ON ANTI-COMMUTATIVE ALGEBRAS AND ANALYTIC LOOPS

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1. Introduction. In **(4)** Malcev generalizes the notion of the Lie algebra of a Lie group to that of an anti-commutative "tangent algebra" of an analytic loop. In this paper we shall discuss these concepts briefly and modify them to the situation where the cancellation laws in the loop are replaced by a unique two-sided inverse. Thus we shall have a set H with a binary operation xy defined on it having the algebraic properties

(1.1) H contains a two-sided identity element e;

(1.2) for every $x \in H$, there exists a unique element $x^{-1} \in H$ such that $xx^{-1} = x^{-1}x = e$;

and H also has the analytic properties that it is an n-dimensional analytic manifold so that

(1.3) $H \times H \rightarrow H : (x, y) \rightarrow xy$ is an analytic mapping;

(1.4) $H \rightarrow H : x \rightarrow x^{-1}$ is an analytic mapping.

Since any system H satisfying (1.1)–(1.4) resembles an analytic H-space which is almost a loop, this system will be called an *analytic hoop.*

We shall show that the tangent algebra of an analytic hoop can be defined using the same definition as that of a tangent algebra of an analytic loop (which is the same as in classical Lie group theory involving tangents to products and commutators of differentiable curves through the identity). Extending the notion of the exponential series to certain non-commutative Jordan algebras **(6; 8)** the main result is the following representation theorem.

THEOREM. *If A is an n-dimensional anti-commutative algebra over the real numbers R, then there exists a power associative flexible analytic hoop K with tangent algebra isomorphic to A. Furthermore, there exist algebra homomorphisms of A if and only if there exist analytic hoop homomorphisms of K.*

In this paper it should be clear when analyticity can be replaced by weaker differentiability conditions.

2. Basic concepts. In this section we shall discuss analytic loops, their tangent algebras **(4)**, and the examples obtained from the split Cayley-Dickson algebra.

Definition. A *loop* $(L, \cdot)$ is a non-empty set L and a binary operation $\cdot$ on L having the following properties:

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(2.1) for any $a, b \in L$, the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

have unique solutions x and y in L;

(2.2) there exists an element $e \in L$ such that for all $x \in L$; $x \cdot e = e \cdot x = x$.

Thus, denoting $x \cdot y$ by xy, (2.1) states that the mappings of L onto L given by $R_a : y \rightarrow ya$ and $L_a : y \rightarrow ay$ are bijections. From this fact and (2.2) we have that for any $x \in L$, there exist $x_r^{-1}, x_l^{-1} \in L$ such that $xx_r^{-1} = e = x_l^{-1}x$; that is, $x_r^{-1}(x_l^{-1})$ is a right (left) inverse of x. In addition, x_r^{-1} need not equal x_l^{-1}.

We shall not consider a general theory of loops but consider an example of a Moufang loop **(5)** which is not a group. Let R denote the field of real numbers and let A denote the three-dimensional Lie algebra defined as follows:

let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3)$, where $a_i, b_i \in R$, and let

$$\alpha \cdot \beta = \sum a_i b_i,$$
$$\alpha \times \beta = (a_2 b_3 - a_3 b_2, a_3 b_1 - b_3 a_1, a_1 b_2 - a_2 b_1).$$

Thus $\cdot$ and $\times$ make R^3 into the usual three-dimensional Lie algebra A of vector analysis. Next let

$$B = \left\{ \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} : a, b \in R \quad \text{and} \quad \alpha, \beta \in A \right\},$$

and for these 2 by 2 matrices define the natural co-ordinate-wise addition and scalar multiplication by elements of R. Defining multiplication of two such matrices by

$$\begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} \begin{bmatrix} c & \gamma \\ \delta & d \end{bmatrix} = \begin{bmatrix} ac + \alpha \cdot \delta & a\gamma + d\alpha - \beta \times \delta \\ c\beta + b\delta + \alpha \times \gamma & bd + \beta \cdot \gamma \end{bmatrix},$$

