemma 4.1, it is clear that $\widetilde{\pi}_\infty$ and $d\chi$ must satisfy the above conditions. By Theorem 5.1, E^q_π is already an isomorphism, if it is not identically zero. Looking up in Tables A7 and A8 (see Appendix A) the possible $w \in W^{P_0}$ that can give rise

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to values $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{0_\mathbb{C}}}$ inside the closed, positive Weyl chamber defined by the positive restricted roots $\Delta(P_0, A_0)$ or recalling Remark 5.2, proves that $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) = 0$ if $q \leqslant 7$. Our Table A8 shows that $W^+(\lambda)$ can actually contain representatives w having $l(w)$ equal to 8, 9, 10, 11, 12, 13 and 14, whence we have to list cohomology in all of these degrees. Again by our Table A8 there is a unique Kostant representative of length 14 in $W^+(\lambda)$ for all λ and its corresponding evaluation point $d\chi = \lambda + \rho|_{\mathfrak{a}_{0_\mathbb{C}}}$ lies in the region $C + \rho_{P_0}$ of absolute convergence of the Eisenstein series $E_{P_0}(f, \Lambda)$, since λ is regular. Hence, we can omit the condition $E_\pi^{14} \neq 0$. This is not true for the other degrees $8 \leqslant q \leqslant 13$, see Table A8. This proves the theorem. □

5.3 The first maximal parabolic subgroup

We explain now which classes of type (π, w), $\pi \in \varphi_{P_1} \in \varphi \in \Psi_{E,P_1}$ and $w \in W^{P_1}$ contribute to the Eisenstein cohomology of G.

Since the highest weight λ of E is supposed to be regular, each irreducible module ${}^\circ F_w$ is also regular [Sch94, Lemma 4.9]. Therefore, $\tilde{\pi}_\infty$ must equal the tensor product of the representation $V = {}^\circ F_w|_{\mathrm{SL}_1(\mathbb{H})}$ as in Lemma 4.1 with one of the two discrete series representations $A^\pm(\mu_w)$, $\mu_w = \varphi_1(w(\lambda+\rho) - \rho|_{(\mathfrak{sp}(1,1)\cap\mathfrak{b}_1)_\mathbb{C}})$, see § 4.4, having non-trivial $(\mathfrak{sp}(1,1), \mathrm{Sp}(1)\times\mathrm{Sp}(1))$-cohomology only in degree two, as is proved in Proposition 4.2. The actual contribution of the first maximal parabolic $\mathbb{Q}$-subgroup to Eisenstein cohomology is given in the next theorem.

Theorem 5.4. *Let E be an irreducible, finite-dimensional, complex-rational representation of $G(\mathbb{R}) = Sp(2,2)$ with regular highest weight $\lambda = \sum_{i=1}^4 c_i\alpha_i$. The summand*

$$H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1} \otimes E) = \bigoplus_{\varphi\in\Psi_{E,P_1}} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1,\varphi} \otimes E)$$

in the Eisenstein cohomology $H^q_{\mathrm{Eis}}(G, E)$ is given as a $G(\mathbb{A}_f)$-module by

$$\begin{aligned} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1} \otimes E) &= \bigoplus_{\substack{w\in W^{P_1}\\ l(w)=q-2}} \bigoplus_{\substack{\pi=\chi\tilde{\pi}\\ \tilde{\pi}_\infty = V\otimes A^\pm(\mu_w),\\ d\chi=-w(\lambda+\rho)|_{\mathfrak{a}_{1_\mathbb{C}}}\\ E_\pi^q\neq 0}} \mathrm{Ind}_{P_1(\mathbb{A}_f)}^{G(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_{P_1}} \otimes \tilde{\pi}_f^{\infty_f}]^{m(\tilde{\pi})} \\ &\qquad \text{for } 8 \leqslant q \leqslant 13 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

All of these spaces are entirely built up by cohomology classes representable by regular values of Eisenstein series.

Proof. The assertions on $d\chi$ and $\tilde{\pi}_\infty$ are already explained. By Theorem 5.1, we only need to sum over those π, which satisfy $E_\pi^q \neq 0$ and for which E_π^q is therefore an isomorphism. Now Lemma 4.1 and Proposition 4.2 imply that we must have $l(w) = q-2$, since $H^r(\mathfrak{sl}_1(\mathbb{H}) \oplus \mathfrak{sp}(1,1), \mathrm{SL}_1(\mathbb{H}) \times \mathrm{Sp}(1)\times\mathrm{Sp}(1), V \otimes A^\pm(\mu_w) \otimes {}^\circ F_w) = 0$ for $r \neq 2$. By Table A3 (see Appendix A) there is no element $w \in W^{P_1}$ of length $l(w) \geqslant 12$ but also $d\chi = \Lambda_w$ does not lie inside the closed, positive Weyl chamber for $l(w) \leqslant 5$. This proves the vanishing of $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1} \otimes E)$ in the degrees $q \leqslant 7$ and $q \geqslant 14$. □

Remark. In fact, Table A3 also shows that all $w \in W^{P_1}$ with $l(w) \geqslant 9$ give rise to evaluation points $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{1_\mathbb{C}}}$ which lie in the region $C + \rho_{P_1}$ of absolute convergence of the Eisenstein series $E_{P_1}(f, \Lambda)$. Hence, we could have omitted the condition $E_\pi^{l(w)+2} \neq 0$ for these w.

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5.4 The second maximal parabolic subgroup

We conclude the analysis of Eisenstein cohomology of G with respect to regular coefficients E describing the remaining summand $H^*(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E)$. Again, since E is supposed to be regular, each representation ${}^\circ F_w$, $w \in W^{P_2}$, of the group $M_2(\mathbb{C})$ is also regular. Recalling Proposition 4.2, there can only be one single cohomological, irreducible, unitary representation of $M_2(\mathbb{R})$ with respect to ${}^\circ F_w$, namely $B_2(\mu_w)$ with $\mu_w = \varphi_2(w(\lambda+\rho) - \rho|_{\mathfrak{b}_{2\mathbb{C}}})$, see § 4.4. Proposition 4.2 now gives us the appropriate tool to decide when $j_2(\mu_w) = 2$, i.e. when $B_2(\mu_w)$ exists. This is the case if and only if the first and the third coefficient of $w(\lambda+\rho) - \rho|_{\mathfrak{b}_{2\mathbb{C}}}$ in its decomposition according to the basis of fundamental weights ω_{21}, ω_{22} and ω_{23} coincide. Our Table A2 (see Appendix A) answers the question of when this happens exactly in detail. Observe that the two conditions $c_1 - c_4 = 1$ and $c_3 - c_4 = c_1$ from Table A2 contradict each other, so they cannot be satisfied at the same time. It can very well happen that they are both not satisfied, e.g., if $c_1 < c_3 - c_4$, or equivalently if the first coefficient of $w(\lambda+\rho) - \rho|_{\mathfrak{b}_{2\mathbb{C}}}$ in its decomposition according to the basis of fundamental weights ω_{21}, ω_{22} and ω_{23} is strictly smaller than the third coefficient. Clearly, there are even regular representations E satisfying $c_1 < c_3 - c_4$. In this case $j_2(\mu_w) = 3$ for *all* $w \in W^{P_2}$, implying that P_2 does not give any contribution to Eisenstein cohomology with respect to such E. This contribution can be described in general as follows.

Theorem 5.5. *Let E be an irreducible, finite-dimensional, complex-rational representation of $G(\mathbb{R}) = Sp(2,2)$ with regular highest weight $\lambda = \sum_{i=1}^4 c_i\alpha_i$. Let us write $W^{P_2}(\lambda) := \{w \in W^{P_2} \mid j(\mu_w) = 2\}$. The summand*

$$H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E) = \bigoplus_{\varphi \in \Psi_{E,P_2}} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2,\varphi} \otimes E)$$

in the Eisenstein cohomology $H^q_{\mathrm{Eis}}(G,E)$ is given as a $G(\mathbb{A}_f)$-module by

$$\begin{aligned} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E) &= \bigoplus_{\substack{w \in W^{P_2}(\lambda) \\ l(w) = q-3}} \bigoplus_{\substack{\pi = \chi\widetilde{\pi} \\ \text{with } \widetilde{\pi}_\infty = B_2(\mu_w), \\ d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{2\mathbb{C}}} \\ E^q_\pi \neq 0}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_2(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_{P_2}} \otimes \widetilde{\pi}^{\infty}_f{}^{f}]^{m(\widetilde{\pi})} \\ &\quad \oplus \bigoplus_{\substack{w \in W^{P_2}(\lambda) \\ l(w) = q-2}} \bigoplus_{\substack{\pi = \chi\widetilde{\pi} \\ \text{with } \widetilde{\pi}_\infty = B_2(\mu_w), \\ d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{2\mathbb{C}}} \\ E^q_\pi \neq 0}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_2(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_{P_2}} \otimes \widetilde{\pi}^{\infty}_f{}^{f}]^{m(\widetilde{\pi})} \\ &\quad \text{for } 8 \leqslant q \leqslant 13 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

All of these spaces are entirely built up by cohomology classes representable by regular values of Eisenstein series.

