LOCAL SPECTRAL THEORY AND SPECTRAL INCLUSIONS

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Introduction. Suppose that T and S are continuous linear operators on complex Banach spaces X and Y, respectively, and that A is a non-zero continuous linear mapping from X to Y. If A intertwines T and S in the sense that $SA = AT$, then a classical result due to Rosenblum implies that the spectra $\sigma(T)$ and $\sigma(S)$ must overlap, see **[12]**. Actually, Davis and Rosenthal **[5]** have shown that the surjectivity spectrum $\sigma_{su}(T)$ will meet the approximate point spectrum $\sigma_{ap}(S)$ in this case (terms to be defined below). Further information about the relations between the two spectra and their finer structure becomes available when the intertwiner A is injective or has dense range, see **[9]**, **[12]**, **[13]**.

Local spectral theory is very well suited to deal with issues of this sort, as witnessed by many results in **[4]** and **[21]**. To mention but one sample, Theorem 2.4.4 of **[4]** shows that $\sigma(T) = \sigma(S)$ if both T and S are decomposable in the sense of Foiaş and the intertwiner A is injective and has dense range. Local spectral theory also yields the tools to handle the more general, but still very natural asymptotic intertwining condition $r_C(A) = 0$, where C stands for the commutator of S and T given by $C(A) := SA - AT$ and

$$r_C(A) := \limsup_{n \to \infty} \|C^n(A)\|^{1/n}$$

denotes the local spectral radius of the commutator C at A.

In a previous paper **[16]**, we have explored the role of asymptotic intertwining in connection with spectral inclusions for quotients and restrictions of decomposable operators. A further step has recently been taken in **[18]**, where the asymptotic intertwining condition was abandoned and instead the closed disc $\nabla(0, r)$ with center 0 and radius $r := r_C(A) \geq 0$ was used to obtain perturbed spectral inclusions of the following type: if A is injective and T is the quotient of a decomposable operator, then $\sigma(T) \subseteq \sigma(S) + \nabla(0, r)$, and dually, if A has dense range and S is the restriction of a decomposable operator, then $\sigma(S) \subseteq \sigma(T) + \nabla(0, r)$.

In Section 2 of the present paper, we shall obtain considerably more general versions of these spectral inclusions, which are of definitive form in the case of quotients and restrictions of decomposable operators. In our main results, Theorems 2.4 and 2.5, we are able to replace the disc $\nabla(0, r)$ by a much smaller set, namely a suitable local spectrum of the commutator. This is done by means of Proposition 2.1, which replaces the power series approach from **[16]** and **[18]** by a more flexible argument involving contour integrals of convolution type.

We are also able to sharpen the spectral inclusions in question to cover the surjectivity and approximate point spectrum. This requires a number of new results and techniques in local spectral theory, which will be developed in Section 1 and should be of independent interest. In particular, we shall show in Theorem 1.1 that the boundary of the local spectrum $\sigma_T(x)$ is always contained in $\sigma_{ap}(T)$. It is somewhat surprising that this result holds in general, not just for restrictions of decomposable operators where it follows easily by localization of the spectrum. For the proof, we introduce the concept of

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the Kato resolvent set as a powerful new tool in this context. Compared to the classical development of local spectral theory in [4] and [21], the emphasis is here on a new type of spectral subspaces, the $\mathfrak{X}_T(F)$-spaces; this allows us to handle operators without the single valued extension property. Further important tools are the recent results on restrictions and quotients of decomposable operators due to Albrecht and Eschmeier [1]. The authors are indebted to Jörg Eschmeier for several helpful comments on an earlier version of this paper and for providing them with the reference [17].