B becomes the eight-dimensional split Cayley–Dickson algebra over R. Next we obtain a loop from B. Let

$$x = \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix}, \quad y = \begin{bmatrix} c & \gamma \\ \delta & d \end{bmatrix} \in B,$$

and let $N(x) = ab - \alpha \cdot \beta$. Then from **(5)** we have $N(xy) = N(x)N(y)$ so that $L = \{x \in B : N(x) \neq 0\}$ is a Moufang loop, and $M' = \{x : N(x) = 1\}/Z$ is actually a simple Moufang loop. Note that any element $x \in L$ has a unique two-sided inverse given by

$$x^{-1} = \frac{1}{N(x)} \begin{bmatrix} b & -\alpha \\ -\beta & a \end{bmatrix}.$$

Next in this example if we give L the relative topology of R^8, then from the various formulas we see that L is a *topological loop*; that is, L has the following properties:

(2.3) L is a topological space,

(2.4) the following maps from the product space $L \times L$ onto L are continuous:

$$(x, y) \to xy, \qquad (x, y) \to xR_y^{-1}, \qquad (x, y) \to xL_y^{-1}.$$

Thus in the equation $xy = z$, any one of the elements x, y, or z depends continuously on the other two elements. Similarly, the simple Moufang loop M' is a topological loop with topological space $S^3 \times S^3 \times R^1$. We note that in both cases the mapping $x \to x^{-1}$ is a continuous operation.

Since topological loops have been discussed at length in **(2; 3)** we shall next consider analytic loops **(4)**. For example, considering the Cayley–Dickson algebra B as an eight-dimensional vector space over R, we can let $X_1, \ldots, X_8$ be any basis of B, then the mapping $\sum^8_{i=1} x_i X_i \to (x_1, \ldots, x_8)$ is an open chart valid on all of B, and the resulting analytic structure is independent of the choice of basis. Now from the formulas, the function $N : L \to R : x \to N(x)$ is continuous, and therefore L is an open submanifold of B with the induced analytic structure. From the formulas for xy and x^{-1} in L, we see that the co-ordinate expressions for the mappings in (2.4) are analytic expressions in terms of the given co-ordinates used in the definition of B. Thus L is an analytic loop according to the following definition.

Definition. A loop L is an *n-dimensional analytic loop* if L is an n-dimensional analytic manifold such that in the equation $xy = z$ any one of the elements x, y, or z depends analytically on the other two elements; that is, the co-ordinate expression for any of the x, y or z depends analytically on the co-ordinates of the other two elements. Thus in particular, the maps $(x, y) \to xy$, $x \to x_r^{-1}$, and $x \to x_l^{-1}$ are analytic.

Similar to the definition of a local Lie group **(1)**, a local analytic loop is an abstraction of the concept of open neighbourhood of the identity in an analytic loop where it is assumed that the identity corresponds to the origin in R^n. We shall identify the neighbourhood of the identity with the region about the origin to which it corresponds.

In **(4)** Malcev discusses the elementary properties of analytic loops similar to those of Lie group theory and makes the definition of a tangent algebra of an analytic loop. As stated in the Introduction, this definition is similar to that in classical Lie group theory. However, Malcev uses the right inverse in the expression for the commutator and this leads to the following modifications. Using notation similar to that in **(1)**, we let V be a local analytic loop which is identified with a neighbourhood of the identity $e = (0, \ldots, 0)$ in R^n; the elements $x \in V$ are identified with the usual co-ordinate vectors $(x_1, \ldots, x_n)$. Let $a(t) = (a_1(t), \ldots, a_n(t))$ be a curve in V such that the $a_i(t)$ are analytic at $t = 0$ and $a(0) = e$. The tangent vector to $a(t)$ at e is $x = (da(t)/dt)_{t=0}$; we note that every vector $x \in R^n$ is a tangent vector to some analytic curve in V, e.g. $a(t) = xt$, t small. The set of tangent vectors as above is an n-dimensional vector space B; i.e., B is the tangent space to V at e. As in the case of Lie groups **(1)**, we can show that if $a(t)$ and $b(t)$ are analytic curves in V with $a(0) = b(0) = e$ and with tangent vectors x and y at $t = 0$, then the curve $a(t)b(t)$ is analytic and has tangent vector $x + y$ at $t = 0$. Next multiplication is defined in B as follows: Let $x, y \in B$ be tangent vectors to $a(t)$ and $b(t)$ at $t = 0$. The product $(x \circ y)_r$ is defined to be the tangent vector to $a(\tau)b(\tau)\cdot[b(\tau)a(\tau)]_r^{-1}$ when $\tau = t^2$ at $\tau = 0$. As in the case of Lie groups **(1)**, B becomes an anti-commutative algebra called the right tangent algebra of V. The left inverse could be used in this definition to obtain the left tangent algebra, but we shall be interested in the *tangent algebra* which is obtained when the left and right inverses in V are equal and use the notation $x \circ y$ for the product in the tangent algebra. In the case of an analytic loop, the various tangent algebras are computed from a local analytic loop (i.e. an open neighbourhood of the identity) just as in the case of a Lie group.

