Epilogue

In this epilogue we describe briefly some results that are closely connected with the theory and tools developed in previous chapters and have been obtained in recent years but, in spite of their importance, could not be fully treated without increasing too much the size of this book.

1 The similarity principle and applications

In this section we will briefly discuss the first-order equation

$$Lu = Au + B\overline{u} \tag{1}$$

where L is a complex vector field in the plane and A and B are bounded, measurable functions. We will also present two applications of equation (1). The first application concerns uniqueness in the Cauchy problem for a class of semilinear equations. The second application will be to the theory of bending of surfaces.

Equation (1) generalizes the classical elliptic equation

$$\frac{\partial u}{\partial \overline{z}} = Au + B\overline{u} \tag{2}$$

which was investigated by numerous researchers (see for example [**Be**], [**CoHi**], [**Re**], and [**V**]). In the literature, solutions of (2) are called pseudo-analytic functions or generalized analytic functions. Such functions share many properties with holomorphic functions of one variable. These properties follow easily from the similarity principle according to which every continuous solution of (2) has the local form

$$u = \exp\{g\}\, h \tag{3}$$

where h is a holomorphic function and g is Hölder continuous. Thus, for example, the zero set of u is the same as that of h. The similarity principle holds for any elliptic vector field L (where the holomorphy of h is replaced by the condition $Lh = 0$) since any such vector field is a multiple of $\frac{\partial}{\partial \overline{z}}$ in appropriate coordinates. In [**Me2**] Meziani explored the validity of the similarity principle for the following three classes of vector fields:

$$L_k = \frac{\partial}{\partial y} - iy^{2k}\frac{\partial}{\partial x}, \quad K_n = \frac{\partial}{\partial y} - ix^n\frac{\partial}{\partial x}, \quad M = \frac{\partial}{\partial y} - iy\frac{\partial}{\partial x}$$

where k and n are non-negative integers. It was proved in [**Me2**] that the similarity principle is valid for the L_k and K_n (under some vanishing assumption on $B(x, y)$ on the characteristic sets of the vector fields) in the following sense: if w is a continuous solution of

$$Lw = Aw + B\overline{w}$$

where $L \in \{L_k, K_n\}$, then w has the local form $w = \exp\{g\}\, h$ where $Lh = 0$ and g is Hölder continuous. It was also shown in [**Me2**] that this principle does not hold for M. The vector fields L_k and K_n are locally solvable while M is not. With this observation as a point of departure, it was shown in [**BHS**] and [**HdaS**] that a weaker version of the similarity principle is valid for all locally solvable vector fields L. In this weaker version, the functions g and h in the representation $w = \exp\{g\}\, h$ may no longer be continuous. However, this representation was still good enough to yield the uniqueness result mentioned below.

1.1 Application to uniqueness in the Cauchy problem

Let the vector field

$$L = \frac{\partial}{\partial t} + i\sum_{k=1}^{n} b_k(x, t)\frac{\partial}{\partial x_k}$$

satisfy condition $\mathcal{P}$ in some neighborhood $\Omega = \Omega_1 \times (-T, T)$ of the origin in $\mathbb{R}^{n+1}$. Here each b_k is real-valued, of class C^{1+r}, $0 < r < 1$. Let $f(x, t, \zeta, \overline{\zeta})$ be a bounded measurable complex-valued function defined for $(x, t) \in \Omega$, $\zeta \in \mathbb{C}$ satisfying the Lipschitz condition in ζ

$$|f(x, t, \zeta, \overline{\zeta}) - f(x, t, \zeta', \overline{\zeta'})| \leq K\,|\zeta - \zeta'|.$$

If L and f are as above, the following result on uniqueness in the Cauchy problem was proved in [**HdaS**] (see also [**BHS**]):

THEOREM 1.1. *Suppose $u(x, t)$, $w(x, t) \in L^p(\Omega)$, $p \geq 2$, satisfy $Lu = f(x, u, \overline{u})$, $Lw = f(x, w, \overline{w})$, and $u(x, 0) = w(x, 0)$. Then $u \equiv w$ in a neighborhood of the origin.*

If the coefficients of L are smooth, Theorem 1.1 was proved in [**BHS**] under the weaker assumption that u and w belong to L^p, $p > 1$. These results were proved by applying the similarity principle to the difference $v = u - w$ which in view of the assumptions satisfies an equation of the form $Lv = Av + B\overline{v}$ with A and B bounded. The fact that L satisfies condition $\mathcal{P}$ is then used to reduce matters to a planar situation.

1.2 Application to infinitesimal bendings of surfaces

In a series of papers (see [**Me3**], [**Me4**], and the references therein) Meziani has demonstrated an intimate link between the study of the equation

$$Lu = Au + B\overline{u}$$

(L a planar vector field) and the study of infinitesimal deformations of surfaces with non-negative curvature. Here we will summarize some of the results in [**Me4**] to indicate this link.