Proof. This is proved in a similar manner to Theorems 5.3 and 5.4, so we will be very brief. Recall from Proposition 4.2 that $B_2(\mu_w)$ has non-trivial $(\mathfrak{sl}_2(\mathbb{H}), \mathrm{Sp}(2))$-cohomology with respect to ${}^\circ F_w$ only in degrees two and three. Therefore, $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E)$ is built up by classes of type (π, w), having $l(w) = q-2$ or $l(w) = q-3$. The rest follows from Table A4 (see Appendix A). □

Remark. The vanishing of $H^q_{\mathrm{Eis}}(G,E)$ for $q \leqslant 7$ is also a consequence of [LS04, Theorem 5.5].

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6. Residual Eisenstein cohomology classes supported by the minimal parabolic

6.1 In § 5 we discussed the contribution of the various standard parabolic $\mathbb{Q}$-subgroups to the Eisenstein cohomology $H^q_{\mathrm{Eis}}(G, E)$, for finite-dimensional irreducible representations E of G with regular highest weight. The regularity condition ensured that residual Eisenstein series would not contribute to cohomology, so we did not really have to check the analytic behavior of Eisenstein series at the various points of evaluation in question.

However, in principle it is possible to give a complete description of Eisenstein cohomology even if the regularity condition is dropped, but we first have to understand the analytic behavior of the Eisenstein series $E_P(f, \Lambda)$ at the points $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}}$. As our parabolics are all self-associate, we can reduce this problem by § 2.3.2 to the following task: *understand the interplay of the various poles of the intertwining operators* $M(\Lambda, \widetilde{\pi}, w)$, $w \in W(A)$.

In order to exemplify the difficulties and some general phenomena that occur during the analysis of residual Eisenstein cohomology, we now consider the space of square-integrable Eisenstein cohomology supported by the minimal parabolic subgroup P_0. We enforce square-integrability because then we only need to consider Eisenstein series which have poles of maximal possible order $\ell = 2$. This allows us to use the results of § 2.3.2, which give a partial answer to the question in which degrees of cohomology maximally residual Eisenstein series contribute.

6.2 When trying to find out the various poles of the intertwining operators $M(\Lambda, \widetilde{\pi}, w)$, $w \in W(A)$, the actual problem is to give a suitable normalization, i.e. to find a function $r(\Lambda, \widetilde{\pi}, w)$ such that $N(\Lambda, \widetilde{\pi}, w) = r(\Lambda, \widetilde{\pi}, w)^{-1} M(\Lambda, \widetilde{\pi}, w)$, be called the *normalized* intertwining operator, is holomorphic and non-vanishing on the open, positive Weyl chamber defined by the pair (P, A). The difficulty lies in the fact that each standard Levi group L of G is a non-quasi-split algebraic group, whence one cannot apply the Langlands–Shahidi method, as developed in [Sha81, Sha88] in order to normalize the local intertwining operators at the non-split places. However, if $L(\mathbb{Q}_p)$ is compact modulo its center, we can use the same trick as in [Gro09, Proposition 3.1] and show that the local intertwining operator at the place p is itself holomorphic and non-vanishing inside the open, positive Weyl chamber defined by $\Delta(P, A)$. Clearly, only the minimal parabolic $P = P_0$ gives a Levi subgroup $L = L_0$ which satisfies the condition to be compact modulo its center at all non-split places.

For the rest of this section let P be the standard minimal parabolic $\mathbb{Q}$-subgroup P_0 of $G = \mathrm{Sp}(2, 2)$ with decompositions $P = LN = MAN$. As already remarked, $L(\mathbb{Q}_p)$ is compact modulo its center at all places $p \in S(B)$, since $M = \mathrm{SL}_1(B) \times \mathrm{SL}_1(B)$. We have $W(A) = W_\mathbb{Q}$.

Let $\pi = \chi\widetilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$, $f \in W_{P,\widetilde{\pi}}$ and identify $\Lambda = x\alpha_2 + y\alpha_4 \in \mathfrak{a}^*_\mathbb{C}$ with $\underline{s} = (s_1, s_2) \in \mathbb{C}^2$ via $s_1 = x/2$ and $s_2 = y - (x/2)$. As in the following, we assume here for sake of simplicity that all roots α_j mean their restriction to $\mathfrak{a}_\mathbb{C}$. Further, observe that since $L = \mathrm{GL}_1(B) \times \mathrm{GL}_1(B)$, each $\widetilde{\pi}$ factors as $\widetilde{\pi} = \theta \hat{\otimes} \tau$, where θ and τ are cuspidal automorphic representations of $\mathrm{GL}_1(B)$.

Now, as mentioned in § 2.3.2, the holomorphic behavior of the Eisenstein series $E_P(f, \Lambda)$ is the same as of its constant term along P, which can be rewritten as

$$E_P(f, \Lambda)_P = \sum_{w \in W_\mathbb{Q}} M(\underline{s}, \widetilde{\pi}, w)(f e^{\langle \Lambda + \rho_P, H_P(\cdot) \rangle}). \tag{8}$$

Therefore, the poles of $E_P(f, \Lambda)$ are determined by the poles of $M(\underline{s}, \widetilde{\pi}, w)$, $w \in W_\mathbb{Q}$. We recall the following fact.

Proposition 6.1 (Shahidi [Sha81, § 2.1] and Muić and Savin [MS00, § 2]). *Let $w \in W_{\mathbb{Q}}$ be an element of the Weyl group with decomposition $w = w_{n_1} \ldots w_{n_k}$ according to the reflections w_{n_i} corresponding to the simple $\mathbb{Q}$-roots α_{n_i}, $n_i \in \{2, 4\}$. Then the local intertwining operator $M(\underline{s}, \widetilde{\pi}_p, w)$ decomposes as*

$$M(\underline{s}, \widetilde{\pi}_p, w) = M(s_k, \widetilde{\pi}_{k,p}, w_{n_k}) \cdots M(s_1, \widetilde{\pi}_{1,p}, w_{n_1})$$

where we put recursively $s_i = (2\langle \underline{s}_i, \alpha_{n_i}\rangle / \langle \alpha_{n_i}, \alpha_{n_i}\rangle)$, $\underline{s}_i = w_{n_{i-1}}(\underline{s}_{i-1})$ with $\underline{s}_1 = \underline{s}$ and $\widetilde{\pi}_{i,p} = w_{n_{i-1}}(\widetilde{\pi}_{i-1,p})$ with $\widetilde{\pi}_{1,p} = \widetilde{\pi}_p$. The action of a Weyl group element on a representation $\widetilde{\pi}_p = \theta_p \hat{\otimes} \tau_p$ is given by $w_2(\widetilde{\pi}_p) = \tau_p \hat{\otimes} \theta_p$ and $w_4(\widetilde{\pi}_p) = \theta_p \hat{\otimes} \check{\tau}_p$.

The point of this proposition is that for each $w \in W_{\mathbb{Q}}$ we can write the local intertwining operator $M(\underline{s}, \widetilde{\pi}_p, w)$ as a finite product of the analogous local intertwining operators $M(s_i, \widetilde{\pi}_{i,p}, w_{n_i})$ attached to the two standard maximal Levi subgroups of G: if $n_i = 4$, the maximal Levi is $\mathrm{Sp}(1, 1)$, while if $n_i = 2$, it is $\mathrm{GL}_2(B)$. Hence, on the one hand, we can apply the following proposition.

Proposition 6.2 (Grobner [Gro09, Proposition 3.1]). *Let $p \in S(B)$. Then $M(s_i, \widetilde{\pi}_{i,p}, w_{n_i})$ is holomorphic and non-vanishing for $\Re e(s_i) > 0$.*

Then we obtain the following corollary.

Corollary 6.3. *The poles of $M(\underline{s}, \widetilde{\pi}, w)$ in the region $\Re e(s_1) > \Re e(s_2) > 0$ are the poles of $\hat{\otimes}'_{p \notin S(B)} M(\underline{s}, \widetilde{\pi}_p, w)$.*

On the other hand, we can *normalize each local operator $M(\underline{s}, \widetilde{\pi}_p, w)$ by normalizing the factors $M(s_i, \widetilde{\pi}_{i,p}, w_{n_i})$ and obtain a global normalization*

$$r(\underline{s}, \widetilde{\pi}, w) = \prod_{i=1}^{k} \hat{\otimes}'_{p \notin S(B)} r(s_{k-i+1}, \widetilde{\pi}_{k-i+1,p}, w_{n_{k-i+1}}). \tag{9}$$

6.2.1 $\mathrm{Sp}(1, 1)$. The corresponding normalizing factors for $n_i = 4$, i.e. our maximal Levi looks like $\mathrm{Sp}(1, 1)$, can be found in [Gro09, § 5], where the whole residual spectrum of $\mathrm{Sp}(1, 1)$ was calculated.[1] For the convenience of the reader, we review these results briefly: recall that we may write $\widetilde{\pi}_i = \theta_i \hat{\otimes} \tau_i$, with θ_i and τ_i being cuspidal automorphic representations of $\mathrm{GL}_1(B)$. The only proper parabolic $\mathbb{Q}$-subgroup inside $\mathrm{Sp}(1, 1)$ has a Levi subgroup isomorphic to $\mathrm{GL}_1(B)$, which is actually the second $\mathrm{GL}_1(B)$-factor of L_0. Hence, we always identify $\widetilde{\pi}_i$ with its second $\mathrm{GL}_1(B)$-factor τ_i, when it comes to $n_i = 4$. Now suppose that $p \notin S(B)$. If τ_i is not one-dimensional, then the required normalization follows from the Gindikin–Karpelevich integral formula, as shown in [Lan71, p. 27] (see also [Sha88, p. 554]) and had been already given in [Kim95, MW89]:

$$r(s_i, \widetilde{\pi}_{i,p}, w_{n_i}) = \frac{L(s_i, \tau_{i,p})}{L(1 + s_i, \tau_{i,p}) \varepsilon(s_i, \tau_{i,p})} \frac{L(2s_i, \widetilde{\chi}_{i,p})}{L(1 + 2s_i, \widetilde{\chi}_{i,p}) \varepsilon(2s_i, \widetilde{\chi}_{i,p})}. \tag{10}$$

Here we wrote $\widetilde{\chi}_{i,p}$ for the central character of $\tau_{i,p}$. The L- and ε-functions are the standard Jacquet–Langlands and Hecke L- and ε-functions of the second $\mathrm{GL}_1(B)$-factor $\tau_{i,p}$ of $\widetilde{\pi}_{i,p}$ and of its central character $\widetilde{\chi}_{i,p}$, respectively.

