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Generalised product integration

David Frederick Paget

This thesis is concerned with the construction of quadrature rules for the numerical evaluation of one dimensional integrals which may be written in the form

$$I(f;\ \lambda) = \int_a^b w(x)f(x)K(x;\ \lambda)dx\ .$$

The function w is assumed to be non-negative with

$$\int_a^b w(x)dx > 0$$

and the moments

$$\int_a^b x^n w(x)dx$$

existing for all non-negative integers n. Associated with w there is a sequence of orthogonal polynomials $\{p_n\}$. It is further assumed that f may be well approximated by a series of the polynomials $\{p_n\}$, and that the function K is such that the integrals

$$\int_a^b w(x)p_n(x)K(x;\ \lambda)dx$$

are known. It is through this last property that the generalised product integration rules attempt to account for any adverse behaviour of the integrand of $I(f;\ \lambda)$.

Three chapters (2, 3, and 4) are devoted to the case when

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$K(x;\ \lambda) = (x-\lambda)^{-1}$, $a < \lambda < b$, and

$$I(f;\ \lambda) = \int_a^b \frac{w(x)f(x)}{x-\lambda}\,dx$$

is a Cauchy principal value integral. The quadrature sum is expressed in two forms:

$$Q_n(f;\ \lambda) = \sum_{k=1}^{n} A_{k,n}(\lambda)f(x_{k,n})$$

and

$$Q_n(f;\ \lambda) = \sum_{k=0}^{n-1} a_{k,n}q_k(\lambda) ,$$

where

$$q_k(x) = \int_a^b \frac{w(t)p_k(t)}{t-x}\,dt .$$

Chapter 2 is devoted to computational aspects of evaluating the quadrature sum $Q_n(f;\ \lambda)$. Convergence of the rule is discussed in Chapter 3 where the following result is proved.

THEOREM. *If* f *is Hölder continuous of order* μ , $0 < \mu \le 1$, *and* $I(f;\ \lambda)$ *exists for* $a < \lambda < b$, *then*

$$\lim_{n\to\infty} \big(I(f;\ \lambda)-Q_n(f;\ \lambda)\big) = 0.$$

In Chapter 4 it is shown that when f is an analytic function then the remainder

$$R_n(f;\ \lambda) = I(f;\ \lambda) - Q_n(f;\ \lambda)$$

may be expressed as a contour integral. From this contour integral asymptotic estimates of the remainder are found. Some examples are given which demonstrate the accuracy of the method. In these three chapters particular attention is given to the case when w is a Jacobi weight function.

The generalised product integration rule for arbitrary K is developed in Chapter 5. Sufficient conditions for convergence of the generalised rule are established for continuous f and integrable K. The implementation and convergence of the generalised rule is further investigated for three functions K of considerable interest, namely:

$$K(x;\ \lambda) = \exp(i\lambda x)\ ,\quad \lambda \text{ real}\ ,\quad |\lambda| \text{ large;}$$

$$K(x;\ \lambda) = \ln|x-\lambda|\ ,\quad -1 < \lambda < 1\ ;$$

$$K(x;\ \lambda) = |x-\lambda|^{s}\ ,\quad s > -1\ ,\quad -1 < \lambda < 1\ .$$

For each of these functions conditions sufficient to ensure convergence of the rule are obtained when w is a Jacobi weight function.