Let S be a surface of class C^l, $l > 2$, embedded in $\mathbb{R}^3$ and given by parametric equations as

$$S = \{R(s,t) = (x(s,t), y(s,t), z(s,t)) \in \mathbb{R}^3, \quad (s,t) \in D \subset \mathbb{R}^2\} \tag{4}$$

with D an open subset of $\mathbb{R}^2$. An infinitesimal bending of S is a deformation

$$S_\epsilon = \{R_\epsilon(s,t) = R(s,t) + \epsilon U(s,t),\ (s,t) \in D\}, \quad -\delta < \epsilon < \delta \tag{5}$$

for some $\delta > 0$ and

$$U(s,t) = (\xi(s,t), \eta(s,t), \zeta(s,t)) \tag{6}$$

satisfying

$$\mathrm{d}R(s,t) \cdot \mathrm{d}U(s,t) = 0 \quad \forall (s,t) \in D. \tag{7}$$

This means that the first fundamental forms of S and S_ϵ satisfy

$$\mathrm{d}R_\epsilon^2 = \mathrm{d}R^2 + O(\epsilon^2).$$

Note that equation (7) is equivalent to the system of three equations

$$R_s \cdot U_s = 0, \quad R_t \cdot U_t = 0, \quad R_s \cdot U_t + R_t \cdot U_s = 0. \tag{8}$$

Recall that the coefficients of the first fundamental form of S are

$$E = R_s \cdot R_s, \qquad F = R_s \cdot R_t, \quad G = R_t \cdot R_t \tag{9}$$

and those of the second fundamental form are

$$e = R_{ss} \cdot N, \qquad f = R_{st} \cdot N, \quad g = R_{tt} \cdot N, \tag{10}$$

where

$$N = \frac{R_s \times R_t}{|R_s \times R_t|}$$

is the unit normal to S. The Gaussian curvature of S is

$$K = \frac{eg - f^2}{EG - F^2}.$$

We will assume that the curvature $K \geq 0$ everywhere on S. The (complex) asymptotic directions of S are given by the quadratic equation

$$\lambda^2 + 2f\lambda + eg = 0.$$

https://doi.org/10.1017/CBO9780511543067.010 Published online by Cambridge University Press

That is,

$$\lambda = -f + i\sqrt{eg - f^2}.$$

Let L be a vector field of asymptotic direction:

$$L = a(s,t)\left(g(s,t)\frac{\partial}{\partial s} + \lambda(s,t)\frac{\partial}{\partial t}\right), \tag{11}$$

where a is any function defined in D. Note that since $K \geq 0$, if $a \neq 0$, then L is an elliptic vector field that degenerates along the set where the curvature $K = 0$.

Let w be the $\mathbb{C}$-valued function defined by

$$w = LR \cdot U \tag{12}$$

where U is as given in (6). In [**Me4**], the following theorem was proved.

THEOREM 1.2. *With w as in (12) and L as in (11), if $U(s,t)$ is a field of infinitesimal bending for the surface S, then the function w satisfies the equation*

$$CLw = Aw + B\overline{w}$$

where A, B, and C are invariants of the surface S.

1.3 Application to uniqueness in the Cauchy problem in elliptic structures

Let $(\mathcal{M}, \mathcal{V})$ define an elliptic structure. If $u \in L^1_{\text{loc}}(\mathcal{M})$ we shall say that u is an *approximate solution* for the structure $\mathcal{V}$ if for any smooth section L of $\mathcal{V}$, Lu has coefficients belonging to $L^1_{\text{loc}}(\mathcal{M})$ and given any point $p \in \mathcal{M}$, there is an open neighborhood U of p and a constant $M > 0$ such that

$$|Lu| \leq M|u| \quad \text{a.e. in } U.$$

In [**Cor2**] the author established a similarity principle for approximate solutions in the following sense: every approximate solution which belongs to $L^p_{\text{loc}}(\mathcal{M})$ with $p > N = \dim \mathcal{M}$ can locally be written as $u = \exp\{S\}h$, where S is Hölder continuous and h is a solution.

This similarity principle was then used to show that every approximate solution that vanishes on a maximally real submanifold $\mathcal{X}$ vanishes identically in a neighborhood of $\mathcal{X}$.

2 Mizohata structures

The vector field in $\mathbb{R}^2$, where the coordinates are denoted (x,t), given by

$$M = \frac{\partial}{\partial t} - it\frac{\partial}{\partial x} \tag{13}$$

is called the (standard) Mizohata vector field (or operator) after the work of S. Mizohata ([**M**]) who studied the analytic hypoellipticity of a class of related operators of which M is the simplest example. A globally defined first integral of M is the function $Z(x,t) = x + it^2/2$. Notice that $t \mapsto t^2$ fails to be monotone in any neighborhood of a point $(x_0, 0)$, i.e., condition $(\mathcal{P})$ in not satisfied at any point of the x-axis and, as discussed in Chapter IV, fails to be locally solvable at those points. Thus, it is the simplest example of a nonlocally solvable operator and, in fact, its lack of local solvability at points of the x-axis can be proved by ad hoc elementary arguments, as shown by L. Nirenberg ([**N1**]). Off the x-axis, M is elliptic. In his Lectures Notes, Nirenberg constructed a perturbation of the Mizohata operator

$$L = \frac{\partial}{\partial t} - it(1+\rho(x,t))\frac{\partial}{\partial x} \tag{14}$$

with $\rho(x,t)$ real-valued and vanishing to infinite order at $t = 0$, which is not locally integrable in any neighborhood of the origin. As a matter of fact, any smooth function u that satisfies the homogeneous equation $Lu = 0$ in a connected open set U that contains the origin must be constant. In spite of the fact that the perturbed vector fields L and M behave differently with respect to local integrability, they have important geometric features in common. We have

(1) M and its conjugate $\overline{M}$ are linearly dependent precisely on the x-axis;
(2) M and $[M, \overline{M}]$ are linearly independent whenever M and $\overline{M}$ are linearly dependent.