1. Local spectral theory. For a given continuous linear operator $T \in L(X)$ on a non-trivial complex Banach space X, recall the following classical subsets of the spectrum $\sigma(T)$: the point spectrum $\sigma_p(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not injective}\}$, the approximate point spectrum $\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \text{there exist unit vectors } x_n \in X \text{ such that } (T-\lambda)x_n \to 0 \text{ as } n \to \infty\}$, and the dual objects, the compression spectrum $\sigma_{com}(T) := \{\lambda \in \mathbb{C}\colon (T-\lambda)X \text{ is not dense in } X\}$ and the surjectivity spectrum $\sigma_{su}(T) := \{\lambda \in \mathbb{C}\colon (T-\lambda)X \neq X\}$. The term 'dual' refers to the well-known facts that, if $T^* \in L(X^*)$ denotes the adjoint of T on the dual space X^*, then $\sigma_p(T) \subseteq \sigma_{com}(T^*)$, $\sigma_{com}(T) = \sigma_p(T^*)$, $\sigma_{ap}(T) = \sigma_{su}(T^*)$, $\sigma_{su}(T) = \sigma_{ap}(T^*)$; see section 57 of [2] for details.

Recall also the notion of local spectrum from [4]: if $x \in X$, then the *local spectrum* $\sigma_T(x)$ of T at x is defined to be the complement of the set of all $\lambda \in \mathbb{C}$ for which there is an analytic function $f: U \to X$ on some open neighborhood U of λ such that $(T-\mu)f(\mu) = x$ for all $\mu \in U$. The local analytic spectral subspace $X_T(F)$, where F is an arbitrary subset of $\mathbb{C}$, is defined to consist of all $x \in X$ for which $\sigma_T(x) \subseteq F$. We shall frequently use the identity $\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x)$ for all $T \in L(X)$, which is an easy consequence of the open mapping principle; see Lemma 2 of [15]. The following general result will play a key role in locating local spectra.

THEOREM 1.1. *For any $T \in L(X)$ and $x \in X$, we have $\partial\sigma_T(x) \subseteq \sigma_{ap}(T)$.*

Proof. Our main tool will be what we call the *Kato resolvent set of T*, namely the set

$$\rho_K(T) := \left\{\lambda \in \mathbb{C} : (T-\lambda)X \text{ is closed and } \ker(T-\lambda) \subseteq \bigcap_{n=1}^{\infty} (T-\lambda)^n X\right\}.$$

In a slightly more general context, this set has been thoroughly investigated by Kato in [14]; further properties of $\rho_K(T)$ may be found in [19]. The Kato resolvent set contains the ordinary resolvent set $\rho(T)$ and is open by Theorem 3 of [14]. For our purposes, the crucial point is that every $\lambda \in \rho_K(T)$ satisfies the identity $X_T(\mathbb{C}\backslash\{\lambda\}) = Z_T(\lambda)$ where $Z_T(\lambda) := \bigcap_{n=1}^{\infty} (T-\lambda)^n X$. Here the inclusion $\subseteq$ is clear from the elementary fact that $(T-\lambda)X_T(\mathbb{C}\backslash\{\lambda\}) = X_T(\mathbb{C}\backslash\{\lambda\})$, which follows from Proposition IV.3.4 of [21]. To prove the reverse inclusion, we consider an arbitrary $x \in Z_T(\lambda)$ and choose $x_n \in X$ such that $x = (T-\lambda)^n x_n$ for all $n \in \mathbb{N}$. Then we have

$$x_1 - (T-\lambda)^n x_{n+1} \in \ker(T-\lambda) \subseteq Z_T(\lambda) \subseteq (T-\lambda)^n X$$

and therefore $x_1 \in (T-\lambda)^n X$ for all $n \in \mathbb{N}$. This observation shows that $x \in (T-\lambda)Z_T(\lambda)$ and hence that $(T-\lambda)Z_T(\lambda) = Z_T(\lambda)$. Moreover, the space $Z_T(\lambda)$ is closed, since it is easily seen that all the powers of $T-\lambda$ have closed range whenever $\lambda \in \rho_K(T)$; cf. Satz 4 of [**19**]. With the notation $S := T|Z_T(\lambda)$, we conclude that $\lambda \notin \sigma_{su}(S)$. On the other hand, we infer from Lemma 2 of [**15**] that $\sigma_S(x)$ is contained in $\sigma_{su}(S)$ for all $x \in Z_T(\lambda)$. Consequently, if $x \in Z_T(\lambda)$, then $\sigma_T(x) \subseteq \sigma_S(x) \subseteq \sigma_{su}(S) \subseteq \mathbb{C}\backslash\{\lambda\}$ and so $x \in X_T(\mathbb{C}\backslash\{\lambda\})$. Hence the subspaces $X_T(\mathbb{C}\backslash\{\lambda\})$ and $Z_T(\lambda)$ do indeed coincide. Now, as noted in Satz 1 of [**19**], Förster has shown that the spaces $Z_T(\lambda)$ are constant on each connected component of $\rho_K(T)$. It follows that, for each component G of $\rho_K(T)$ and every $\lambda \in G$, we have