If $\tau_i = \widetilde{\chi}_i$ is one-dimensional, then we used the concrete normalization of [Grb07], where the idea of [MW89, Lemme I.8] had been applied, i.e. induction from generic representations of

[1] As we were preparing this article, the residual spectrum of $\mathrm{Sp}(1, 1)$ was, in even greater generality, calculated independently in [Yas07].

smaller parabolic subgroups:

$$r(s_i, \widetilde{\pi}_{i,p}, w_{n_i}) = \frac{L(s_i - \frac{1}{2}, \widetilde{\chi}_{i,p})L(2s_i, \widetilde{\chi}^2_{i,p})}{L(s_i + \frac{3}{2}, \widetilde{\chi}_{i,p})\varepsilon(s_i - \frac{1}{2}, \widetilde{\chi}_{i,p})\varepsilon(s_i + \frac{1}{2}, \widetilde{\chi}_{i,p})L(1 + 2s_i, \widetilde{\chi}^2_{i,p})\varepsilon(2s_i, \widetilde{\chi}^2_{i,p})}. \tag{11}$$

6.2.2 $\mathrm{GL}_2(\mathrm{B})$. Here we have to distinguish three cases: suppose that $\widetilde{\pi}_i = \theta_i \hat{\otimes} \tau_i$, with θ_i and τ_i being cuspidal automorphic representations of $\mathrm{GL}_1(B)$, satisfies $\dim \theta_i > 1$ and $\dim \tau_i > 1$. Then, again after having used the Gindikin–Karpelevich integral formula, our local normalizing factor at $p \notin S(B)$ looks like

$$r(s_i, \widetilde{\pi}_{i,p}, w_{n_i}) = \frac{L(s_i, \theta_{i,p} \times \check{\tau}_{i,p})}{L(1 + s_i, \theta_{i,p} \times \check{\tau}_{i,p})\varepsilon(s_i, \theta_{i,p} \times \check{\tau}_{i,p})}. \tag{12}$$

Here, the L-functions and the ε-factor are of Rankin–Selberg type. Again see [MW89].

Suppose now that one factor of $\widetilde{\pi}_i$ is one-dimensional, without loss of generality say $\dim \theta_i = 1$. We can use [Grb09] to normalize $M(s_i, \widetilde{\pi}_{i,p}, w_{n_i})$ and obtain

$$r(s_i, \widetilde{\pi}_{i,p}, w_{n_i}) = \frac{L(s_i - \frac{1}{2}, \theta_{i,p}\check{\tau}_{i,p})}{L(s_i + \frac{3}{2}, \theta_{i,p}\check{\tau}_{i,p})\varepsilon(s_i - \frac{1}{2}, \theta_{i,p}\check{\tau}_{i,p})\varepsilon(s_i + \frac{1}{2}, \theta_{i,p}\check{\tau}_{i,p})}. \tag{13}$$

In the third case, i.e. both factors θ_i and τ_i are one-dimensional, again [Grb09] provides a normalization by

$$\begin{aligned} &r(s_i, \widetilde{\pi}_{i,p}, w_{n_i}) \\ &= \frac{L(s_i, \theta_{i,p}\tau^{-1}_{i,p})L(s_i - 1, \theta_{i,p}\tau^{-1}_{i,p})}{L(s_i + 2, \theta_{i,p}\tau^{-1}_{i,p})L(s_i + 1, \theta_{i,p}\tau^{-1}_{i,p})\varepsilon(s_i, \theta_{i,p}\tau^{-1}_{i,p})^2\varepsilon(s_i - 1, \theta_{i,p}\tau^{-1}_{i,p})\varepsilon(s_i + 1, \theta_{i,p}\tau^{-1}_{i,p})}. \end{aligned} \tag{14}$$

Therefore, we have defined recursively the global normalization factor $r(\underline{s}, \widetilde{\pi}, w)$ as in (9) for each $w \in W_{\mathbb{Q}}$. We finally conclude as follows.

PROPOSITION 6.4. *Let $\underline{s}$ be inside the open region $\Re e(s_1) > \Re e(s_2) > 0$. Then there is an $f \in W_{P,\widetilde{\pi}}$ such that the Eisenstein series $E_P(f, \Lambda)$ has a double pole at $\underline{s}$ if and only if*

$$r(\underline{s}, \widetilde{\pi}) = \sum_{w \in W_{\mathbb{Q}}} r(\underline{s}, \widetilde{\pi}, w)$$

has a double pole at $\underline{s}$.

Proof. Suppose that $r(\underline{s}, \widetilde{\pi})$ has a double pole at $\underline{s}$, $\Re e(s_1) > \Re e(s_2) > 0$. Then there is a $w \in W_{\mathbb{Q}}$ such that $r(\underline{s}, \widetilde{\pi}, w)$ has a double pole at $\underline{s}$. By our discussion of the normalizing factors we know that $N(\underline{s}, \widetilde{\pi}, w) = r(\underline{s}, \widetilde{\pi}, w)^{-1} M(\underline{s}, \widetilde{\pi}, w)$ is holomorphic and non-vanishing at $\underline{s}$. So there is an $f \in W_{P,\widetilde{\pi}}$, which is not sent to zero by $N(\underline{s}, \widetilde{\pi}, w)$ and therefore $M(\underline{s}, \widetilde{\pi}, w)f = r(\underline{s}, \widetilde{\pi}, w)N(\underline{s}, \widetilde{\pi}, w)f$ really has a double pole at $\underline{s}$. By the decomposition (8), the constant term $E_P(f, \Lambda)_P$ has a double pole at $\underline{s}$, wherefrom it finally follows that $E_P(f, \Lambda)$ has a double pole at $\underline{s}$. □

6.3 Double poles of normalizing factors

Recall the well-known facts on the analytic behavior of Jacquet–Langlands, Hecke and Rankin–Selberg L-functions, summarized in our next result.

Lemma 6.5 (Jacquet–Langlands [JL70], Tate [Tat67] and Jacquet [Jac72]). *We have the following results.*

(i) *Let $\sigma = \hat{\bigotimes}'_p \sigma_p$ be a cuspidal automorphic representation of $GL_1(B)$ with central character $\chi_\sigma = \hat{\bigotimes}'_p \chi_{\sigma_p}$, assuming that $\dim \sigma > 1$. Then the local Jacquet–Langlands L-function $L(s, \sigma_p)$ is holomorphic and non-zero on $\Re e(s) > 1$ at each place p. For the infinite place, in particular, we obtain $L(s, \sigma_\infty) = 2(2\pi)^{-s-n-\frac{1}{2}}\Gamma(s + n + \frac{1}{2})$ if σ_∞ is the nth symmetric power $\bigodot^n \mathbb{C}^2$ and hence this local L-factor is holomorphic and non-vanishing for $\Re e(s) \geqslant 0$. The global Jacquet–Langlands L-function $L(s, \sigma)$ is an entire function and has no zeros for $\Re e(s) \geqslant 1$.*

(ii) *The local Hecke L-function $L(s, \chi_{\sigma_p})$ has a simple pole at $s = 0$ if $\chi_{\sigma_p} = \mathbf{1}_p$ and is entire otherwise. It vanishes nowhere. The global Hecke L-function $L(s, \chi_\sigma)$ has simple poles at $s = 0$ and $s = 1$ if $\chi_\sigma = \mathbf{1}$ (and $L(s, \mathbf{1}) = \pi^{-(s/2)}\Gamma(s/2)\zeta(s)$) and is entire otherwise. It is non-zero for $\Re e(s) \geqslant 1$.*

(iii) *Let $\rho = \hat{\bigotimes}'_p \rho_p$, $\eta = \hat{\bigotimes}'_p \eta_p$ be two cuspidal automorphic representations of $GL_2(\mathbb{A})$. Then the local Rankin–Selberg L-function $L(s, \rho_p \times \eta_p)$ is holomorphic and non-vanishing for $\Re e(s) \geqslant 1$. If ρ_p and η_p are both square integrable, then $L(s, \rho_p \times \eta_p)$ is holomorphic and non-zero in $\Re e(s) > 0$. The global Rankin–Selberg L-function $L(s, \rho \times \eta)$ has simple poles at $s = 0$ and $s = 1$ if and only if $\rho \cong \check{\eta}$ and is entire otherwise. It has no zeros in $\Re e(s) \geqslant 1$.*

Proposition 6.6. *For an Eisenstein series $E_P(f, \Lambda)$ to have a double pole at $\underline{s} = (s_1, s_2)$ inside the region $\Re e(s_1) > \Re e(s_2) > 0$ it is necessary that one of the following three conditions holds:*

(A) $\dim \theta > 1$ *and* $\dim \tau > 1$,

$\underline{s} = A := (\frac{3}{2}, \frac{1}{2})$, $\widetilde{\pi} = \tau \mathbin{\hat{\otimes}} \tau$, $\chi_\tau = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$;

(B) $\dim \theta = 1$, $\dim \tau > 1$,

$\underline{s} = B := (\frac{3}{2}, \frac{1}{2})$, $\chi_\tau = \mathbf{1}$, $\theta = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$;

(C) $\dim \theta = \dim \tau = 1$,

(1) $\underline{s} = C_1 := (\frac{3}{2}, \frac{1}{2})$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$, $\theta = \check{\tau}$ *or* $\theta = \mathbf{1}$;

(2) $\underline{s} = C_2 := (\frac{5}{2}, \frac{1}{2})$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$, $\theta = \tau$;

(3) $\underline{s} = C_3 := (\frac{7}{2}, \frac{3}{2}) = \rho_P$, $\widetilde{\pi} = \mathbf{1} \otimes \mathbf{1}$.