These properties are shared by L in a neighborhood of the origin.

Definition 2.1. *A vector field L defined on a connected 2-manifold Ω is called a Mizohata vector field if for a nonempty subset $\Sigma \subset \Omega$ the following holds:*

(1) *L and $\overline{L}$ are linearly dependent precisely on Σ;*
(2) *L and $[L, \overline{L}]$ are linearly independent on Σ.*

We also say that a Mizohata vector field L is of standard type at $p \in \Sigma$ if there exist local coordinates (x,t) in a neighborhood of p in terms of which Σ is given by $\{t = 0\}$ and $\mathcal{L}$ has the form (13)*. A Mizohata structure $\mathcal{L}$ on Ω is a structure which is locally generated in the neighborhood of every point by a Mizohata vector field.*

Notice that (1) means that Σ is the image of the characteristic set $\{(p,\xi) \in T^*(\Omega) : \ell(p,\xi) = 0\}$, ℓ being the symbol of L, under the canonical projection $\Pi : T^*(\Omega) \longrightarrow \Omega$. With this terminology, the vector field (13) is a Mizohata vector field of standard type and (14) is also a Mizohata vector field but not of standard type. Indeed, (14) cannot be of standard type because it is not locally integrable.

Notice that a Mizohata vector field is elliptic on $\Omega \backslash \Sigma$, which is a relatively small set, since an application of the implicit function theorem shows that Σ is an embedded curve. The following question was considered by Treves [**T7**]: when is a Mizohata vector field L of standard type at a given point? Of course, since this is a local question, it is enough to study the case when L is defined in a neighborhood of the origin in $\mathbb{R}^2$. He showed that local coordinates can be found so that L becomes of the form (14) with $\rho(x,t)$ real-valued and vanishing to infinite order at $t = 0$, in other words, every Mizohata vector field has this form locally and it will be of standard type if we are able to take $\rho \equiv 0$. Furthermore, L is of standard type at the origin if and only if it is locally integrable. Then Sjöstrand ([**Sj2**]) took a closer look into the nonlocally integrable case. To describe his results, let us consider the problem of finding a smooth function $Z^+(x,t)$ satisfying $\mathrm{d}Z(0,0) \neq 0$ and $LZ^+ = 0$ on $U^+ = U \cap \{t \geq 0\}$, where U is a small disk centered at the origin. By the proof of Lemma I.13.4, to find Z^+ it is enough to find a smooth function u that satisfies $Lu = t\rho_x$ on U^+. This is, in fact, possible because L satisfies condition $(\mathcal{P})$ for $t > 0$ ([**BH6**]). Similarly, shrinking U if necessary, we can also find a smooth function $Z^-(x,t)$ satisfying $\mathrm{d}Z^-(0,0) \neq 0$ and $LZ^- = 0$ on $U^- = U \cap \{t \leq 0\}$. We can always choose Z^+ and Z^- satisfying $Z^\pm(0,0) = 0$, $\Im Z^\pm_x(0,0) = 0$, and $\Re Z^\pm_x(0,0) > 0$ and we will do so. If we are so lucky that $Z^+(x,0) = Z^-(x,0)$, $(x,0) \in U$, we may patch Z^+ and Z^- to get a single continuous solution Z of $LZ = 0$ on U and it is easy to see using the equation that Z is actually smooth. So the obstruction to the local integrability of L is related to the difficulty of finding a pair (Z^+, Z^-) such that $LZ^\pm = 0$ on $U^\pm$ and $Z^+ = Z^-$ on $U^+ \cap U^-$. Given such a pair, it can be shown that the range of $Z^\pm$ lies on one side of the smooth curve $\{Z^\pm(x,0)\}$ (in fact, above the curve because $\Re Z^\pm_x(0,0) > 0$), so let $H^\pm(z)$ be a smooth function defined on the range of $Z^\pm$ and holomorphic in its interior with $H^\pm(0) = 0$, $(H^\pm)'(0) = \Re(H^\pm)'(0) > 0$. Then, $\tilde{Z}^\pm = H^\pm \circ Z^\pm$ satisfies $\mathrm{d}\tilde{Z}^\pm(0,0) \neq 0$ and $L\tilde{Z}^\pm = 0$ on $U^\pm$. By the Riemann mapping theorem we may find H^+ and H^- so that the range of $\tilde{Z}^+$ and $\tilde{Z}^-$ is the upper half-plane. In other words, we may restrict ourselves to consider pairs (Z^+, Z^-) such that $Z^\pm(U^\pm) = \{\Im z \geq 0\}$ and $Z^\pm(U^+ \cap U^-) = \mathbb{R}$. Given such a pair and a smooth function H defined on $\Im z \geq 0$, holomorphic on $\Im z > 0$, real for z real and satisfying $H(0) = 0$, $H'(0) > 0$, a new pair $(Z^+, \tilde{Z}^-) = (Z^+, H \circ Z^-)$ may be considered and L will be locally integrable if $Z^+(x,0) = \tilde{Z}^-(x,0)$. It turns out that L is locally integrable if and only if there exists a pair (Z^+, Z^-) such that $H(z) = Z^+ \circ (Z^-)^{-1}(z)$ is holomorphic for $\Im z > 0$ and smooth up to $\Im z = 0$. Since $H(z)$ is real for z real, H has, by the reflection principle, an extension to a holomorphic function. By uniqueness, $H(x+iy)$ is determined by its trace $bH(x) = H(x+i0)$ so it is enough to look at the restrictions $bZ^\pm(x) = Z^\pm(x,0)$ and check whether $\kappa \doteq bZ^+ \circ (bZ^-)^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ has a holomorphic extension to a neighborhood. Summing up, to each Mizohata vector field L we have associated an increasing diffeomorphism $\kappa : \mathbb{R} \longrightarrow \mathbb{R}$ such that L is locally integrable if an only if $\kappa = bH$ for some $H \in \mathcal{H}(\mathbb{C})$, i.e., κ has a holomorphic extension. More generally, we may consider the following question: given two Mizohata vector fields L_1, L_2, when are they equivalent in the sense that one can be locally transformed into a multiple of the other by a change of variables? The answer, due to Sjöstrand, can be stated as follows. Consider the associated diffeomorphisms $\kappa_1 = bZ_1^+ \circ (bZ_1^-)^{-1}$ and $\kappa_2 = bZ_2^+ \circ (bZ_2^-)^{-1}$, then L_1 and L_2 are equivalent if and only if there are holomorphic functions, $H_1(z)$, $H_2(z)$, real and increasing for z real, such that $\kappa_1(H_1(x)) = \kappa_2(H_2(x))$, $x \in \mathbb{R}$.

