Bull. Aust. Math. Soc. **85** (2012), 456–462
doi:10.1017/S0004972712000093

A VARIATIONAL McSHANE INTEGRAL CHARACTERISATION OF THE WEAK RADON–NIKODYM PROPERTY

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(Received 23 June 2011)

Abstract

We present a characterisation of Banach spaces possessing the weak Radon–Nikodym property in terms of finitely additive interval functions whose McShane variational measures are absolutely continuous with respect to Lebesgue measure.

2010 *Mathematics subject classification*: primary 28B05; secondary 46G10.

Keywords and phrases: weak Radon–Nikodym property, Banach space, finitely additive interval function, McShane variational measure.

1. Introduction

In [1], Bongiorno *et al.* have shown characterisations of Banach spaces possessing the weak Radon–Nikodym property (WRNP) in terms of finitely additive interval functions. They proved that a Banach space X has the WRNP if and only if, for every X-valued finitely additive interval function φ that has absolutely continuous Henstock variational measure, there is a Henstock–Kurzweil–Pettis integrable function $f : [0, 1] \to X$ such that

$$\varphi(I) = (\mathrm{HKP}) \int_I f \quad \text{for every interval } I \subset [0, 1], \tag{1.1}$$

where (HKP) $\int_I f$ denotes the Henstock–Kurzweil–Pettis integral of f over I; see [1, Definition 2.2].

In this paper, we present a characterisation of Banach spaces possessing the WRNP in terms of finitely additive interval functions whose McShane variational measures are absolutely continuous with respect to Lebesgue measure. We prove that a Banach space X has the WRNP if and only if, for every X-valued finitely additive interval function φ that has absolutely continuous McShane variational measure, there is a weakly McShane integrable function $f : [0, 1] \to X$ such that (1.1) holds true for every interval $I \subset [0, 1]$ (but now the integral is the weak McShane integral).

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https://doi.org/10.1017/S0004972712000093 Published online by Cambridge University Press

Henstock and McShane variational measures have been used extensively for studying the primitives (indefinite integrals) of real functions. See, for example, the papers by Di Piazza [3] and Lee [4] and the book Pfeffer [5] for relations to integration; see also the fundamental general work by Thomson [10].

2. Basic definitions

Throughout this paper, X denotes a real Banach space with its norm $\|\cdot\|$. By X^* we denote the dual to X. Given a functional $x^* \in X^*$ its value on the element $x \in X$ will be denoted by $x^*(x)$.

Let S be the unit interval $[0, 1]$ of the real line equipped with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of S and by $\mathscr{S}$ the family of all nondegenerate closed subintervals of S. The intervals I and J are said to be *nonoverlapping* if $\mathrm{int}(I) \cap \mathrm{int}(J) = \emptyset$, where $\mathrm{int}(I)$ denotes the interior of I.

A mapping $\nu : \mathcal{L} \to X$ is said to be an X-*valued measure* if ν is countable additive in the norm topology of X. An X-valued measure is said to be λ-*continuous* if $\lambda(E) = 0$ implies $\nu(E) = 0$. The *variation* of an X-valued measure ν is denoted by $|\nu|$.

A function $\varphi : \mathscr{S} \to X$ is said to be an *interval function*. An interval function $\varphi : \mathscr{S} \to X$ is said to be *finitely additive* if $\varphi(I \cup J) = \varphi(I) + \varphi(J)$ for all nonoverlapping intervals $I, J \in \mathscr{S}$ with $I \cup J \in \mathscr{S}$. We denote by Φ the family of all finitely additive interval functions $\varphi : \mathscr{S} \to X$

A function $\varphi \in \Phi$ is said to be *strongly absolutely continuous* (or briefly *sAC*) if for every $\varepsilon > 0$ there exists $\eta > 0$ such that, for every finite collection $\{I_i : i = 1, 2, \ldots, n\}$ of nonoverlapping intervals in $\mathscr{S}$,

$$\sum_{i=1}^{n} \lambda(I_i) < \eta \Rightarrow \sum_{i=1}^{n} \|\varphi(I_i)\| < \varepsilon.$$