Sketch of a proof. As the determination of these necessary conditions is easy (by the concrete form of our normalizing factors $r(\underline{s}, \widetilde{\pi})$ and Lemma 6.5) but rather cumbersome, we confine ourselves in exemplifying the general procedure in the case (A). There is no loss of generality if we assume that $\underline{s} \in \mathbb{R}^2$, since this can be achieved by just twisting a cuspidal automorphic representation of $L(\mathbb{A})$ with an appropriate imaginary power of the absolute value of the reduced norm of the determinant. So let $\widetilde{\pi}$ be a cuspidal automorphic representation whose two cuspidal factors θ and τ are both not one-dimensional. We need to regard the global function $r(\underline{s}, \widetilde{\pi})$. Each of its summands $r(\underline{s}, \widetilde{\pi}, w)$, $w \in W_{\mathbb{Q}}$, is according to (9) a finite product of some of the following five functions $r_1(\underline{s}, \widetilde{\pi}) = r(s_1 - s_2, \widetilde{\pi}, w_2)$, $r_2(\underline{s}, \widetilde{\pi}) = r(s_2, \tau, w_4)$, $r_3(\underline{s}, \widetilde{\pi}) = r(s_1, \theta, w_4)$, $r_4(\underline{s}, \widetilde{\pi}) = r(s_1 + s_2, \tau \mathbin{\hat{\otimes}} \check{\theta}, w_2)$ or $r_5(\underline{s}, \widetilde{\pi}) = r(s_1 + s_2, \theta \mathbin{\hat{\otimes}} \check{\tau}, w_2)$. Here we already calculated the various infinite products $\hat{\bigotimes}'_{p \notin S(B)} r(s_{k-i+1}, \widetilde{\pi}_{k-i+1,p}, w_{n_{k-i+1}})$ according to the rule of Proposition 6.1. By the concrete form of $r_1(\underline{s}, \widetilde{\pi})$, given by (12), Lemma 6.5 now gives that the poles of $r_1(\underline{s}, \widetilde{\pi})$ are those of $L(s_1 - s_2, JL(\theta) \times JL(\check{\tau}))$, where $JL(\sigma)$ denotes the global Jacquet–Langlands lift of the cuspidal automorphic representation σ of $\mathrm{GL}_1(B)$ to a cuspidal automorphic representation $JL(\sigma)$

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of GL_2, see [GJ79, Theorem (8.3)]. Therefore, again by the above lemma, $r_1(\underline{s}, \widetilde{\pi})$ has simple poles in the region $\Re e(s_1) > \Re e(s_2) > 0$ if and only if $s_1 - s_2 = 1$ and $\theta = \tau$. Analogously, the poles of $r_2(\underline{s}, \widetilde{\pi})$ are by its concrete form given in (10) those of $L(s_2, \tau)L(2s_2, \chi_\tau)$. We apply Lemma 6.5 and see that $r_2(\underline{s}, \widetilde{\pi})$ has simple poles in the region $\Re e(s_1) > \Re e(s_2) > 0$ if and only if $2s_2 = 1$, $\chi_\tau = \mathbf{1}$ and $L(\frac{1}{2}, \tau) \neq 0$. An analog and easy observation shows that $r_3(\underline{s}, \widetilde{\pi})$ has simple poles along $2s_1 = 1$ if $\chi_\theta = \mathbf{1}$ and $L(\frac{1}{2}, \theta) \neq 0$ and that the poles of $r_j(\underline{s}, \widetilde{\pi})$, $j = 4, 5$ lie along $s_1 + s_2 = 1$ for $\theta \cong \check{\tau}$. Only the singular hyperplanes of $r_1(\underline{s}, \widetilde{\pi})$ and $r_2(\underline{s}, \widetilde{\pi})$ intersect in the region $\Re e(s_1) > \Re e(s_2) > 0$ and they intersect in $\underline{s} = (\frac{3}{2}, \frac{1}{2})$. □

Remark 6.7. Only the longest element $w_0 = w_2w_4w_2w_4$ in $W_\mathbb{Q}$ gives rise to a normalizing factor $r(\underline{s}, \widetilde{\pi}, w)$ which carries $r_1(\underline{s}, \widetilde{\pi})$ and $r_2(\underline{s}, \widetilde{\pi})$. The remaining other two factors $r_3(\underline{s}, \widetilde{\pi})$ and $r_4(\underline{s}, \widetilde{\pi})$ showing up in the decomposition of $r(\underline{s}, \widetilde{\pi}, w_0)$ have no zero at $A = (\frac{3}{2}, \frac{1}{2})$. So for any cuspidal automorphic representation $\widetilde{\pi}$ of $L(\mathbb{A})$, satisfying $\widetilde{\pi} = \tau \hat{\otimes} \tau$, $\chi_\tau = \mathbf{1}$ and $L(\frac{1}{2}, \tau) \neq 0$, the necessary condition given above is also sufficient to ensure that there will be an $f \in W_{P,\widetilde{\pi}}$ such that the Eisenstein series $E_P(f, \Lambda)$ has a double pole at A.

In fact, by the same argument the points C_2 and C_3 will actually give rise to double poles of Eisenstein series attached to cuspidal automorphic representations $\widetilde{\pi}$ of the Levi L that satisfy the indicated condition. So for these points the given conditions on $\widetilde{\pi}$ will also be sufficient for an appropriate choice of $f \in W_{P,\widetilde{\pi}}$.

PROPOSITION 6.8. *For an Eisenstein series $E_P(f, \Lambda)$ to have a double pole at $\underline{s} = (s_1, s_2)$ on the boundary of the closed, positive Weyl chamber, i.e. either $\Re e(s_1) - \Re e(s_2) = 0$ or $\Re e(s_1) = 0$, it is necessary that:*

(A) $\dim \theta > 1$ *and* $\dim \tau > 1$,

$\underline{s} = (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0)$ *or* $(1, 0)$;

(B) $\dim \theta = 1$, $\dim \tau > 1$,

$\underline{s} = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$ *or* $(\frac{3}{2}, 0)$;

(B′) $\dim \theta > 1$, $\dim \tau = 1$,

$\underline{s} = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, *or* $(\frac{1}{2}, 0)$;

(C) $\dim \theta = \dim \tau = 1$,

$\underline{s} = (\frac{1}{2}, \frac{1}{2})$, $(1, 1)$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$ *or* $(2, 0)$.

Sketch of a proof. Again this is easy, but not very instructive, so we will again confine ourselves to case (A). We may also assume that $\underline{s} \in \mathbb{R}^2$. We cannot decide by our means chosen here if the root hyperplanes $R_1 := \{\underline{s} \in \mathbb{R}^2 \mid s_1 - s_2 = 0\}$ and $R_2 := \{\underline{s} \in \mathbb{R}^2 \mid s_1 = 0\}$ forming the boundary of the closed, positive Weyl chamber are actually singular hyperplanes for Eisenstein series attached to cuspidal automorphic representations $\widetilde{\pi}$ of $L(\mathbb{A})$. However, in order to have a double pole at a point $\underline{s}$ on this boundary, we need to have one of our singular root hyperplanes, given by the five factors $r_i(\underline{s}, \widetilde{\pi})$, $1 \leqslant i \leqslant 5$, to cross R_1 or R_2 in $\underline{s}$. From the proof above we know that these singular hyperplanes are $s_1 - s_2 = 1$, $2s_2 = 1$, $2s_1 = 1$ and $s_1 + s_2 = 1$. Their intersection points with one of the boundary hyperplanes R_1 and R_2 are precisely the points we claim to be the only candidates for double poles of Eisenstein series on the boundary of the positive Weyl chamber in case (A). □

6.4 Square-integrable Eisenstein cohomology

We now determine the square-integrable Eisenstein cohomology supported by P. Therefore, let

$$L_{E,P,\varphi} \subseteq \mathcal{A}_{E,P,\varphi}$$

be the subspace of $\mathcal{A}_{E,P,\varphi}$ which consists of square-integrable automorphic forms. By [Lan76] or [MW95] it is spanned by all twice-iterated, square-integrable residues of Eisenstein series $E_P(f,\Lambda)$, $f \in W_{P,\tilde{\pi}}$, $\pi = \chi\tilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$ at the point $d\chi$ inside the closed, positive Weyl chamber defined by $\Delta(P,A)$. Hence, it is spanned by the square-integrable residues at $d\chi$ of those Eisenstein series attached to a cuspidal automorphic representation $\tilde{\pi}$ of $L(\mathbb{A})$ which have a double pole there. This is because simple poles integrate to zero. Put

$$L_{E,P} := \bigoplus_{\varphi \in \Psi_{E,P}} L_{E,P,\varphi}.$$

We define the space of *square-integrable Eisenstein cohomology* (supported by P) by

$$H^q(\mathfrak{g}, K, L_{E,P} \otimes E) = \bigoplus_{\varphi \in \Psi_{E,P}} H^q(\mathfrak{g}, K, L_{E,P,\varphi} \otimes E).$$

Combining our previous results, Propositions 6.6 and 6.8, with the Langlands' 'square integrability criterion' (cf. [MW95, Lemma I.4.11]) we conclude as follows.