The local questions of standardness and equivalence for Mizohata vector fields have their global counterpart. For instance, it was established in [**BCH**] that a locally standard Mizohata planar vector field has a first integral globally defined in a tubular neighborhood of the characteristic set Σ. The standardness of a particular class of Mizohata structures on the sphere S^2 was proved in [**Ho4**] and Jacobowitz ([**J2**]) studied Mizohata structures on compact surfaces Ω, in particular, he proved that the existence of a first integral is equivalent to the fact that the genus is even. In the case of the sphere, he gave a classification of Mizohata structures in the spirit of Sjöstrand's result, proving in particular the existence of nonstandard Mizohata structures. These topics were developed further by Meziani in [**Me5**] and [**Me6**].

2.1 Mizohata structures in higher-dimensional manifolds

The questions discussed in the previous section admit natural generalization to higher dimension. A formally integrable structure $\mathcal{V}$ defined on a manifold Ω of dimension N is said to be a Mizohata structure if the following holds:

(1) $\mathcal{V}$ has rank $n = N - 1$;
(2) the characteristic set $T^0 = T' \cap T^*(\Omega)$ is not empty;
(3) the Levi form is nondegenerate at every point of $T^0 \setminus \{0\}$.

Example 2.2. Denote by $t = (t_1, \ldots, t_n)$ the variables in $\mathbb{R}^n$, $n \geq 1$, and write $t = (t', t'')$, $t' = (t_1, \ldots, t_\nu)$, $t'' = (t_{\nu+1}, \ldots, t_n)$, for some $1 \leq \nu \leq n$. Consider the function $Z(x, t) = x + i(|t'|^2 - |t''|^2)/2$ defined on $\mathbb{R}_x \times \mathbb{R}_t$ and the locally integrable structure $\mathcal{V}$ determined by imposing that T' is spanned by $\mathrm{d}Z(x, t)$. Then, $\mathcal{V}$ is spanned by the vector fields

$$M_j = \frac{\partial}{\partial t_j} - i\varepsilon_j t_j \frac{\partial}{\partial x}, \quad j = 1, \ldots, n, \tag{15}$$

with $\varepsilon_j = 1$ for $1 \leq j \leq \nu$ and $\varepsilon_j = -1$ for $\nu + 1 \leq j \leq n$. Then $\mathcal{V}$ is a Mizohata structure such that at every characteristic point its Levi form has ν eigenvalues with one sign and $n - \nu$ eigenvalues with the opposite sign and when this happens we say that $\mathcal{V}$ has type $\{\nu, n - \nu\}$. Thus, we have examples of Mizohata structures with all possible types. Notice that the projection of the characteristic set is the curve $\Sigma = \{t = 0\}$, i.e., the x-axis. A Mizohata structure with type $\{\nu, n - \nu\}$ is standard if for any point lying in the projection of the characteristic set we can choose local coordinates (x, t) so that the vector fields (15) span $\mathcal{V}$ in a neighborhood of that point. Let $\mathcal{V}$ be a Mizohata structure with type $\{\nu, n - \nu\}$. By analogy with the case $n = 1$, it turns out that for any $n \in \mathbb{N}$ and $1 \leq \nu \leq n$ ([**T5**]) it is possible to find local coordinates in a neighborhood U of a generic point p in the projection of Σ such that $x(p) = t(p) = 0$ and $\mathcal{V}$ is generated over U by the vector fields

$$L_j = \frac{\partial}{\partial t_j} - i\varepsilon_j t_j (1 + \rho_j(x, t)) \frac{\partial}{\partial x}, \quad j = 1, \ldots, n, \tag{16}$$

where the functions $\rho_j(x, t)$, $j = 1, \ldots, n$, vanish to infinite order at $t = 0$. In other words, every Mizohata structure has at a given point a contact of infinite order with a standard Mizohata structure of the same type. In particular, if we can take all the functions ρ_j identically zero $\mathcal{V}$ will have a first integral in U and will be standard in U. Conversely, if $\mathcal{V}$ has a first integral it is possible to choose the coordinates so that $\mathcal{V}$ is generated by the vector fields (15).