We denote by $\langle a, b\rangle$ the closed interval $[\min\{a, b\}, \max\{a, b\}]$, $a, b \in \mathbb{R}$. A function $\varphi \in \Phi$ is said to be *differentiable* at $s \in S$, if there exists $x \in X$ such that

$$\lim_{h\to 0} \frac{\varphi(\langle s, s+h\rangle)}{|h|} = x.$$

We write $x = \varphi'(s)$ to denote the *derivative* of φ at s.

We say that a function $\varphi \in \Phi$ is *pseudodifferentiable* on S if there exists a function $\varphi'_p : E \to X$ such that, for every $x^* \in X^*$,

$$\lim_{h\to 0} \frac{x^*\varphi(\langle s, s+h\rangle)}{|h|} = x^*\varphi'_p(s),$$

for almost all $s \in S$. (The exceptional sets depend on x^*.) The function φ'_p is said to be a *pseudoderivative* of φ.

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A pair (I, s) of an interval $I \in \mathscr{S}$ and a point $s \in S$ is said to be the *McShane tagged interval*; s is said to be the *tag* of I. Requiring $s \in I$ for the tag of I we get the concept of a *Henstock–Kurzweil tagged interval*.

A *McShane partition* (or *$\mathcal{M}$-partition*) *π in S* is a finite collection of McShane tagged intervals (I, s) whose corresponding intervals are nonoverlapping. Similarly, a *Henstock–Kurzweil partition* (or *$\mathcal{HK}$-partition*) *π in S* is a finite collection of Henstock–Kurzweil tagged intervals (I, s) whose corresponding intervals are nonoverlapping.

A function $\delta : E \to (0, +\infty)$ is said to be a *gauge* on E, where E is a subset of S. We say that an $\mathcal{M}$-partition π in S ($\mathcal{HK}$-partition π in S) is:

- an *$\mathcal{M}$-partition* of S (*$\mathcal{HK}$-partition* of S) if $\bigcup_{(I,s)\in\pi} I = S$;
- *E-tagged* if, for all $(I, s) \in \pi$, $s \in E$;
- *δ-fine*, if, for every tagged interval $(I, s) \in \pi$, $I \subset (s - \delta(s), s + \delta(s))$.

Definition 2.1. A function $f : S \to X$ is said to be *McShane integrable* on S and $w_S \in X$ is its *McShane integral* on S if, for every $\varepsilon > 0$, there exists a gauge δ on S such that, for every δ-fine $\mathcal{M}$-partition π of S,

$$\left\| \sum_{(I,s)\in\pi} f(s)\lambda(I) - w_S \right\| < \varepsilon.$$

We write $w_S = (M) \int_S f$. A function $f : S \to X$ is said to be *McShane integrable on* $E \subset S$ if the function $f \cdot \chi_E : S \to X$ is McShane integrable on S, where χ_E is the characteristic function of the set E. The McShane integral of f over E will be denoted by $(M) \int_E f$. Thus we have

$$(M) \int_E f = (M) \int_S f \cdot \chi_E.$$

If f is McShane integrable on S then we obtain by [7, Theorem 4.1.6] that for every $E \in \mathcal{L}$ the function f is McShane integrable on E.

Definition 2.2. We say that a function $f : S \to X$ is *strongly McShane integrable* (or briefly $\mathcal{SM}$-integrable) on S if there exists $\varphi \in \Phi$ such that, for every $\varepsilon > 0$, there exists a gauge δ on S such that, for every δ-fine $\mathcal{M}$-partition π of S,

$$\sum_{(I,s)\in\pi} \|f(s)\lambda(I) - \varphi(I)\| < \varepsilon.$$

By [7, Proposition 3.6.16] we obtain $\varphi(I) = (M) \int_I f$, for each $I \in \mathscr{S}$.