$$X_T(\mathbb{C}\backslash\{\lambda\}) = \bigcap_{n=1}^{\infty} (T-\lambda)^n X = \bigcap_{\mu \in G} \bigcap_{n=1}^{\infty} (T-\mu)^n X = X_T(\mathbb{C}\backslash G).$$

From this the proof of Theorem 1.1 can now be easily completed. Suppose that there exists a $\lambda_0 \in \partial\sigma_T(x)\backslash\sigma_{ap}(T)$. Then $T-\lambda_0$ is bounded below and consequently λ_0 belongs to $\rho_K(T)$. Now, if we let G denote the connected component of $\rho_K(T)$ which contains λ_0, then we have $G \subseteq \sigma_T(x)$. In fact, if there were a $\lambda \in G\backslash\sigma_T(x)$, then $x \in X_T(\mathbb{C}\backslash\{\lambda\}) = X_T(\mathbb{C}\backslash G)$ and therefore $\sigma_T(x) \cap G = \emptyset$, which is impossible because of $\lambda_0 \in \sigma_T(x) \cap G$. Thus $\lambda_0 \in G \subseteq \sigma_T(x)$. Since G is open, this contradicts our assumption that $\lambda_0 \in \partial\sigma_T(x)$.

Note that the preceding proof shows that actually $\partial\sigma_T(x) \subseteq \mathbb{C}\backslash\rho_K(T)$. However, not too much is gained by this stronger inclusion, since we shall see in Remark 1.6 that the identity $\mathbb{C}\backslash\rho_K(T) = \sigma_{ap}(T)$ holds under a mild assumption on T.

The following corollary applies, for instance, to any non-invertible isometry $T \in L(X)$, since in this case $\sigma(T)$ is the unit disc, while $\sigma_{ap}(T)$ is the unit circle.

Corollary 1.2. *Suppose that the operator $T \in L(X)$ has a spectrum with connected interior G for which $\sigma(T) = G^-$. If $\sigma_{ap}(T) = \partial\sigma(T)$, then for every $x \in X$ we have either $\sigma_T(x) = \sigma(T)$ or $\sigma_T(x) \subseteq \partial\sigma(T)$.*

Proof. If $\sigma_T(x) \neq \sigma(T)$ and $\sigma_T(x) \nsubseteq \partial\sigma(T)$, then there exist $\lambda_1 \in G\backslash\sigma_T(x)$ and $\lambda_2 \in \sigma_T(x) \cap G$. By the connectedness of G, there must be a point $\lambda_3 \in \partial\sigma_T(x) \cap G$. By Theorem 1.1, it follows that $\lambda_3 \in \sigma_{ap}(T) \cap G$, which contradicts our assumption that G and $\sigma_{ap}(T)$ are disjoint.