THEOREM 6.9. *Let $P = LN$ be the minimal standard parabolic $\mathbb{Q}$-subgroup of $G = Sp(2,2)$ and E any irreducible, finite-dimensional complex-rational representation of $G(\mathbb{R})$. Then the square-integrable Eisenstein cohomology supported by P, $H^*(\mathfrak{g}, K, L_{E,P} \otimes E)$, is spanned by cohomology classes which are Eisenstein lifts of a class of type (π, w), $\pi = \chi\tilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$, $w \in W^P$, such that necessarily one of the following conditions holds.*

If $d\chi$ is inside the open, positive Weyl chamber defined by $\Delta(P,A)$:

(A) *if* $\dim\theta > 1$ *and* $\dim\tau > 1$,
$\tilde{\pi} = \tau \hat{\otimes} \tau$, $\chi_\tau = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$;

(B) *if* $\dim\theta = 1$, $\dim\tau > 1$,
$\tilde{\pi} = \mathbf{1} \otimes \tau$, $\chi_\tau = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$;

(C) *if* $\dim\theta = \dim\tau = 1$,
 (1) $\tilde{\pi} = \mathbf{1} \hat{\otimes} \tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$;
 (2) $\tilde{\pi} = \tau \hat{\otimes} \tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$ *and* $d\chi = (\frac{5}{2}, \frac{1}{2})$;
 (3) $\tilde{\pi} = \mathbf{1} \otimes \mathbf{1}$ *and* $d\chi = (\frac{7}{2}, \frac{3}{2}) = \rho_P$.

If $d\chi$ is on the boundary of the closed, positive Weyl chamber defined by $\Delta(P,A)$:

(A) *if* $\dim\theta > 1$ *and* $\dim\tau > 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0)$ *or* $(1, 0)$;

(B) *if* $\dim\theta = 1$, $\dim\tau > 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$ *or* $(\frac{3}{2}, 0)$;

(B′) *if* $\dim\theta > 1$, $\dim\tau = 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, *or* $(\frac{1}{2}, 0)$;

(C) *if* $\dim\theta = \dim\tau = 1$,
$d\chi = (\frac{1}{2}, \frac{1}{2})$, $(1, 1)$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$ *or* $(2, 0)$.

Remark. By the square integrability criterion, i.e. [MW95, Lemma I.4.11], (non-zero) iterated residues of Eisenstein series at $C_1 = (\frac{3}{2}, \frac{1}{2})$ cannot be square integrable, if they come from a representation $\widetilde{\pi} = \check{\tau} \,\hat{\otimes}\, \tau$ of $L(\mathbb{A})$. Hence, we excluded them from the list in Theorem 6.9.

We also have the following vanishing theorem.

THEOREM 6.10. *If $E \neq \mathbb{C}$, then square-integrable Eisenstein cohomology supported by P vanishes below degree three*

$$H^q(\mathfrak{g}, K, L_{E,P} \otimes E) = 0 \quad \text{for } q \leqslant 3.$$

If $E = \mathbb{C}$, then there is an epimorphism

$$H^0(\mathfrak{g}, K, L_{\mathbb{C},P}) \twoheadrightarrow H^0(G, \mathbb{C}) = \mathbb{C}$$

and

$$H^q(\mathfrak{g}, K, L_{\mathbb{C},P}) = 0 \quad \text{for } 1 \leqslant q \leqslant 3.$$

Proof. For any E, $L_{E,P}$ is a direct summand of the residual spectrum of $G(\mathbb{A})$, so $L_{E,P}$ is the direct Hilbert sum of certain residual automorphic representations. So in order to give a non-trivial cohomological contribution in the degrees $0 \leqslant q \leqslant 3$, it is necessary that there is an irreducible, unitary representation $\pi = \hat{\otimes}'_p \pi_p$ of $G(\mathbb{A})$ with π_∞ cohomological with respect to E. However, if $\pi_\infty \neq \mathbb{C}$, then $H^q(\mathfrak{g}, K, \pi_\infty \otimes E) = 0$ for $q \leqslant 3$, as it follows from [VZ84, Theorem 8.1]. Conversely, we can only have $\pi_\infty = \mathbb{C}$ if $E = \mathbb{C}$ itself, and then we know that

$$H^q(\mathfrak{g}, K, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } q = 0, 4, 12, 16 \\ \mathbb{C}^2 & \text{if } q = 8 \end{cases} \tag{15}$$

and vanishes in all other degrees. This happens, since $H^q(\mathfrak{g}, K, \mathbb{C})$ equals the de Rham cohomology of the quaternionic Grassmannian $G_2(\mathbb{H}^4)$ of two-dimensional $\mathbb{H}$-subspaces in $\mathbb{H}^4$. Further, identifying $H^0(G, \mathbb{C})$ with the de Rham cohomology of $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ proves $H^0(G, \mathbb{C}) = \mathbb{C}$. Observe that we have now shown every assertion except that there is a surjection $H^0(\mathfrak{g}, K, L_{\mathbb{C},P}) \twoheadrightarrow H^0(G, \mathbb{C})$. This is certainly well-known and follows from general theory, but for the convenience of the reader we give a direct argument here.

Therefore, recall that for $E = \mathbb{C}$ the longest element in W^P will give the evaluation point $d\chi = (\frac{7}{2}, \frac{3}{2}) = \rho_P$ and $\widetilde{\pi}_\infty = \mathbf{1}_\infty \otimes \mathbf{1}_\infty$ and consider the image of the local normalized intertwining operator $N((\frac{7}{2}, \frac{3}{2}), \mathbf{1}_\infty \otimes \mathbf{1}_\infty, w_0)$. As $\infty \in S(B)$, $\mathbf{1}_\infty \otimes \mathbf{1}_\infty$ is compactly supported modulo the center, whence tempered, and so the image of the local normalized operator is the Langlands quotient of the local trivial representation. As $C_3 = (\frac{7}{2}, \frac{3}{2}) = \rho_P$, this quotient is the local trivial representation of $G(\mathbb{R})$. A twice-iterated residue at $d\chi = \rho_P$ of a singular Eisenstein series will be square-integrable, by [MW95, Lemma I.4.11]. Therefore, there is a global residual automorphic representation $\pi \subset L_{\mathbb{C},P}$, namely the image of the global normalized intertwining operator $N((\frac{7}{2}, \frac{3}{2}), \mathbf{1} \otimes \mathbf{1}, w_0)$, such that $\pi_\infty = \mathbb{C}$. (In fact, by the above local argument, one can also easily see that the image of the global operator is the global trivial representation $\mathbf{1}$ of $G(\mathbb{A})$.) By (15) we are done. □

Remark. Of course we could also have gained the result in degrees $q = 0, 1$ by the following: in degree $q = 0$ we could have used the equality $H^0(G, E) = \varinjlim_{K_\Gamma} E^\Gamma$ and Borel's 'density theorem' (cf., e.g., [PR93]) or in degree $q = 1$ we could have referred to the well-known vanishing results of Margulis and Raghunathan, which give $H^1(G, E) = 0$ (see [Mar91, Rag67]). In particular,

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for $E=\mathbb{C}$ we could also have used [Bor74], which shows that $H^q(\mathfrak{g},K,\mathbb{C})\to H^q(G,\mathbb{C})$ is an isomorphism in low degrees together with our computations of the line (15).