For the case $\nu = 1$, i.e., if the type is $\{1, n - 1\}$, Treves showed the existence of functions $\rho_j(x, t)$ vanishing to infinite order at $t = 0$ such that the structure $\mathcal{V}$ spanned by (16) is formally integrable (i.e., $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$) and not locally integrable. On the other hand, Meziani proved in [**Me7**] that Mizohata structures of all other types $\{\nu, n - \nu\} \neq \{1, n - 1\}$ are always locally integrable. His proof is delicate and beyond the scope of this book: he first constructs first integrals on the connected components of $\{(x, t', t'') \in \mathbb{R}_x \times \mathbb{R}_t : |t'|^2 \neq |t''|^2\}$ which can be 2 (if $n > 2$ and $\nu < n-2$), 3 (if $n > 2$ and $\nu = n-2$), or 4 (if $n = 2$ and $\nu = 0$). When the components are 2 or 4, these first integrals can be patched together to yield a globally defined first integral of class C^1 which, by the hypoellipticity of the structure, is in fact smooth. The possibility of patching together these partially defined first integrals depends on a careful analysis of the holonomy of a certain foliation with leaves of dimension $n-1$ defined by the structure. For the case of type $\{1, n-1\}$ he gives a classification of Mizohata structures analogous to Sjöstrand's result for a single vector field. The local integrability for Mizohata structures of type $\{0, n\}$, $n \geq 3$, was first proved in [**HMa2**], by techniques akin to those used in the proof of Kuranishi's embedding theorem for CR structures ([**Ku1**], [**Ku2**], [**Ak**], [**W2**], [**W3**]), which also fall beyond the scope of this book. The restriction $n \geq 3$ comes from a technical fact: Kuranishi's approach depends on the existence of certain so-called homotopy formulas that do not exist when $n = 2$ ([**HMa3**]). However, the local integrability of Mizohata structures of type $\{0, n\}$ in $\mathbb{R}^{n+1}$, $n \geq 2$, can be proved by elementary methods. Consider a system of n commuting vector fields

$$L_j = \frac{\partial}{\partial t_j} - it_j(1 + \rho_j(x, t))\frac{\partial}{\partial x}, \quad j = 1, \dots, n. \tag{17}$$

Here a generic point is described by coordinates $(x, t_1, \dots, t_n)$ and the smooth functions $\rho_j(x, t)$ vanish to infinite order at $\Gamma = \{t = 0\} = \mathbb{R}_x \times \{0\}$. We regard the L_j's as perturbations of the Mizohata vector fields

$$M_j = \frac{\partial}{\partial t_j} - it_j\frac{\partial}{\partial x}, \quad j = 1, \dots, n.$$

A simple computation using polar coordinates, $t = r\theta$, $r > 0$, $\theta \in S^{n-1}$ shows that the standard Mizohata structure spanned by the M_j's is also spanned on $\mathbb{R}^{n+1} \backslash \Gamma$ by

$$\begin{cases} M = \dfrac{\partial}{\partial r} - ir\dfrac{\partial}{\partial x} \\[2ex] \partial_k = \dfrac{\partial}{\partial \theta_k} \qquad \text{k=1,...,n } -1, \end{cases}$$

with $(\theta_1, \dots, \theta_{n-1})$ angular variables in S^{n-1}. Then, the change of variables $s = r^2/2$ (x and θ are kept unchanged) takes M into a multiple of the Cauchy–Riemann operator

$$\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial s}\right), \quad z = x + is, \ s > 0,$$

and does not change ∂_k. If we perform the same operations on the perturbed system (17) we may find a set of generators of $\mathcal{V}$ in the variables $(x, s, \theta) \in \mathbb{R}_x \times \mathbb{R}_s^+ \times \mathcal{S}^{n-1}$ of the form

$$\begin{cases} \tilde{L}_1 = \dfrac{\partial}{\partial \bar{z}} + \sigma_1 \dfrac{\partial}{\partial z} \\ \tilde{L}_k = \dfrac{\partial}{\partial \theta_{k-1}} + \sigma_k \dfrac{\partial}{\partial z} \quad k{=}2,\ldots,n, \end{cases} \tag{18}$$

with smooth coefficients $\sigma_j(x, s, \theta)$, $j = 1, \ldots, n$, that converge to zero when $s \searrow 0$ together with their derivatives of any order. Thus, we may smoothly extend the coefficients σ_j as zero for $\Im z = s \leq 0$ and obtain an elliptic system defined on $\mathbb{C} \times S^{n-1} \simeq \mathbb{R}_x \times \mathbb{R}_s \times S^{n-1}$ which for $\Im z < 0$ has the first integral $z = x + is$. The process that produced an elliptic system starting from a nonelliptic one was obtained by a combination of singular changes of variables (polar coordinates that are singular at the origin of $\mathbb{R}^n_t$ and $s = r^2/2$ which is singular at $r = 0$) and blows up the line $\mathbb{R}_x \times \{t = 0\}$ to the n-manifold $\mathbb{R}_x \times S^{n-1}$. Although we know from Theorem I.12.1 that elliptic structures are locally integrable, applying that result to (18) would only give us a first integral defined in a neighborhood of a point $s = 0$, $x = 0$, $\theta = \theta_0 \in S^{n-1}$ while only a first integral defined for all $\theta \in S^{n-1}$ can give us a first integral defined in a neighborhood of the origin of the original variables (x, t). Let's consider first the case $n = 2$, that is the system of two vector fields