We now discuss another class of analytic spectral subspaces [**1**]: for any closed $F \subseteq \mathbb{C}$, let $\mathfrak{X}_T(F)$ consist of all $x \in X$ for which there exists an analytic function $f: \mathbb{C}\backslash F \to X$ such that $(T-\lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash F$. The spaces $\mathfrak{X}_T(F)$ may be called *glocal* analytic spectral subspaces because the analytic functions in their definition are globally defined, but depend on $x \in X$; in fact $f(\lambda) = (T-\lambda)^{-1}x$ for all $\lambda \in \rho(T) \cap (\mathbb{C}\backslash F)$. Evidently, $\mathfrak{X}_T(F) = X_T(F)$ for all closed $F \subseteq \mathbb{C}$, if T has the single valued extension property (SVEP); i.e. if, for every open $U \subseteq \mathbb{C}$, the only analytic solution of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$, see [**4**]. On the other hand, Liouville's theorem implies that $\mathfrak{X}_T(\emptyset) = \{0\}$ for any T, and since, by Proposition IV.3.6 of [**21**], $X_T(\emptyset) = \{0\}$ if and only if T has SVEP, we see that the two classes of spectral subspaces are identical precisely when T has SVEP. As witnessed by the recent development of local spectral theory in [**1**], [**7**], [**16**], [**18**], for operators without SVEP the spaces $\mathfrak{X}_T(F)$ are often more appropriate than their classical counterparts $X_T(F)$. We collect some basic properties.

Proposition 1.3. *For all $T \in L(X)$ and closed $F \subseteq \mathbb{C}$, the following properties hold.*

(a) $\mathfrak{X}_T(\emptyset) = \{0\}$, $\mathfrak{X}_T(F) = \mathfrak{X}_T(\sigma(T) \cap F)$, $\mathfrak{X}_T(F) \subseteq X_T(F)$, *and* $\ker T \subseteq \mathfrak{X}_T(\{0\})$.

(b) $\mathfrak{X}_T(F) = X$ *if and only if* $\sigma_{su}(T) \subseteq F$.

(c) *If* T *has SVEP, then* $\mathfrak{X}_T(F) = X$ *if and only if* $\sigma(T) \subseteq F$.

(d) $\mathfrak{X}_T\left(\bigcap_{\alpha \in A} F_\alpha\right) = \bigcap_{\alpha \in A} \mathfrak{X}_T(F_\alpha)$ *for any collection of closed convex sets* F_α, $(\alpha \in A)$.

(e) $\mathfrak{X}_{f(T)}(F) \subseteq \mathfrak{X}_T(\sigma(T) \cap f^{-1}(F))$ *for every analytic function* $f: U \to \mathbb{C}$ *on an open neighborhood* U *of* $\sigma(T)$.

Proof. Assertion (a) and one of the implications of assertion (b) are straightforward. The other one follows from the identity $X = \mathfrak{X}_T(\sigma_{su}(T))$, which, as pointed out to the authors by Jörg Eschmeier, is contained in a deep result due to Leiterer; see Theorem 5.1 of [**17**]. Since Lemma 3 of [**15**] shows that $\sigma_{su}(T) = \sigma(T)$ when T has SVEP, assertion (c) is, of course, a special case of (b), but actually this result can also be easily verified directly. To prove assertion (d), we may assume, after intersection with a compact disc containing $\sigma(T)$ if necessary, that each of the convex sets F_α is compact. Now, let $x \in \bigcap_{\alpha \in A} \mathfrak{X}_T(F_\alpha)$, and choose, for each $\alpha \in A$, an analytic function $f_\alpha: \mathbb{C} \backslash F_\alpha \to X$ such that $(T - \lambda) f_\alpha(\lambda) = x$ for all $\lambda \in \mathbb{C} \backslash F_\alpha$. For arbitrary α, $\beta \in A$, we obtain that $f_\alpha(\lambda) = (T - \lambda)^{-1} x = f_\beta(\lambda)$ for all $\lambda \in \rho(T) \cap (\mathbb{C} \backslash F_\alpha) \cap (\mathbb{C} \backslash F_\beta)$. Since the latter set is both non-empty and open and since the convexity of F_α and F_β implies that $(\mathbb{C} \backslash F_\alpha) \cap (\mathbb{C} \backslash F_\beta)$ is connected, the identity theorem for analytic functions yields that $f_\alpha \equiv f_\beta$ on $(\mathbb{C} \backslash F_\alpha) \cap (\mathbb{C} \backslash F_\beta)$ for all $\alpha, \beta \in A$. Therefore, if $F := \bigcap_{\alpha \in A} F_\alpha$, then the function $f: \mathbb{C} \backslash F \to X$ given by $f(\lambda) := f_\alpha(\lambda)$ for all $\lambda \in \mathbb{C} \backslash F_\alpha$ and $\alpha \in A$ is a well-defined analytic solution of the equation $(T - \lambda) f(\lambda) = x$ on $\mathbb{C} \backslash F$. This shows that $x \in \mathfrak{X}_T(F)$, which establishes the non-trivial part of the desired identity in (d). Finally, assertion (e) can be verified like Theorem 1.1.6 of [**4**].