Remark. As $H^4(\mathfrak{g},K,\mathbb{C})=\mathbb{C}$, $q=3$ is in fact a sharp upper bound for the vanishing of $(\mathfrak{g},K)$-cohomology of $G=\mathrm{Sp}(2,2)$ in low degrees. However, there is also another residual representation which has $(\mathfrak{g},K)$-cohomology in degree four. This is a consequence of our Theorem 2.1: in fact, if we consider $q=4$, then for all λ the element $w=w_4w_2w_3w_4w_1w_2w_3w_1w_2w_1$ of length 10 is in $W^+(\lambda)$ as indicated in our Table A8 (see Appendix A). We obtain $\widetilde{\pi}_\infty=\bigodot^{4+c_1-c_3+2c_4}\mathbb{C}^2\otimes\bigodot^{4-c_1+c_3}\mathbb{C}^2$, whence $\dim\theta\geqslant 5$ and $\dim\tau\geqslant 5$ and we are in case (A). The corresponding evaluation point $d\chi=-w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$ reads as $\underline{s}=((3+c_1+c_3-2c_4/2),(1+2c_2-c_1-c_3/2))$, satisfies $s_1\geqslant\frac{3}{2}$, $s_2\geqslant\frac{1}{2}$ and is always inside the open, positive Weyl chamber. By Proposition 6.6 and Remark 6.7 there will really be an Eisenstein series $E_P(f,\Lambda)$ which has a double pole at $A=(\frac{3}{2},\frac{1}{2})$ for all cuspidal automorphic representations $\widetilde{\pi}=\theta\,\hat{\otimes}\,\tau$ of $L(\mathbb{A})$ subject to the condition $\theta=\tau$, $\chi_\tau=\mathbf{1}$ and $L(\frac{1}{2},\tau)\neq 0$. According to [MW95, Lemma I.4.11], the space of twice-iterated residues at $A=(\frac{3}{2},\frac{1}{2})$ of such Eisenstein series consists of square-integrable automorphic forms. Playing around with the concrete form of $\widetilde{\pi}_\infty$ and $\underline{s}$ given above yields $\lambda=k\omega_4$, $k=0,1,2,3\ldots$ and ω_4 the fundamental weight of $\mathfrak{g}_\mathbb{C}$ which corresponds to the fourth simple root α_4. By Theorem 2.1 the twice-iterated residue of $E_P(f,\Lambda)$ will therefore contribute to square-integrable Eisenstein cohomology with respect to $E=E_{k\omega_4}$ in degree $\dim N(\mathbb{R})-l(w)=14-10=4$.

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Appendix A. Tables for the three standard parabolic $\mathbb{Q}$-subgroups

As before, E denotes a finite-dimensional, irreducible, complex-rational representation of $G(\mathbb{R})=\mathrm{Sp}(2,2)$ with highest weight $\lambda=\sum_{i=1}^4 c_i\alpha_i$. As λ is algebraically integral and dominant, we can easily see that we obtain the following relations among the coefficients:

$$c_4\geqslant\frac{c_3}{2}\geqslant\frac{c_2}{2}\geqslant\frac{c_3}{3}\geqslant 0\quad\text{and}\quad c_1\geqslant\frac{c_2}{2}. \tag{16}$$

Let ω_{01} and ω_{02} be the two fundamental weights of $M_0(\mathbb{C})$. Analogously, ω_{ij}, $j=1,2,3$, denotes the jth fundamental weight of $M_i(\mathbb{C})$, $i=1,2$.

Tables A1 and A2 give the values $w(\lambda+\rho)-\rho|_{\mathfrak{b}_{i_\mathbb{C}}}$, $w\in W^{P_i}$, $i=1,2$, in terms of the fundamental weights ω_{ij}. Recall that $\rho=4\alpha_1+7\alpha_2+9\alpha_3+5\alpha_4$. Table A2 additionally says

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TABLE A1. ${}^{\circ}F_w$ for P_1.

	$w(\lambda+\rho)-\rho\vert_{\mathfrak{b}_{1\mathbb{C}}}$
id	$(2c_1-c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
w_2	$(1+c_1+c_2-c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
w_2w_1	$(-c_1+2c_2-c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
w_2w_3	$(2+c_1+c_3-2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$w_2w_3w_4w_3$	$(5+c_1-c_2+c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_3w_1$	$(4-c_1+c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$w_2w_3w_4w_3w_2$	$(6+c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_3w_2w_1$	$(6+c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_3w_1w_2$	$(4-c_1+c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$w_2w_3w_4w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_4w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$(-c_1+2c_2-c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$

TABLE A2. ${}^{\circ}F_w$ for P_2.

	$w(\lambda+\rho)-\rho\vert_{\mathfrak{b}_{2\mathbb{C}}}$	
id	$(2c_1-c_2)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(-c_2+2c_3-2c_4)\omega_{23}$	
w_4	$(2c_1-c_2)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(2-c_2+2c_4)\omega_{23}$	
w_4w_3	$(2c_1-c_2)\omega_{21}+(1-c_1+c_2+c_3-2c_4)\omega_{22}+(2-c_2+2c_4)\omega_{23}$	
$w_4w_3w_2$	$(1+c_1+c_2-c_3)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(3-c_1+c_2-c_3+2c_4)\omega_{23}$	
$w_4w_3w_4$	$(2c_1-c_2)\omega_{21}+(3-c_1+c_2-c_3+2c_4)\omega_{22}+(-c_2+2c_3-2c_4)\omega_{23}$	
$w_4w_3w_4w_2$	$(1+c_1+c_2-c_3)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(1-c_1+c_2+c_3-2c_4)\omega_{23}$	
$w_4w_3w_2w_1$	$(-c_1+2c_2-c_3)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(4+c_1-c_3+2c_4)\omega_{23}$	$\nexists$
$w_4w_3w_4w_2w_1$	$(-c_1+2c_2-c_3)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(2+c_1+c_3-2c_4)\omega_{23}$	$\nexists$
$w_4w_3w_4w_2w_3$	$(2+c_1+c_3-2c_4)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(-c_1+2c_2-c_3)\omega_{23}$	$\nexists$
$w_4w_3w_4w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(-c_1+2c_2-c_3)\omega_{23}$	$\nexists$
$w_4w_3w_4w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(1+c_1+c_2-c_3)\omega_{23}$	*
$w_4w_3w_4w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(1+c_1+c_2-c_3)\omega_{23}$	**
$w_4w_3w_4w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{21}+(3-c_1+c_2-c_3+2c_4)\omega_{22}+(2c_1-c_2)\omega_{23}$	*
$w_4w_3w_4w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{21}+(1-c_1+c_2+c_3-2c_4)\omega_{22}+(2c_1-c_2)\omega_{23}$	**
$w_4w_3w_4w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(2c_1-c_2)\omega_{23}$	**
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(2c_1-c_2)\omega_{23}$	*

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TABLE A3. Λ_w for P_1.

	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{1_\mathbb{C}}}, \alpha_2\vert_{\mathfrak{a}_{1_\mathbb{C}}}\rangle$
id	$2(-7-c_2) \leqslant -14$
w_2	$2(-6-c_1+c_2-c_3) \leqslant -12$
w_2w_1	$2(-5+c_1-c_3) \leqslant -10$
w_2w_3	$2(-5-c_1+c_3-2c_4) \leqslant -10$
$w_2w_3w_1$	$2(-4+c_1-c_2+c_3-2c_4) \leqslant -8$
$w_2w_3w_4$	$2(-3-c_1-c_3+2c_4) \leqslant -6$
$w_2w_3w_1w_2$	$2(-3+c_2-2c_4) \leqslant -6$
$w_2w_3w_4w_1$	$2(-2+c_1-c_2-c_3+2c_4) \leqslant -4$
$w_2w_3w_4w_3$	$2(-2-c_1-c_2+c_3) \leqslant -4$
$w_2w_3w_4w_3w_1$	$2(-1+c_1-2c_2+c_3) \leqslant -2$
$w_2w_3w_4w_1w_2$	$2(-1+c_2-2c_3+2c_4) \leqslant -2$
$w_2w_3w_4w_3w_2$	$2(-1-2c_1+c_2) \leqslant -2$
$w_2w_3w_4w_3w_2w_1$	$2(1+2c_1-c_2) \geqslant 2$
$w_2w_3w_4w_3w_1w_2$	$2(1-c_1+2c_2-c_3) \geqslant 2$
$w_2w_3w_4w_1w_2w_3$	$2(1-c_2+2c_3-2c_4) \geqslant 2$
$w_2w_3w_4w_3w_1w_2w_1$	$2(2+c_1+c_2-c_3) \geqslant 4$
$w_2w_3w_4w_1w_2w_3w_2$	$2(2-c_1+c_2+c_3-2c_4) \geqslant 4$
$w_2w_3w_4w_1w_2w_3w_4$	$2(3-c_2+2c_4) \geqslant 6$
$w_2w_3w_4w_1w_2w_3w_2w_1$	$2(3+c_1+c_3-2c_4) \geqslant 6$
$w_2w_3w_4w_1w_2w_3w_4w_2$	$2(4-c_1+c_2-c_3+2c_4) \geqslant 8$
$w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$2(5+c_1-c_3+2c_4) \geqslant 10$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$2(5-c_1+c_3) \geqslant 10$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$2(6+c_1-c_2+c_3) \geqslant 12$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$2(7+c_2) \geqslant 14$

TABLE A4. Λ_w for P_2.

	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{2_\mathbb{C}}}, \alpha_4\vert_{\mathfrak{a}_{2_\mathbb{C}}}\rangle$
id	$4(-5-c_4) \leqslant -20$
w_4	$4(-4-c_3+c_4) \leqslant -16$
w_4w_3	$4(-3-c_2+c_3-c_4) \leqslant -12$
$w_4w_3w_2$	$4(-2-c_1+c_2-c_4) \leqslant -8$
$w_4w_3w_4$	$4(-2-c_2+c_4) \leqslant -8$
$w_4w_3w_4w_2$	$4(-1-c_1+c_2-c_3+c_4) \leqslant -4$
$w_4w_3w_2w_1$	$4(-1+c_1-c_4)$?
$w_4w_3w_4w_2w_1$	$4(c_1-c_3+c_4)$?
$w_4w_3w_4w_2w_3$	$4(-c_1+c_3-c_4)$?
$w_4w_3w_4w_2w_3w_4$	$4(1-c_1+c_4)$?
$w_4w_3w_4w_2w_3w_1$	$4(1+c_1-c_2+c_3-c_4) \geqslant 4$
$w_4w_3w_4w_2w_3w_4w_1$	$4(2+c_1-c_2+c_4) \geqslant 8$
$w_4w_3w_4w_2w_3w_1w_2$	$4(2+c_2-c_4) \geqslant 8$
$w_4w_3w_4w_2w_3w_4w_1w_2$	$4(3+c_2-c_3+c_4) \geqslant 12$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3$	$4(4+c_3-c_4) \geqslant 16$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4$	$4(5+c_4) \geqslant 20$

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Table A5. $^{\circ}F_w$ for the lower-half representatives.