Of course, owing to the lack of associativity in an analytic loop, the tangent algebras need not be Lie algebras. For example, consider the analytic Moufang loop L. Using the notation in the construction of L, we see that for t small enough and for any X and Y in the Cayley–Dickson algebra B,

$$x(t) = I + tX \quad \text{and} \quad y(t) = I + tY$$

are analytic curves belonging to L which have tangent vectors X and Y at $t = 0$ and pass through the identity I (we shall not recoordinatize so that I corresponds to the origin in R^8). A straightforward computation shows that

$$x(t)y(t)\cdot[y(t)x(t)]^{-1} = I + t^2(XY - YX) + O(t^3)$$

where

$$\lim_{t\to 0} O(t^3)/t^2 = 0$$

and the product $X \circ Y = XY - YX$ occurs in the algebra B. Thus we see that the tangent algebra to L is actually the eight-dimensional Cayley–Dickson algebra B with multiplication $X \circ Y = XY - YX$. Let us denote this anti-commutative algebra by B^-; it is known that it is not a Lie algebra and in general it is shown in **(4)** that the tangent algebra to an analytic Moufang loop will be a Malcev algebra **(7)**. It can also be shown that the tangent algebra to the simple analytic Moufang loop M' is isomorphic to the simple Malcev algebra B^-/IR, where IR is the ideal generated by the identity I in B^-. It is noted in **(4)** that these examples are special cases of analytic disassociative loops **(2; 3)** and binary Lie algebras **(9)**. An analytic loop in which every two elements generate a Lie subgroup is an analytic disassociative loop. In addition analytic Moufang loops have this property. The tangent algebra of an analytic disassociative loop is a binary Lie algebra (i.e. every two elements generate a Lie subalgebra). Malcev algebras also have this property.

3. Proof of the main theorem. The formulas used from **(1)** in deriving the properties of the right tangent algebra of an analytic loop L utilize only the following facts:

(3.1) L is an n-dimensional analytic manifold with an analytic binary operation $L \times L \to L : (x, y) \to xy$,

(3.2) L has a two-sided identity e such that $xe = ex = x$ for all $x \in$ L,

(3.3) for every $x \in L$ there exists a unique right inverse x_r^{-1} such that the mapping $L \to L : x \to x_r^{-1}$ is analytic.

Thus we see from the definition of an analytic hoop given in the Introduction that a tangent algebra of an analytic hoop exists. The tangent algebra is an anti-commutative algebra, and we now consider the problem of representing a given anti-commutative algebra over R as a tangent algebra of an analytic hoop.