REMARK 1.4. In connection with assertion (d), it is interesting to note that, in contrast to the case of the $X_T(F)$-spaces, intersections of arbitrary closed sets need not be preserved by the $\mathfrak{X}_T(F)$-spaces. More precisely, if T does not have SVEP, then there exist disjoint closed sets F, $G \subseteq \mathbb{C}$ for which $\mathfrak{X}_T(F) \cap \mathfrak{X}_F(G) \neq \{0\}$, while of course $\mathfrak{X}_T(F \cap G) = \mathfrak{X}_T(\emptyset) = \{0\}$. In fact, because of the lack of SVEP, there exists a non-trivial analytic function $f: U \to X$ on an open set $U \subseteq \mathbb{C}$ such that $(T - \lambda) f(\lambda) = 0$ for all $\lambda \in U$. Let $G := \mathbb{C} \backslash U$ and $F := \{\mu\}$ for some $\mu \in U$ with $f(\mu) \neq 0$. Then $f(\mu)$ belongs to $\ker(T - \mu)$ and consequently to $\mathfrak{X}_T(F)$. On the other hand, with the analytic function $g: U \to X$ given by $g(\mu) := f'(\mu)$ and $g(\lambda) := (\lambda - \mu)^{-1}(f(\lambda) - f(\mu))$ for all $\lambda \in U \backslash \{\mu\}$, we have $(T - \lambda) g(\lambda) = f(\mu)$ for all $\lambda \in U$ and therefore $f(\mu) \in \mathfrak{X}_T(G)$. Thus $0 \neq f(\mu) \in \mathfrak{X}_T(F) \cap \mathfrak{X}_T(G)$, which establishes the claim. Note that this shows in full generality what Sun has recently illustrated by a class of examples of bilateral weighted shifts on Hilbert spaces; see Proposition 1 of [**20**]. On the other hand, it would be interesting to know if the intersection property in part (d) of Proposition 1.3 holds for an arbitrary collection of polynomially convex compact sets. Here, the obstacle is, of course, that the union of two polynomially convex sets need not be polynomially convex.

It should be noted that, in general, the identity $\mathfrak{X}_T(F) = \mathfrak{X}_T(\sigma_{ap}(T) \cap F)$ will not be true for arbitrary closed $F \subseteq \mathbb{C}$. In fact, if this identity holds for $F = \sigma(T)$, then Proposition 1.3 shows that necessarily $\sigma_{su}(T) \subseteq \sigma_{ap}(T)$, which rules out, for instance, all non-invertible isometries. On the other hand, we have the following positive result.

THEOREM 1.5. *If $T \in L(X)$ and F is a closed subset of $\mathbb{C}$ such that $F \cap \sigma_{ap}(T) = \varnothing$, then $\mathfrak{X}_T(F) = \{0\}$.*