	$w(\lambda+\rho)-\rho\|_{\mathfrak{b}_{0_{\mathbb{C}}}}$
id	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
w_2	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
w_4	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
w_2w_1	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
w_4w_2	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
w_2w_3	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
w_4w_3	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_1$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_3w_2$	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_3w_4$	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_3w_2w_1$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_2$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2$	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_2w_3w_2w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_1$	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_3w_1$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_2w_3w_4w_3w_2$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3$	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_1w_2w_1$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_2w_3w_4w_3w_2w_1$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_3w_1$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_3w_2$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_2w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_2w_3w_4w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_1$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_3w_2w_1$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_1$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$

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Table A6. ${}^{\circ}F_w$ for the upper-half representatives.

	$w(\lambda+\rho)-\rho\|_{\mathfrak{b}_{0\mathbb{C}}}$
$w_4w_2w_3w_4w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_2$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_2w_1$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_1$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_1$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_2$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_2w_1$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_1w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_1w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$

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TABLE A7. Λ_w for the lower-half representatives.

	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0\mathbb{C}}}, \alpha_2\vert_{\mathfrak{a}_{0\mathbb{C}}}\rangle$	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0\mathbb{C}}}, \alpha_4\vert_{\mathfrak{a}_{0\mathbb{C}}}\rangle$
id	$2(-2-c_2+c_4) \leqslant -4$	$2(-3+c_2-2c_4) \leqslant -6$
w_2	$2(-1-c_1+c_2-c_3+c_4) \leqslant -2$	$2(-4+c_1-c_2+c_3-2c_4) \leqslant -8$
w_4	$2(-3-c_2+c_3-c_4) \leqslant -6$	$2(-1+c_2-2c_3+2c_4) \leqslant -2$
w_2w_1	$2(c_1-c_3+c_4)$?	$2(-5-c_1+c_3-2c_4) \leqslant -10$
w_4w_2	$2(-2-c_1+c_2-c_4) \leqslant -4$	$2(-2+c_1-c_2-c_3+2c_4) \leqslant -4$
w_2w_3	$2(-c_1+c_3-c_4)$?	$2(-5+c_1-c_3) \leqslant -10$
w_4w_3	$2(-4-c_3+c_4) \leqslant -8$	$2(1-c_2+2c_3-2c_4) \geqslant 2$
$w_4w_2w_1$	$2(-1+c_1-c_4)$?	$2(-3-c_1-c_3+2c_4) \leqslant -6$
$w_2w_3w_1$	$2(1+c_1-c_2+c_3-c_4) \geqslant 2$	$2(-6-c_1+c_2-c_3) \leqslant -12$
$w_4w_3w_2$	$2(-4-c_3+c_4) \leqslant -8$	$2(2-c_1+c_2+c_3-2c_4) \geqslant 4$
$w_4w_2w_3$	$2(-2-c_1+c_2-c_4) \leqslant -4$	$2(-1+c_1-2c_2+c_3) \leqslant -2$
$w_2w_3w_4$	$2(1-c_1+c_4)$?	$2(-5+c_1-c_3) \leqslant -10$
$w_4w_3w_4$	$2(-5-c_4) \leqslant 10$	$2(3-c_2+2c_4) \geqslant 6$
$w_4w_3w_2w_1$	$2(-4-c_3+c_4) \leqslant -8$	$2(3+c_1+c_3-2c_4) \geqslant 6$
$w_4w_2w_3w_1$	$2(-1+c_1-c_4)$?	$2(-2-c_1-c_2+c_3) \leqslant -4$
$w_2w_3w_4w_1$	$2(2+c_1-c_2+c_4) \geqslant 4$	$2(-6-c_1+c_2-c_3) \leqslant -12$
$w_2w_3w_1w_2$	$2(2+c_2-c_4) \geqslant 4$	$2(-7-c_2) \leqslant -14$
$w_4w_2w_3w_2$	$2(-3-c_2+c_3-c_4) \leqslant -6$	$2(1-c_1+2c_2-c_3) \geqslant 2$
$w_4w_3w_4w_2$	$2(-5-c_4) \leqslant -10$	$2(4-c_1+c_2-c_3+2c_4) \geqslant 8$
$w_2w_3w_4w_3$	$2(1-c_1+c_4)$?	$2(-4+c_1-c_2+c_3-2c_4) \leqslant -8$
$w_4w_2w_3w_4$	$2(-1-c_1+c_2-c_3+c_4) \leqslant -2$	$2(-1+c_1-2c_2+c_3) \leqslant -2$
$w_4w_2w_3w_2w_1$	$2(-3-c_2+c_3-c_4) \leqslant -6$	$2(2+c_1+c_2-c_3) \geqslant 4$
$w_4w_3w_4w_2w_1$	$2(-5-c_4) \leqslant -10$	$2(5+c_1-c_3+2c_4) \geqslant 10$
$w_2w_3w_4w_3w_1$	$2(2+c_1-c_2+c_4) \geqslant 4$	$2(-5-c_1+c_3-2c_4) \leqslant -10$
$w_4w_2w_3w_4w_1$	$2(c_1-c_3+c_4)$?	$2(-2-c_1-c_2+c_3) \leqslant -4$
$w_4w_2w_3w_1w_2$	$2(-1+c_1-c_4)$?	$2(-1-2c_1+c_2) \leqslant -2$
$w_2w_3w_4w_1w_2$	$2(3+c_2-c_3+c_4) \geqslant 6$	$2(-7-c_2) \leqslant -14$
$w_2w_3w_4w_3w_2$	$2(1-c_1+c_4)$?	$2(-3+c_2-2c_4) \leqslant -6$
$w_4w_2w_3w_4w_2$	$2(-2-c_2+c_4) \leqslant -4$	$2(1-c_1+2c_2-c_3) \geqslant 2$
$w_4w_3w_4w_2w_3$	$2(-5-c_4) \leqslant -10$	$2(5-c_1+c_3) \geqslant 10$
$w_4w_2w_3w_4w_3$	$2(-c_1+c_3-c_4)$?	$2(-2+c_1-c_2-c_3+2c_4) \leqslant -4$
$w_4w_2w_3w_1w_2w_1$	$2(-2-c_1+c_2-c_4) \leqslant -4$	$2(1+2c_1-c_2) \geqslant 2$
$w_2w_3w_4w_3w_2w_1$	$2(2+c_1-c_2+c_4) \geqslant 4$	$2(-3+c_2-2c_4) \leqslant -6$
$w_4w_2w_3w_4w_2w_1$	$2(-2-c_2+c_4) \leqslant -4$	$2(2+c_1+c_2-c_3) \geqslant 4$
$w_4w_3w_4w_2w_3w_1$	$2(-5-c_4) \leqslant -10$	$2(6+c_1-c_2+c_3) \geqslant 12$
$w_4w_2w_3w_4w_3w_1$	$2(1+c_1-c_2+c_3-c_4) \geqslant 2$	$2(-3-c_1-c_3+2c_4) \leqslant -6$
$w_2w_3w_4w_3w_1w_2$	$2(3+c_2-c_3+c_4) \geqslant 6$	$2(-5-c_1+c_3-2c_4) \leqslant -10$
$w_4w_2w_3w_4w_1w_2$	$2(c_1-c_3+c_4)$?	$2(-1-2c_1+c_2) \leqslant -2$
$w_4w_2w_3w_4w_3w_2$	$2(-c_1+c_3-c_4)$?	$2(-1+c_2-2c_3+2c_4) \leqslant -2$
$w_2w_3w_4w_1w_2w_3$	$2(4+c_3-c_4) \geqslant 8$	$2(-7-c_2) \leqslant -14$
$w_4w_2w_3w_4w_2w_3$	$2(-2-c_2+c_4) \leqslant -4$	$2(2-c_1+c_2+c_3-2c_4) \geqslant 4$
$w_4w_3w_4w_2w_3w_4$	$2(-4-c_3+c_4) \leqslant -8$	$2(5-c_1+c_3) \geqslant 10$
$w_2w_3w_4w_3w_1w_2w_1$	$2(3+c_2-c_3+c_4) \geqslant 6$	$2(-4+c_1-c_2+c_3-2c_4) \leqslant -8$
$w_4w_2w_3w_4w_1w_2w_1$	$2(-1-c_1+c_2-c_3+c_4) \leqslant -2$	$2(1+2c_1-c_2) \geqslant 2$
$w_4w_2w_3w_4w_3w_2w_1$	$2(1+c_1-c_2+c_3-c_4) \geqslant 2$	$2(-1+c_2-2c_3+2c_4) \leqslant -2$
$w_4w_2w_3w_4w_2w_3w_1$	$2(-2-c_2+c_4) \leqslant -4$	$2(3+c_1+c_3-2c_4) \geqslant 6$
$w_4w_3w_4w_2w_3w_4w_1$	$2(-4-c_3+c_4) \leqslant -8$	$2(6+c_1-c_2+c_3) \geqslant 12$
$w_4w_3w_4w_2w_3w_1w_2$	$2(-5-c_4) \leqslant -10$	$2(7+c_2) \geqslant 14$

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Table A8. Λ_w for the upper-half representatives.