Skvortsov and Solodov defined the *McShane variational integrability* of functions $f : I \to X$, where I is a nondegenerate compact interval of $\mathbb{R}^m$, $m \in \mathbb{N}$; see [8]. This notion coincides with $\mathcal{SM}$-integrability from Definition 2.2.

If X is a finite dimensional Banach space then we obtain by [7, Theorem 5.2.2] that Definitions 2.1 and 2.2 are equivalent.

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Definition 2.3. A function $f : S \to X$ is said to be *weakly McShane integrable* (or briefly $\mathcal{WM}$-integrable) on S if, for every $x^* \in X^*$, the real function x^*f is McShane integrable on S and, for every $I \in \mathscr{S}$, there exists $w_I \in X$ such that $(M) \int_I x^* f = x^*(w_I)$. We call w_I the weak McShane integral of f over I and we write $w_I = (WM) \int_I f$. The additive interval function $F(I) = (WM) \int_I f$ is said to be the $\mathcal{WM}$-primitive of f.

According to [7, Theorem 5.2.3] a real-valued function is McShane integrable if and only if it is Lebesgue integrable. It follows that, if a function $f : S \to X$ is Pettis integrable, then the function f is $\mathcal{WM}$-integrable and, for every $I \in \mathscr{S}$,

$$(P) \int_I f = (WM) \int_I f,$$

where $(P) \int_I f$ denotes the Pettis integral of f on I. In [11], Ye and Schwabik have shown that there exists a $\mathcal{WM}$-integrable function that is not Pettis integrable.

Given $\varphi \in \Phi$, a subset $E \subset S$ and gauge δ on E, we define

$$V_\varphi^{\mathcal{M}}(E, \delta) = \sup \sum_{(I,t)\in\pi} \|\varphi(I)\|,$$

where the supremum is taken over all E-tagged, δ-fine, $\mathcal{M}$-partitions π in S. Then we set

$$V_\varphi^{\mathcal{M}}(E) = \inf\{V_\varphi^{\mathcal{M}}(E, \delta) : \delta \text{ is a gauge on } E\}.$$

The set function $V_\varphi^{\mathcal{M}}$ is said to be the *McShane variational measure* (or $\mathcal{M}$-variational measure) generated by φ. According to Thomson's results from [9], it is known that $V_\varphi^{\mathcal{M}}$ is a Borel metric outer measure on S. We say that the McShane variational measure $V_\varphi^{\mathcal{M}}$ is absolutely continuous with respect to Lebesgue measure (or briefly $V_\varphi^{\mathcal{M}}(E) \ll \lambda$), if $\lambda(E) = 0$ implies that $V_\varphi^{\mathcal{M}}(E) = 0$.

If we replace $\mathcal{M}$-partitions by $\mathcal{HK}$-partitions in the definition of McShane variational measure we obtain the definition of *Henstock variational measure*, [1, Definition 3.1]. We denote by $V_\varphi^{\mathcal{H}}$ the Henstock variational measure generated by $\varphi \in \Phi$.

3. The main result

The following lemma was proved by Di Piazza in [3, Proposition 1]. (There she considers real-valued functions, but the proof works also for vector valued functions, after trivial changes.)

Lemma 3.1. *If $\varphi \in \Phi$, then $V_\varphi^{\mathcal{M}} \ll \lambda$ if and only if φ is sAC.*

We now present the main theorem.

Theorem 3.2. *Let X be a Banach space and let $\varphi \in \Phi$. Then the following statements are equivalent.*

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(i) *X has the WRNP.*
(ii) *If $V_\varphi^{\mathcal{M}} \ll \lambda$, then φ is pseudodifferentiable on S.*
(iii) *If $V_\varphi^{\mathcal{M}} \ll \lambda$, then there exists a function $f : S \to X$ such that f is $\mathcal{WM}$-integrable on S and, for every $I \in \mathscr{S}$,*

$$\varphi(I) = (WM) \int_I f.$$

Proof. (i) ⇒ (ii). Assume that $V_\varphi^{\mathcal{M}} \ll \lambda$. Since each $\mathcal{HK}$-partition is an $\mathcal{M}$-partition, we obtain $V_\varphi^{\mathcal{H}} \ll \lambda$. Therefore the statement (v) of Theorem 4.5 in [1] implies that φ is pseudodifferentiable on S.