Proof. Choose an open neighborhood U of F with $U \cap \sigma_{ap}(T) = \varnothing$, and consider an arbitrary $x \in \mathfrak{X}_T(F)$. Theorem 1.1 implies that $\partial\sigma_T(x) \subseteq F \cap \sigma_{ap}(T) = \varnothing$ and therefore $\sigma_T(x) = \varnothing$. Hence, for every $\lambda \in U$, there exist an open neighborhood U_λ of λ contained in U and an analytic function $f_\lambda : U_\lambda \to X$ such that $(T - \mu)f_\lambda(\mu) = x$ for all $\mu \in U_\lambda$. Also, since $x \in \mathfrak{X}_T(F)$, there is an X-valued analytic function f_∞ on $U_\infty := \mathbb{C}\backslash F$ such that $(T - \mu)f_\infty(\mu) = x$ for all $\mu \in U_\infty$. Now, for every $\lambda \in U$, we have $f_\lambda \equiv f_\infty$ on $U_\lambda \cap U_\infty$. Indeed, for each $\mu \in U_\lambda \cap U_\infty$, we obtain $(T - \mu)(f_\lambda(\mu) - f_\infty(\mu)) = 0$ and therefore $f_\lambda(\mu) - f_\infty(\mu) = 0$, since μ belongs to U and U is disjoint from $\sigma_{ap}(T)$ and hence contains no eigenvalue of T. A similar argument shows that $f_\kappa \equiv f_\lambda$ on $U_k \cap U_\lambda$ for all $\kappa, \lambda \in U$. We conclude that there exists an analytic function f defined on the entire complex plane $\mathbb{C}$ for which $(T - \mu)f(\mu) = x$ for all $\mu \in \mathbb{C}$. Thus $x \in \mathfrak{X}_T(\varnothing)$ and therefore, by Liouville's theorem, $x = 0$.

The authors have been informed by Jörg Eschmeier that an alternative short proof of Theorem 1.5 can also be obtained from part (b) of Proposition 1.3 via duality theory. Indeed, by a general annihilator result due to Frunză, which is contained in the proof of Lemma 2 of **[11]**, the assumption $F \cap \sigma_{ap}(T) = \varnothing$ implies that

$$\mathfrak{X}_T(F) \subseteq \mathfrak{X}^*_{T^*}(\sigma_{ap}(T))^\perp = \mathfrak{X}^*_{T^*}(\sigma_{su}(T^*))^\perp = \{0\},$$

since we know from Propositions 1.3 that $\mathfrak{X}^*_{T^*}(\sigma_{su}(T)) = X^*$. Since this argument depends on the techniques developed in **[17]**, its brevity is, of course, somewhat deceptive.

REMARK 1.6. Since the proof of Theorem 1.1 shows that actually $\partial\sigma_T(x) \subseteq \mathbb{C}\backslash\rho_K(T)$ for all $T \in L(X)$ and $x \in X$, one might suspect that the conclusion of Theorem 1.5 also extends to the case of the Kato resolvent set. However, it is not true in general that $\mathfrak{X}_T(F) = \{0\}$ for all closed $F \subseteq \mathbb{C}$ with $F \subseteq \rho_K(T)$. Indeed, let $T \in L(X)$ be any surjective, but not injective operator, for instance the left shift on the Hilbert space $\ell^2(\mathbb{N})$. Then $\mathfrak{X}_T(\{0\})$ is non-trivial since $\{0\} \neq \ker T \subseteq \mathfrak{X}_T(\{0\})$, but $0 \in \rho_K(T)$ since $TX = X$. Note that such examples can only occur for operators without SVEP, since the identity $\rho_K(T) = \mathbb{C}\backslash\sigma_{ap}(T)$ holds whenever T has SVEP. In fact, the inclusion $\supseteq$ is standard, and to show the converse, let $\lambda \in \rho_K(T)$ be arbitrarily given. As noted in the proof of Theorem 1.1, the space $Z_T(\lambda)$ is closed and satisfies $(T - \lambda)Z_T(\lambda) = Z_T(\lambda)$. Since the restriction $S := T|Z_T(\lambda)$ has SVEP, we also know from Lemma 3 of **[15]** that $\sigma_{su}(S) = \sigma(S)$. Thus $S - \lambda$ is invertible on $Z_T(\lambda)$. Because $\lambda \in \rho_K(T)$, we have $\ker(T - \lambda) \subseteq Z_T(\lambda)$ and therefore $\ker(T - \lambda) = \{0\}$. Consequently $T - \lambda$ is bounded below on X, which shows that $\lambda \notin \sigma_{ap}(T)$ and hence completes the proof of the identity $\rho_K(T) = \mathbb{C}\backslash\sigma_{ap}(T)$. By duality, if the adjoint T^* has SVEP, then $\rho_K(T) = \mathbb{C}\backslash\sigma_{su}(T)$, since $\rho_K(T) = \rho_K(T^*) = \mathbb{C}\backslash\sigma_{ap}(T^*) = \mathbb{C}\backslash\sigma_{su}(T)$. In particular, it follows that $\rho_K(T) = \rho(T)$ if both T and T^* have SVEP.