Let A be an n-dimensional anti-commutative algebra over R, and let

$$B = \left\{ \begin{bmatrix} a & \alpha \\ \alpha & a \end{bmatrix} : a \in R \quad \text{and} \quad \alpha \in A \right\}.$$

Thus B is an $(n + 1)$-dimensional vector space of 2 by 2 matrices with the obvious definitions of addition and scalar multiplication and we give B the topology of R^{n+1}. B becomes a non-commutative Jordan algebra **(6; 8)** when multiplication is defined by

$$\begin{bmatrix} a & \alpha \\ \alpha & a \end{bmatrix} \begin{bmatrix} b & \beta \\ \beta & b \end{bmatrix} = \begin{bmatrix} ab & a\beta + b\alpha + \tfrac{1}{2}\alpha\beta \\ a\beta + b\alpha + \tfrac{1}{2}\alpha\beta & ab \end{bmatrix}.$$

Since B is power-associative, we use induction to show that if

$$X = \begin{bmatrix} a & \alpha \\ \alpha & a \end{bmatrix}, \qquad Y = \begin{bmatrix} b & \beta \\ \beta & b \end{bmatrix} \in B,$$

then

$$X^k = \begin{bmatrix} a^k & ka^{k-1}\alpha \\ ka^{k-1}\alpha & a^k \end{bmatrix}, \qquad \text{for } k = 0, 1, \ldots .$$

Next we investigate $\exp X$ by considering the partial sum

$$S_n = \sum_{k=0}^{n} \frac{X^k}{k!} = \begin{bmatrix} a(n) & \alpha(n) \\ \alpha(n) & a(n) \end{bmatrix},$$

where

$$a(n) = \sum_{k=0}^{n} \frac{a^k}{k!},$$

$$\alpha(n) = \sum_{k=0}^{n} \frac{ka^{k-1}\alpha}{k!} = \left(\sum_{k=0}^{n-1} \frac{a^k}{k!} \right) \alpha.$$

Thus the sequence (S_n) converges in B, and from the formulas we see that

$$\exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!} = e^a \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}. \tag{3.4}$$

By (3.4), the mapping $\exp : B \to \exp B \equiv H \subset B$ has the following properties:

(3.5) exp is injective.

(3.6) exp is continuous and open.

Since the topology in R^n is independent of basis, we choose a fixed basis $X_1, \ldots, X_n$ for A (with the topology of R^n) and write $\alpha = \sum x_i X_i$ in terms of co-ordinates $x_1, \ldots, x_n$ relative to $X_1, \ldots, X_n$. Let

$$X = x_0 I + \sum x_i \begin{bmatrix} 0 & X_i \\ X_i & 0 \end{bmatrix} \in B,$$

and let

$$Y = \exp X = \phi_0(x_0, \ldots, x_n) I + \sum \phi_j(x_0, \ldots, x_n) \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix}.$$

Then we see from (3.4) that

$$\phi_0(x_0, \ldots, x_n) = e^{x_0},$$
$$\phi_j(x_0, \ldots, x_n) = e^{x_0} x_j,$$

and the results now follow. In particular, $H = \exp B$ is an open subset of B.

(3.7) $H = \exp B$ is a hoop.

For let $X, Y \in B$; then from the various formulas we obtain

$$\exp X \exp Y = e^{a+b} \begin{bmatrix} 1 & \alpha + \beta + \frac{1}{2}\alpha\beta \\ \alpha + \beta + \frac{1}{2}\alpha\beta & 1 \end{bmatrix}. \tag{3.8}$$

But

$$[X, Y] = XY - YX = \begin{bmatrix} 0 & \alpha\beta \\ \alpha\beta & 0 \end{bmatrix}$$

and

$$\exp(X + Y + \tfrac{1}{2}[X, Y]) = \exp \begin{bmatrix} a + b & \alpha + \beta + \frac{1}{2}\alpha\beta \\ \alpha + \beta + \frac{1}{2}\alpha\beta & a + b \end{bmatrix}$$
$$= e^{a+b} \begin{bmatrix} 1 & \alpha + \beta + \frac{1}{2}\alpha\beta \\ \alpha + \beta + \frac{1}{2}\alpha\beta & 1 \end{bmatrix}.$$