$$\begin{cases} \tilde{L}_1 = \dfrac{\partial}{\partial \bar{z}} + \sigma_1 \dfrac{\partial}{\partial z}, \quad z = x + is \in \mathbb{C}, \\ \tilde{L}_2 = \dfrac{\partial}{\partial \theta} + \sigma_2 \dfrac{\partial}{\partial z}, \quad 0 \leq \theta \leq 2\pi, \end{cases} \tag{$*$}$$

defined in $\mathbb{C} \times S^1$, where the $\sigma_j(x, s, \theta)$, $j = 1, 2$, are C^∞ functions, 2π-periodic in θ, and vanish for $s = \Im z \leq 0$. Choose a smooth function $\beta = \beta(x, s, \theta)$ such that $X = \tilde{L}_2 + \beta \tilde{L}_1$ is a real vector. It is easy to check that this is possible if $|\Re \sigma_1| < 1$ (in particular for (x, s) close to the origin). Thus, X is a real generator of the structure $\tilde{\mathcal{V}}_2$ spanned by $\tilde{L}_1$ and $\tilde{L}_2$ for $|x| < 1$, $|s| < \varepsilon$ and $0 \leq \theta \leq 2\pi$. It is clear that $X = \partial/\partial\theta$ for $s \leq 0$, and that the orbits of X stemming from points $(x_0, s_0, 0)$, $s_0 \leq 0$, are the closed circles $\tau \to (x_0, s_0, \tau)$, $0 \leq \tau \leq 2\pi$. Notice also that the component of X along $\partial/\partial\theta$ is 1, i.e.,

$$X = \frac{\partial}{\partial \theta} + \rho_1 \frac{\partial}{\partial x} + \rho_2 \frac{\partial}{\partial s}$$

for some smooth functions ρ_1 and ρ_2 which are 2π-periodic in θ and vanish for $s \leq 0$. Since the commutator $[X, \tilde{L}_1] \in \tilde{\mathcal{V}}_2$ it must be a linear combination of $\tilde{L}_1$ and $\tilde{L}_2$; on the other hand, it does not contain derivations with respect to θ so it has to be proportional to $\tilde{L}_1$. This shows that there exists a smooth function $\lambda = \lambda(x, s, \theta)$ such that

$$[X, \tilde{L}_1] = \lambda \tilde{L}_1. \tag{19}$$

Now pick once and for all a local solution $W(x, s)$ of

$$\begin{aligned} &\tilde{L}_{10} W \doteq \frac{\partial W}{\partial \bar{z}} + \sigma_1(x, s, 0) \frac{\partial W}{\partial z} = 0, \\ &W_x(0,0) \neq 0. \end{aligned} \tag{20}$$

We may assume that in a neighborhood of the origin any other solution $W^\flat(x,s)$ of $\tilde{L}_{10}W^\flat=0$ is a holomorphic function of W, in fact, W is a local diffeomorphism that takes $\tilde{L}_{10}$ into a multiple of the Cauchy–Riemann operator. Let γ denote the closed orbit of X stemming from $(0,0,0)$, given by $\tau\to(0,0,\tau)$, $0\le\tau\le 2\pi$. We now solve the Cauchy problem

$$\begin{aligned} XV&=0, \\ V(x,s,0)&=W(x,s) \end{aligned} \tag{21}$$

in a tubular neighborhood of γ made up of orbits of X. Let us set $U=\tilde{L}_1V$ and observe that it follows from (19), (20) and (21) that U satisfies the Cauchy problem

$$\begin{aligned} XU-\lambda U&=0, \\ U(x,s,0)&=0 \end{aligned}$$

so it must vanish identically in a tubular neighborhood of γ. This proves that $\mathrm{d}V$ is orthogonal to $\tilde{\mathcal{V}}_2$ because $\tilde{L}_1$ and X form a basis of $\tilde{\mathcal{V}}_2$. Differentiating (21) with respect to x and setting $s=x=0$ it is easy to conclude that $V_{x\theta}(0,0,\theta)=0$, $0\le\theta\le 2\pi$, so $V_x(0,0,\theta)=W_x(0,0)$ is constant, in particular it does not vanish in a neighborhood of γ. This already implies that $\mathrm{d}V$ is a generator of the orthogonal of $\tilde{\mathcal{V}}_2$, but we do not know yet that V is 2π-periodic in θ. Since the coefficients of $\tilde{L}_1$ are 2π-periodic we have that $V(x,s,2\pi)$ satisfies $\tilde{L}_{10}V(x,s,2\pi)=0$ and therefore, there exists a holomorphic function G such that $V(x,s,2\pi)=G\circ V(x,s,0)=G\circ W(x,s)$ hold for (x,s) in a neighborhood of the origin. But $X=\partial/\partial\theta$ for $s\le 0$, which implies that $V(x,s,0)=V(x,s,2\pi)$ for $s\ge 0$, and it turns out that $G(z)=z$. Thus, $V(x,s,0)=V(x,s,2\pi)$ in a neighborhood of $x=s=0$. This proves that V is well-defined in $\mathbb{C}\times S^1$ and is a first integral globally defined in $\theta\in S^1$ of the system $(*)$. Furthermore, using $\theta'=\theta$, $x'=\Re V$ and $s'=\Im V$ as local coordinates in a neighborhood of the origin we see that $\partial_{x'}+i\partial_{s'}$ and $\partial_{\theta'}$ generate the same structure as $\tilde{L}_1$, $\tilde{L}_2$.