Recall that an operator $T \in L(X)$ is said to be *decomposable* if, for every open covering $\{U, V\}$ of $\mathbb{C}$, there exist T-invariant closed linear subspaces Y and Z of X such that $\sigma(T|Y) \subseteq U$, $\sigma(T|Z) \subseteq V$, and $Y + Z = X$; see **[4]** and **[21]**. In the following, we shall be concerned with certain weaker notions, which have been fundamental to recent progress in local spectral theory **[1]**. The operator T is said to have *Bishop's property* (β) if, for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(T - \lambda)f_n(\lambda) \to 0$ locally uniformly on U, it follows that $f_n(\lambda) \to 0$ locally uniformly on U; see [**3**]. By Lemma IV.4.16 of [**21**], every decomposable operator has Bishop's property (β), and it is easily seen that this latter property implies that T has *Dunford's property* (C); i.e. that $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. Note that (C) implies SVEP by Proposition 1.2 of [**16**]. Moreover, it follows from Remark 1.6 and the duality theory for decomposable operators [**21**] that $\rho_K(T) = \rho(T)$ whenever T is decomposable. Finally recall from [**1**] that T is said to have the *decomposition property* (δ) if $X = \mathfrak{X}_T(U^-) + \mathfrak{X}_T(V^-)$ holds for every open covering $\{U, V\}$ of $\mathbb{C}$.

It is not difficult to see that an operator $T \in L(X)$ is decomposable if and only if it has both properties (β) and (δ). Much deeper are the recent results due to Albrecht and Eschmeier [**1**]: they establish the complete duality between (β) and (δ) and show that (β) characterizes, up to similarity, the restrictions of decomposable operators onto closed invariant subspaces, while (δ) characterizes, up to similarity, the quotients of decomposable operators. For related results and applications, for instance to the invariant subspace problem, we refer to [**7**], [**8**], [**16**], [**18**].

Theorem 1.5 allows a simple direct proof of the following fact, which generalizes a classical result on decomposable operators; cf. Corollary 2.1.4 of [**4**]. Note that Corollary 1.7 also follows from the duality between the properties (β) and (δ), but the argument given here avoids this heavy machinery.

Corollary 1.7. *If $T \in L(X)$ has property (δ), then $\sigma_{ap}(T) = \sigma(T)$.*

Proof. Given any $\lambda \in \mathbb{C} \backslash \sigma_{ap}(T)$, we choose closed sets F, $G \subseteq \mathbb{C}$ with $F \cap \sigma_{ap}(T) = \varnothing$ and $\lambda \notin G$ such that $\mathbb{C}$ is covered by the interiors of F and G. By property (δ), we have $X = \mathfrak{X}_T(F) + \mathfrak{X}_T(G)$. Since Theorem 1.5 shows that $\mathfrak{X}_T(F) = \{0\}$, we see that $X = \mathfrak{X}_T(G)$ and therefore $\sigma_{su}(T) \subseteq G$. Since λ is not an eigenvalue and does not belong to G, it follows that $\lambda \in \rho(T)$. Thus $\sigma_{ap}(T) = \sigma(T)$.