Thus

$$\exp X \exp Y = \exp(X + Y + \tfrac{1}{2}[X, Y]) \in H \tag{3.9}$$

and $I = \exp 0 \in H$. Next for any $V = \exp X$, $U = \exp(-X)$ satisfies

$VU = UV = I$. Furthermore, the two-sided inverse is unique; for if $W = \exp Y$ is such that $WV = VW = I$, then we have from (3.8) that $e^{a+b} = 1$ so that $b = -a$ and

$$\alpha + \beta + \tfrac{1}{2}\alpha\beta = 0, \qquad \beta + \alpha + \tfrac{1}{2}\beta\alpha = 0.$$

Therefore using the anti-commutativity of A, $\alpha + \beta = 0$ so that $\beta = -\alpha$ and

$$Y = -X; \qquad \text{i.e. } W = \exp(-X) = U.$$

We note that from the identities in B, H is power-associative and satisfies the flexible identity $ST \cdot S = S \cdot TS$ for all $S, T \in H$. Also from (3.8) it is easy to see that H satisfies the following weak cancellation laws: If for any $S, T \in H$ we have

$$\exp tX \cdot S = \exp tX \cdot T$$

or

$$S \cdot \exp tX = T \cdot \exp tX$$

for all t in some interval of R, then $S = T$.

Next we assume the algebra A is given the usual analytic structure of R^n. Then addition, scalar multiplication, and multiplication are all analytic operations in A, since relative to a fixed basis of A the co-ordinate expressions are just first or second degree polynomials in the given co-ordinates. We next show property

(3.10) H is an analytic hoop.

For if B is given the usual analytic structure of R^{n+1}, H is an open submanifold with the analytic structure induced from B. Next from the various formulas (including $(\exp X)^{-1} = \exp(-X)$) and from the analyticity of the operations in A, we see that the maps $H \times H \to H : (S, T) \to ST$ and $H \to H : S \to S^{-1}$ are analytic.

We now complete the proof of the main theorem by considering an analytic subhoop of the analytic hoop constructed in (3.10). Let

$$C = \left\{ X \in B : X = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \text{ and } \alpha \in A \right\};$$

then from the computations following (3.8) we see that the map $X \to \alpha$ is an isomorphism of C^- (the vector space C with multiplication $[X, Y] = XY - YX$) onto A. Let

$$K = \{\exp X : X \in C^-\};$$

then K is an n-dimensional submanifold of H with the usual analytic structure of R^n, and from (3.8) and (3.9), we see that K is actually an analytic hoop.

Next let $X \in C^-$, then the analytic curve $x(t) = \exp tX$ is such that

$$x(0) = I \quad \text{and} \quad (dx(t)/dt)_{t=0} = X.$$

Thus since C^- is n-dimensional, C^- is the tangent space to K at I. Now let $X, Y \in C^-$ and consider the tangent-algebra multiplication in C^-. We note that

$$\begin{aligned} z(t^2) &= (\exp tX \exp tY)(\exp tY \exp tX)^{-1} \\ &= \exp(tX + tY + \tfrac{1}{2}t^2[X, Y])\exp(-tX - tY - \tfrac{1}{2}t^2[Y, X]) \\ &= \exp(t^2[X, Y] + \tfrac{1}{2}t^3[X + Y, [X, Y]]) \end{aligned} \tag{3.11}$$

by (3.9). Then with the change of variable $\tau = t^2$, we see that

$$X \circ Y = (dz(\tau)/d\tau)_{\tau=0} = [X, Y].$$

Thus the tangent algebra to K is C^-, which is isomorphic to A.

For the statement concerning homomorphisms, let $\phi : \alpha \to \alpha'$ be a homomorphism of A onto an algebra A'. Then defining C'^- for the algebra A', we note that the mapping

$$C^- \to C'^- : X \to X',$$

where

$$X = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \quad \text{and} \quad X' = \begin{bmatrix} 0 & \alpha' \\ \alpha' & 0 \end{bmatrix}$$

is an algebra homomorphism. Next let $K' = \exp C'^-$; then the mapping

$$\eta : K \to K' : \exp X \to \exp X'$$

is a homomorphism of the hoop K onto the hoop K'. For using (3.9) and the homomorphism $C^- \to C'^- : X \to X'$, we have

$$\begin{aligned} \exp X \exp Y &= \exp(X + Y + \tfrac{1}{2}[X, Y]) \\ &\to \exp(X' + Y' + \tfrac{1}{2}[X', Y']) \\ &= \exp X' \cdot \exp Y'. \end{aligned}$$