In the case of the system (18) with $n>2$ the arguments above can be applied to the first two equations keeping the variables $\theta_2,\dots,\theta_{n-1}$ as parameters. Thus, after a change of variables $(x,s)\mapsto(x',s')$, we may now assume that $\sigma_1\equiv 0$ in (18). But then we have $\sigma_k\equiv 0$ for all values of k. Indeed, since $\tilde{L}_1$ commutes with $\tilde{L}_k$, it follows that σ_k, $k\ge 2$, depends holomorphically on z and then has to be identically zero because it vanishes for $\Im z\le 0$. Thus, all the σ_k are identically zero in the new variables and $z=x+is$ is a first integral of the system. Returning to the original variables (x,s,θ) this shows the existence of a solution $V(x,s,\theta)$ of system (18) for $|x|$ and $|s|$ small and $\theta\in S^{n-1}$ that satisfies $V_x(0,0,0)=V_x(0,0,\theta)\ne 0$. Finally, the function $(x,t)\mapsto V(x,|t|^2)/2,\theta(t))$ is smooth in a neighborhood of the origin and its differential spans $\mathcal{V}$.

3 Hypoanalytic structures

Let Ω be a smooth manifold of dimension N. By a *hypoanalytic structure on* Ω (*cf.* [**T5**]) we mean a collection of pairs $\mathcal{A}=\{(U_\ell,Z_\ell)\}$, with U_ℓ an open subset of Ω and

https://doi.org/10.1017/CBO9780511543067.010 Published online by Cambridge University Press

$Z_\ell = (Z_{\ell,1}, \dots, Z_{\ell,m}) : U_\ell \to \mathbb{C}^m$ a smooth map, where $1 \le m \le N$ is independent of ℓ, such that the following conditions are satisfied:

- $(\mathrm{H})_1$ $\{U_\ell\}$ is an open covering of Ω;
- $(\mathrm{H})_2$ $\mathrm{d}Z_{\ell,1}, \dots, \mathrm{d}Z_{\ell,m}$ are $\mathbb{C}$-linearly independent at each point of U_ℓ;
- $(\mathrm{H})_3$ if $\ell \ne \ell'$ and if $p \in U_\ell \cap U_{\ell'}$ there exists a biholomorphism $F^\ell_{\ell',p}$ of an open neighborhood of $Z_\ell(p)$ in $\mathbb{C}^m$ onto one of $Z_{\ell'}(p)$ such that $Z_{\ell'} = F^\ell_{\ell',p} \circ Z_\ell$ in a neighborhood of p in $U_\ell \cap U_{\ell'}$.

A complex-valued function f defined on an open subset U of Ω is called *hypoanalytic* if in a neighborhood of any point p of U we can write $f = h_\ell \circ Z_\ell$, where ℓ is such that $p \in U_\ell$ and h_ℓ is a holomorphic function in a neighborhood of $Z_\ell(p)$ in $\mathbb{C}^m$. By a *hypoanalytic chart* we shall mean a pair (U, Z) where $U \subset X$ is open, $Z = (Z_1, \dots, Z_m) : U \to \mathbb{C}^m$ has hypoanalytic components and $\mathrm{d}Z_1 \wedge \dots \wedge \mathrm{d}Z_m \ne 0$ in U.

If $\mathcal{A} = \{(U_\ell, Z_\ell)\}$ is a hypoanalytic structure on Ω and if $\Omega_\bullet \subset \Omega$ is open then we can induce a hypoanalytic structure $\mathcal{A}_{\Omega_\bullet}$ by the rule

$$\mathcal{A}_{\Omega_\bullet} = \{(U_\ell \cap \Omega_\bullet, Z_\ell|_{U_\ell \cap \Omega_\bullet})\}.$$

To each hypoanalytic structure $\mathcal{A} = \{(U_\ell, Z_\ell)\}$ on Ω we can canonically associate a locally integrable structure $\mathcal{V}$ on Ω in the following way: for each ℓ its orthogonal on U_ℓ is defined by

$$T'|_{U_\ell} = \mathrm{span}\,\{\mathrm{d}Z_{\ell,1}, \dots, \mathrm{d}Z_{\ell,m}\}.$$

By properties $(\mathrm{H})_1$, $(\mathrm{H})_2$, and $(\mathrm{H})_3$ it follows that T' is indeed a subbundle of $\mathbb{C}T^*\Omega$ of rank m.

Notice however that two different hypoanalytic structures can define the same locally integrable structure. Indeed, to give an example it suffices to take $\Omega = \mathbb{R}$ and consider the hypoanalytic structure $\{(\mathbb{R}, \mathrm{Id})\}$, where $\mathrm{Id}(x) = x$, and the hypoanalytic structure $\{(\mathbb{R}, f)\}$, where $f : \mathbb{R} \to \mathbb{R}$ is smooth but *not* real-analytic and $f' \ne 0$ at each point.

By a *hypoanalytic manifold* we shall mean a pair $(\Omega, \mathcal{A})$, where Ω is a smooth manifold and $\mathcal{A}$ is a hypoanalytic structure on Ω. Notice that if $(\Omega, \mathcal{A})$ is a hypoanalytic manifold, endowed with the hypoanalytic structure $\mathcal{A} = \{(U_\ell, Z_\ell)\}$, if Ω' is another smooth manifold and if $f : \Omega' \to \Omega$ is a smooth submersion, then we can pull back the hypoanalytic structure $\mathcal{A}$ to a hypoanalytic structure $f^*\mathcal{A}$ on Ω' by defining

$$f^*\mathcal{A} = \{(f^{-1}(U_\ell), Z_\ell \circ f)\}.$$

Finally we shall say that two hypoanalytic manifolds $(\Omega' \mathcal{A}')$ and $(\Omega, \mathcal{A})$ are equivalent if there is a smooth diffeomorphism $f : \Omega' \to \Omega$ such that $f^*\mathcal{A} = \mathcal{A}'$.