	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0_\mathbb{C}}}, \alpha_2\vert_{\mathfrak{a}_{0_\mathbb{C}}}\rangle$	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0_\mathbb{C}}}, \alpha_4\vert_{\mathfrak{a}_{0_\mathbb{C}}}\rangle$
$w_4w_2w_3w_4w_3w_1w_2$	$2(2+c_2-c_4)\geqslant 4$	$2(-3-c_1-c_3+2c_4)\leqslant -6$
$w_2w_3w_4w_1w_2w_3w_2$	$2(4+c_3-c_4)\geqslant 8$	$2(-6-c_1+c_2-c_3)\leqslant -12$
$w_4w_2w_3w_4w_2w_3w_2$	$2(-1-c_1+c_2-c_3+c_4)\leqslant -2$	$2(1-c_2+2c_3-2c_4)\geqslant 2$
$w_4w_2w_3w_4w_1w_2w_3$	$2(1+c_1-c_2+c_3-c_4)\geqslant 2$	$2(-1-2c_1+c_2)\leqslant -2$
$w_4w_3w_4w_2w_3w_4w_3$	$2(-3-c_2+c_3-c_4)\leqslant -6$	$2(4-c_1+c_2-c_3+2c_4)\geqslant 8$
$w_2w_3w_4w_1w_2w_3w_4$	$2(5+c_4)\geqslant 10$	$2(-7-c_2)\leqslant -14$
$w_4w_2w_3w_4w_3w_1w_2w_1$	$2(2+c_2-c_4)\geqslant 4$	$2(-2+c_1-c_2-c_3+2c_4)\leqslant -4$
$w_2w_3w_4w_1w_2w_3w_2w_1$	$2(4+c_3-c_4)\geqslant 8$	$2(-5+c_1-c_3)\leqslant -10$
$\underline{w_4w_2w_3w_4w_2w_3w_2w_1}$	$2(c_1-c_3+c_4)$?	$2(1-c_2+2c_3-2c_4)\geqslant 2$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_1}$	$2(-c_1+c_3-c_4)$?	$2(1+2c_1-c_2)\geqslant 2$
$w_4w_3w_4w_2w_3w_4w_3w_1$	$2(-3-c_2+c_3-c_4)\leqslant -6$	$2(5+c_1-c_3+2c_4)\geqslant 10$
$w_4w_2w_3w_4w_2w_3w_1w_2$	$2(-1-c_1+c_2-c_3+c_4)\leqslant -2$	$2(3+c_1+c_3-2c_4)\geqslant 6$
$w_4w_3w_4w_2w_3w_4w_1w_2$	$2(-4-c_3+c_4)\leqslant -8$	$2(7+c_2)\geqslant 14$
$w_4w_2w_3w_4w_1w_2w_3w_2$	$2(2+c_2-c_4)\geqslant 4$	$2(-2-c_1-c_2+c_3)\leqslant -4$
$w_4w_3w_4w_2w_3w_4w_3w_2$	$2(-2-c_1+c_2-c_4)\leqslant -4$	$2(3-c_2+2c_4)\geqslant 6$
$w_2w_3w_4w_1w_2w_3w_4w_2$	$2(5+c_4)\geqslant 10$	$2(-6-c_1+c_2-c_3)\leqslant -12$
$w_4w_2w_3w_4w_1w_2w_3w_4$	$2(2+c_1-c_2+c_4)\geqslant 4$	$2(-1-2c_1+c_2)\leqslant -2$
$\underline{w_4w_2w_3w_4w_2w_3w_1w_2w_1}$	$2(c_1-c_3+c_4)$?	$2(2-c_1+c_2+c_3-2c_4)\geqslant 4$
$w_4w_2w_3w_4w_1w_2w_3w_2w_1$	$2(2+c_2-c_4)\geqslant 4$	$2(-1+c_1-2c_2+c_3)\leqslant -2$
$\underline{w_4w_3w_4w_2w_3w_4w_3w_2w_1}$	$2(-1+c_1-c_4)$?	$2(3-c_2+2c_4)\geqslant 6$
$w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$2(5+c_4)\geqslant 10$	$2(-5+c_1-c_3)\leqslant -10$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1}$	$2(1-c_1+c_4)$?	$2(1+2c_1-c_2)\geqslant 2$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_1w_2}$	$2(-c_1+c_3-c_4)$?	$2(2+c_1+c_2-c_3)\geqslant 4$
$w_4w_3w_4w_2w_3w_4w_3w_1w_2$	$2(-2-c_1+c_2-c_4)\leqslant -4$	$2(5+c_1-c_3+2c_4)\geqslant 10$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$2(3+c_2-c_3+c_4)\geqslant 6$	$2(-2-c_1-c_2+c_3)\leqslant -4$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3$	$2(-3-c_2+c_3-c_4)\leqslant -6$	$2(7+c_2)\geqslant 14$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$2(5+c_4)\geqslant 10$	$2(-5-c_1+c_3-2c_4)\leqslant -10$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_1w_2w_1}$	$2(1+c_1-c_2+c_3-c_4)\geqslant 2$	$2(1-c_1+2c_2-c_3)\geqslant 2$
$\underline{w_4w_3w_4w_2w_3w_4w_3w_1w_2w_1}$	$2(-1+c_1-c_4)$?	$2(4-c_1+c_2-c_3+2c_4)\geqslant 8$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$2(3+c_2-c_3+c_4)\geqslant 6$	$2(-1+c_1-2c_2+c_3)\leqslant -2$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$2(5+c_4)\geqslant 10$	$2(-4+c_1-c_2+c_3-2c_4)\leqslant -8$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2}$	$2(1-c_1+c_4)$?	$2(2+c_1+c_2-c_3)\geqslant 4$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2$	$2(-2-c_1+c_2-c_4)\leqslant -4$	$2(6+c_1-c_2+c_3)\geqslant 12$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$2(4+c_3-c_4)\geqslant 8$	$2(-3-c_1-c_3+2c_4)\leqslant -6$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4$	$2(-2-c_2+c_4)\leqslant -4$	$2(7+c_2)\geqslant 14$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_1}$	$2(2+c_1-c_2+c_4)\geqslant 4$	$2(1-c_1+2c_2-c_3)\geqslant 2$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2w_1}$	$2(-1+c_1-c_4)$?	$2(5-c_1+c_3)\geqslant 10$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$2(4+c_3-c_4)\geqslant 8$	$2(-2+c_1-c_2-c_3+2c_4)\leqslant -4$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$2(5+c_4)\geqslant 10$	$2(-3+c_2-2c_4)\leqslant -6$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$2(-1-c_1+c_2-c_3+c_4)\leqslant -2$	$2(6+c_1-c_2+c_3)\geqslant 12$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3}$	$2(1-c_1+c_4)$?	$2(3+c_1+c_3-2c_4)\geqslant 6$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1}$	$2(c_1-c_3+c_4)$?	$2(5-c_1+c_3)\geqslant 10$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1}$	$2(2+c_1-c_2+c_4)\geqslant 4$	$2(2-c_1+c_2+c_3-2c_4)\geqslant 4$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$2(4+c_3-c_4)\geqslant 8$	$2(-1+c_2-2c_3+2c_4)\leqslant -2$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3}$	$2(-c_1+c_3-c_4)$?	$2(5+c_1-c_3+2c_4)\geqslant 10$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1}$	$2(1+c_1-c_2+c_3-c_4)\geqslant 2$	$2(4-c_1+c_2-c_3+2c_4)\geqslant 8$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1w_2}$	$2(3+c_2-c_3+c_4)\geqslant 6$	$2(1-c_2+2c_3-2c_4)\geqslant 2$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2}$	$2(2+c_2-c_4)\geqslant 4$	$2(3-c_2+2c_4)\geqslant 6$

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for which $w \in W^{P_2}$ the principal series representation $B_2(\mu_w)$ exists. If some condition is added, then it is necessary and sufficient for the existence of $B_2(\mu_w)$: '$*$' means '$c_3 - c_4 = c_1$' and '$**$' means '$c_1 - c_4 = 1$', while '$\nexists$' indicates that for these w the representation $B_2(\mu_w)$ never exists. Tables A3 and A4 give the values of the inner product of the point $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{i_{\mathbb{C}}}}$ of evaluation of Eisenstein series and the only simple root within $\Delta(P_i, A_i)$. Using (16), we can then read off which points $d\chi$ lie inside the closed, positive Weyl chamber defined by the above system.

Tables A5–A8 give the previous data for the standard minimal parabolic $\mathbb{Q}$-subgroup P_0. Owing to a lack of space we divided the set of Kostant representatives W^{P_0} into a 'lower' and an 'upper' half, according to the length of the elements $w \in W^{P_0}$. In Table A8 the Kostant representatives w which *can* give rise to values $\Lambda_w = -w(\lambda+\rho)|_{\mathfrak{a}_{0_{\mathbb{C}}}} \in \overline{C}$ are underlined. No point Λ_w in Table A7 will lie inside $\overline{C}$.

All lists were compiled by a computer program, developed by Jakub Orbán.

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