(ii) ⇒ (iii). Assume that $V_\varphi^{\mathcal{M}} \ll \lambda$ and let φ'_p be a pseudoderivative of φ. We will prove that the function $f = \varphi'_p$ is $\mathcal{WM}$-integrable with $\mathcal{WM}$-primitive φ.

Assume that an arbitrary $I \in \mathscr{S}$ and an arbitrary vector $x^* \in X^*$ are given. Note that $x^*\varphi$ is sAC and $(x^*\varphi)'(s) = x^*(f(s))$ almost everywhere in S. Therefore, Theorem 7.4.13 together with [7, Theorem 5.2.2] yields that the real-valued function x^*f is McShane integrable on S with the primitive $x^*\varphi$. Thus,

$$(M) \int_I x^* f = (x^*\varphi)(I) = x^*(\varphi(I)),$$

and, since I and x^* are arbitrary, we obtain that f is $\mathcal{WM}$-integrable on S and, for every $I \in \mathscr{S}$,

$$\text{(WM)} \int_I f = \varphi(I).$$

(iii) ⇒ (i). Let $\nu : \mathcal{L} \to X$ be a λ-continuous countable additive measure of bounded variation. We define a function $\varphi \in \Phi$ as follows:

$$\varphi(I) = \nu(I), \quad I \in \mathscr{S}.$$

Since ν is λ-continuous, its variation $|\nu|$ is also λ-continuous, and since $|\nu|$ is a bounded measure we obtain by [6, Theorem 6.11] that to a given $\varepsilon > 0$ there exists $\eta > 0$ such that, for every $E \in \mathcal{L}$,

$$\lambda(E) < \eta \Rightarrow |\nu|(E) < \varepsilon.$$

Let D be a finite collection of nonoverlapping intervals in $\mathscr{S}$ such that

$$\bigcup_{I \in D} \lambda(I) < \eta.$$

Then

$$\sum_{I \in D} \|\varphi(I)\| = \sum_{I \in D} \|\nu(I)\| \le \sum_{I \in D} |\nu|(I) = |\nu|\Big(\bigcup_{I \in D} I\Big) < \varepsilon.$$

This means that φ is sAC and therefore we obtain by Lemma 3.1 that $V_\varphi^{\mathcal{M}} \ll \lambda$. Hence, by (iii), there exists a function $f : S \to X$ such that f is $\mathcal{WM}$-integrable on S and, for every $I \in \mathscr{S}$,

$$\nu(I) = \varphi(I) = \text{(WM)} \int_I f.$$

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Now we will show that f is Pettis integrable on S. Since each real-valued McShane integrable function is Lebesgue integrable we obtain that for each $x^* \in X^*$ the real function $x^* f$ is Lebesgue integrable. Thus it remains to prove that for every $E \in \mathcal{L}$ there exists $x_E \in X$ such that, for every $x^* \in X^*$,

$$x^*(x_E) = (L)\int_E x^* f,$$

where $(L)\int_E x^* f$ denotes the Lebesgue integral of $x^* f$ over E.

First we consider an open subinterval I of S. We denote by $\bar{I}$ the closure of I in S. Note that

$$(L)\int_I x^* f = (L)\int_{\bar{I}} x^* f = x^*\Big((\mathrm{WM})\int_{\bar{I}} f\Big) = x^*(\nu(\bar{I})).$$

Thus we have $x_I = \nu(\bar{I}) = \nu(I)$.