4 The local model for a hypoanalytic manifold

Let $N \ge 1$ and write $N = m + n$. The variable in $\mathbb{C}^N = \mathbb{C}^m \times \mathbb{C}^n$ will be denoted by (z, z') with $z = (z_1, \dots, z_m)$, $z' = (z'_1, \dots, z'_n)$. In this space we consider the hypoana-

https://doi.org/10.1017/CBO9780511543067.010 Published online by Cambridge University Press

lytic structure defined by $\mathcal{A}^\bullet = \{(\mathbb{C}^N, (z_1, \dots, z_m))\}$. The corresponding hypoanalytic functions are just the holomorphic functions of z that are locally independent of z'.

Let Ω and $\{(U_\ell, Z_\ell)\}$ be as in Section 3. An arbitrary point p of Ω has an open neighborhood U_p in which there are defined hypoanalytic functions $Z_1, \dots, Z_m$ and a complementary number of C^∞ functions $Z'_1, \dots, Z'_n$, with $m + n = N$, such that

$$\mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m \wedge \mathrm{d}Z'_1 \wedge \cdots \wedge \mathrm{d}Z'_n \neq 0 \text{ at } p.$$

Possibly after contracting U_p about p we may assume that

$$\lambda : (Z, Z') \doteq (Z_1, \dots, Z_m, Z'_1, \dots, Z'_n)$$

is a smooth diffeomorphism of U_p onto a smooth, *maximally real* submanifold Σ_p of $\mathbb{C}^m \times \mathbb{C}^n$. We refer to the triplet (U_p, Z, Z') as an *extended hypoanalytic chart*.

The hypoanalytic $\mathcal{A}^\bullet$ induces a hypoanalytic structure $\mathcal{A}^\#$ on Σ_p, simply by setting

$$\mathcal{A}^\# = \{(\Sigma_p, (z_1|_{\Sigma_p}, \dots, z_m|_{\Sigma_p}))\},$$

and it is easily seen that

$$\mathcal{A}_{U_p} = \lambda^* \mathcal{A}^\#. \tag{22}$$

This remark is crucial for what follows.

5 The sheaf of hyperfunction solutions on a hypoanalytic manifold

The sheaf of hyperfunctions can be introduced on any real-analytic manifold. This is a fundamental result, due to M. Sato ([**Sa**]). It is also possible to extend such a concept to hypoanalytic manifolds where no real-analyticity is required, but in order to obtain an invariant meaning, we must restrict ourselves to the hyperfunctions that are solutions in some sense. We give now a brief description of this theory.

It is a consequence of a result due to Harvey ([**Ha**]) that over any maximally real submanifold $\mathcal{M}$ of $\mathbb{C}^N$ it is also possible to define the sheaf of hyperfunctions $\mathcal{B}_\mathcal{M}$. Moreover, the following description is valid: given $q \in \mathcal{M}$ there is an open neighborhood V of q in $\mathcal{M}$ such that the following is true: if $W \subset\subset V$ is open then

$$\mathcal{B}_\mathcal{M}(W) = \mathcal{O}'(\overline{W})/\mathcal{O}'(\partial W). \tag{23}$$

Here the boundary of W is taken in $\mathcal{M}$ and for a compact subset K of $\mathbb{C}^N$ we are denoting by $\mathcal{O}'(K)$ the space of analytic functionals of $\mathbb{C}^N$ carried by K.

We return to the discussion of Section 4. We fix $p \in \mathcal{M}$ and Σ_p as described. Since the holomorphic derivatives act on $\mathcal{O}'(K)$ by transposition we can consider the space of hyperfunctions u on Σ_p which satisfy the system

$$\frac{\partial u}{\partial z'_j} = 0, \quad j = 1, \dots, n. \tag{24}$$

The main result presented in the monograph [**CorT2**] states that the sheaf of these hyperfunctions on Σ_p, when pulled back to U_p, gives rise to a *well-defined sheaf* Sol_Ω

on Ω*, which is furthermore a hypoanalytic invariant*. The proof of this fundamental result relies on (22). We call Sol_Ω the sheaf of germs of *hyperfunction solutions* on Ω. This sheaf contains, as a subsheaf, the sheaf of germs of distribution solutions with respect to the associated locally integrable structure $\mathcal{V}$. Moreover, if Ω and the maps Z_ℓ are real-analytic then Sol_Ω equals the sheaf of hyperfunctions on Ω that are annihilated by the (real-analytic) sections of $\mathcal{V}$.

Many of the basic results that were proved in this book remain valid within this more general concept of solution, as for instance the propagation of the support of solutions by the orbits of the underlying structure and the uniqueness in the Cauchy problem ([**CorT2**]). Another important feature is that a certain class of infinite-order operators, which are local in the sense of Sato, act as endomorphisms of Sol_Ω ([**Cor1**]). It can then be proved that every hyperfunction solution can be obtained, locally, as the action of one such operator on a *smooth solution* and then, as a consequence, a version of the approximation formula for hyperfunction solutions can be derived (*cf.* [**Cor1**]).