The mapping η is analytic; for if

$$\alpha = \sum_{i=1}^{m} y_i n_i + \sum_{i=1}^{r} x_i m_i \in A = N \oplus M,$$

where N is the kernel of ϕ, then as in the proof of (3.6) we see that the coordinate expressions for $\exp X'$ relative to the basis $\phi(m_i)$, $i = 1, \ldots, r$, of A' are just the analytic expressions x_i, $i = 1, \ldots, r$. Also if $\phi : A \to A'$ is a proper homomorphism, so is $\eta : K \to K'$.

Conversely, let η be an analytic homomorphism of $K = \exp C^-$ into an analytic hoop K' and let A' be the tangent algebra of K'. We shall construct a homomorphism of C^- into A' and therefore a homomorphism of A into A'. Let $X \in C^-$ and let $x(t)$ be a curve in K with X as its tangent vector, e.g. $x(t) = \exp tX$. Then $x'(t) = \eta(x(t))$ defines a curve in K' with tangent vector

X' which depends only on X and not on the choice of the curve $x(t)$; so we choose $x(t) = \exp tX$. Now the map

$$\phi : C^- \to A' : X \to X'$$

is an algebra homomorphism. For if $X, Y \in C^-$, then from (3.11) we have

$$\begin{aligned}\phi([X, Y]) &= \phi(X \circ Y)\\ &= (d\eta(z(\tau))/d\tau)_{\tau=0}\\ &= (dz'(\tau)/d\tau)_{\tau=0}\\ &= X' \circ Y',\end{aligned}$$

where $z'(t^2) = \eta z(t^2) = [x'(t) \cdot y'(t)] \cdot [y'(t) \cdot x'(t)]^{-1}$. Similarly,

$$\begin{aligned}\phi(X + Y) &= (d\eta(x(t)y(t))/dt)_{t=0}\\ &= (d\eta(x(t)) \cdot \eta(y(t))/dt)_{t=0}\\ &= X' + Y'\end{aligned}$$

by the remarks of Section 2 concerning the sum of tangent vectors. Also, for $a \in R$, $\phi(aX) = aX'$.

Next we note that ϕ is a proper homomorphism if η is a proper homomorphism which satisfies the following condition:

$\eta : \exp X \to e'$ implies that $\eta : \exp tX \to e'$ for all $t \in J$, where J is an interval containing 0 and e' is the identity of K'.

For if η is proper, then there exists $X \neq 0$ in C^- such that $\eta : \exp X \to e'$ and therefore $\eta : \exp tX \to e'$ for all $t \in J$. Therefore we can compute the tangent vector X' to $x'(t) = \eta\,(\exp tX) = e'$ and obtain $X' = 0$; i.e. $\phi : X \to X' = 0$.

References

1. H. W. Guggenheimer, *Differential geometry* (New York, 1963), Chap. 6.
2. K. H. Hoffmann, *Topologische loops, Topologische Loops mit schwachen Assoziativitätsforderungen*, Math. Z., *70* (1958), 13–37, 125–155.
3. S. Hudson, *Topological loops with invariant uniformities*, Trans. Amer. Math. Soc., *109* (1963), 181–190.
4. A. Malcev, *Analytic loops*, Mat. Sbornik, *78* (1955), 569–576.
5. L. J. Paige, *A class of simple Moufang loops*, Proc. Amer. Math. Soc., *7* (1956), 471–482.
6. ——— *A note on noncommutative Jordan algebras*, Portugal. Math., *16* (1957), 15–18.
7. A. Sagle, *Simple Malcev algebras over fields of characteristic zero*, Pacific J. Math., *12* (1962), 1057–1078.
8. ——— *On anti-commutative algebras with an invariant form*, Can. J. Math., *16* (1964), 370–378.
9. ——— *On simple extended Lie algebras over fields of characteristic zero*, to appear.

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