Secondly, let G be an open subset of S. There exists a sequence (I_k) of pairwise disjoint open subintervals of S such that $G = \bigcup_{k=1}^{\infty} I_k$. Then

$$\begin{aligned}(L)\int_G x^* f &= (L)\int_{\bigcup_{k=1}^{\infty} I_k} x^* f = \sum_{k=1}^{\infty}(L)\int_{I_k} x^* f\\ &= \sum_{k=1}^{\infty} x^*(x_{I_k}) = \sum_{k=1}^{\infty} x^*(\nu(I_k)) = \lim_{n\to\infty}\sum_{k=1}^{n} x^*(\nu(I_k))\\ &= \lim_{n\to\infty} x^*\Big(\sum_{k=1}^{n}\nu(I_k)\Big) = x^*\Big(\nu\Big(\bigcup_{k=1}^{\infty} I_k\Big)\Big) = x^*(\nu(G)).\end{aligned}$$

Hence, $x_G = \nu(G)$.

Finally, we consider a measurable set $E \in \mathcal{L}$. There exists a sequence (G_n) of open subsets of S such that, for every $n \in \mathbb{N}$,

$$E \subset G_n, \quad G_{n+1} \subset G_n$$

and $\lambda(G_\delta \setminus E) = 0$, where $G_\delta = \bigcap_{n=1}^{\infty} G_n$. Since ν is a λ-continuous countable additive measure and $\lim_{n\to\infty}\lambda(G_n) = \lambda(G_\delta)$ we obtain by [2, Theorem I.2.1] that $\lim_{n\to\infty}\nu(G_n) = \nu(G_\delta)$. Therefore,

$$\begin{aligned}(L)\int_E x^* f &= (L)\int_{G_\delta} x^* f - (L)\int_{G_\delta\setminus E} x^* f = (L)\int_{G_\delta} x^* f\\ &= \lim_{n\to\infty}(L)\int_{G_n} x^* f = \lim_{n\to\infty} x^*(x_{G_n}) = \lim_{n\to\infty} x^*(\nu(G_n)) = x^*(\nu(G_\delta)),\end{aligned}$$

and so $x_E = \nu(G_\delta) = \nu(E)$.

Consequently the function f is Pettis integrable and, for every $E \in \mathcal{L}$,

$$\nu(E) = (P)\int_E f.$$

This proves that X has the WRNP. □

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References

[1] B. Bongiorno, L. Di Piazza and K. Musial, 'A characterization of the Weak Radon–Nikodym property by finitely additive interval functions', *Bull. Aust. Math. Soc.* **80** (2009), 476–485.
[2] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, 15 (American Mathematical Society, Providence, RI, 1977).
[3] L. Di Piazza, 'Variational measures in the theory of the integration in $\mathbb{R}^n$', *Czechoslovak Math. J.* **51**(126) (2001), 95–110.
[4] T. Y. Lee, 'A full descriptive definition of the Henstock–Kurzweil integral in the Euclidean space', *Proc. Lond. Math. Soc.* (3) **87** (2003), 677–700.
[5] W. F. Pfeffer, *Derivation and Integration* (Cambridge University Press, Cambridge, 2001).
[6] W. Rudin, *Real and Complex Analysis*, 2nd edn (McGraw-Hill, New York, NY, 1974).
[7] Š. Schwabik and G. Ye, *Topics in Banach Space Integration*, Series in Real Analysis, 10 (World Scientific, Hackensack, NJ, 2005).
[8] V. A. Skvortsov and A. P. Solodov, 'A variational integral for Banach-valued functions', *Real Anal. Exchange* **24** (1998/9), 799–806.
[9] B. S. Thomson, 'Derivates of interval functions', *Mem. Amer. Math. Soc.* (1991), 452.
[10] B. S. Thomson, 'Differentiation', in: *Handbook of Measure Theory* (ed. E. Pap) (Elsevier Science B.V., 2002), pp. 179–247.
[11] G. Ye and S. Schwabik, 'The McShane and the weak McShane integrals of Banach space-valued functions defined on $\mathbb{R}^m$', *Math. Notes (Miskolc)* **2** (2001), 127–136.

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