There are two general obstacles in the local spectral theory for an operator without SVEP, namely: the local spectrum $\sigma_T(x)$ will be empty for certain non-zero $x \in X$, and there need not exist a globally defined analytic solution of the equation $(T - \lambda)f(\lambda) = x$ outside $\sigma_T(x)$. To overcome the difficulties which arise from these problems, we permit ourselves to introduce the term *glocal spectrum* $\tau_T(x)$ for an arbitrary operator $T \in L(X)$ at the point $x \in X$ as the set $\tau_T(x) := \sigma(T) \cap \pi_T(x)$, where $\pi_T(x)$ denotes the intersection of all closed convex sets $F \subseteq \mathbb{C}$ for which $x \in \mathfrak{X}_T(F)$; for instance, F may be taken as any closed disc which contains $\sigma(T)$. The next theorem collects some basic properties of the glocal spectrum $\tau_T(x)$. In the following, let $V(\lambda, \delta)$ and $\nabla(\lambda, \delta)$ denote the open (respectively closed) disc in $\mathbb{C}$ with center $\lambda \in \mathbb{C}$ and radius $\delta \geq 0$. Also, for a not necessarily convex set $F \subseteq \mathbb{C}$, let $\mathrm{ex}(F)$ denote the extreme points of F, i.e. the set of all elements $\lambda \in F$ which cannot be represented as a non-trivial convex combination of finitely many elements of F.

Theorem 1.8. *For every $T \in L(X)$ and $x \in X$, the following properties hold.*
(a) *$x \in \mathfrak{X}_T(\tau_T(x))$ and $\sigma_T(x) \subseteq \tau_T(x)$.*
(b) *$\tau_T(x) = \varnothing$ if and only if $x = 0$.*
(c) *$\mathrm{ex}(\tau_T(x)) = \mathrm{ex}(\pi_T(x)) \subseteq \sigma_{ap}(T)$.*

Proof. From part (d) of Proposition 1.3 we see that $\pi_T(x)$ is the smallest closed convex set $F \subseteq \mathbb{C}$ which satisfies $x \in \mathfrak{X}_T(F)$. We conclude that $x \in \mathfrak{X}_T(\pi_T(x)) = \mathfrak{X}_T(\pi_T(x) \cap \sigma(T)) = \mathfrak{X}_T(\tau_T(x))$ and therefore $\sigma_T(x) \subseteq \tau_T(x)$. Assertion (b) follows from

(a) and the fact that $\mathfrak{X}_T(\emptyset)=\{0\}$. Finally, to prove assertion (c), let $\lambda \in \mathrm{ex}(\pi_T(x))$ be arbitrarily given. Then $\lambda \in \partial\pi_T(x)$ and, since $\sigma_T(x)\subseteq \pi_T(x)$, λ does not belong to the interior of $\sigma_T(x)$. Since we know from Theorem 1.1 that $\partial\sigma_T(x)\subseteq\sigma_{\mathrm{ap}}(T)$, it remains to consider the case that $\lambda \notin \sigma_T(x)$. In this case, there exists a $\delta>0$ and an analytic function $f: V(\lambda,\delta)\to X$ such that $(T-\mu)f(\mu)=x$ for all $\mu \in V(\lambda,\delta)$. Moreover, from $x \in \mathfrak{X}_T(\pi_T(x))$ we obtain an X-valued analytic function g on $U:=\mathbb{C}\backslash\pi_T(x)$ such that $(T-\mu)g(\mu)=x$ for all $\mu \in U$. Now, for $\epsilon>0$, the convex hull G_ϵ of the compact set $\pi_T(x)\backslash V(\lambda,\epsilon)$ is both compact and convex. Also, since $\lambda\in \mathrm{ex}(\pi_T(x))$, we have $\lambda \notin G_\epsilon$ so that G_ϵ is strictly contained in $\pi_T(x)$. By the minimality of $\pi_T(x)$, we conclude that $x \notin \mathfrak{X}_T(G_\epsilon)$. Consequently, for every $0<\epsilon\le\delta$, the analytic functions f and g cannot be identical on the non-empty open set $U\cap V(\lambda,\epsilon)$. Hence there exists a sequence of points $\lambda_n \in U$ such that $f(\lambda_n)\neq g(\lambda_n)$ for all $n\in\mathbb{N}$ and $\lambda_n\to\lambda$ as $n\to\infty$. Since $x_n:=f(\lambda_n)-g(\lambda_n)\neq 0$ and $(T-\lambda_n)x_n=0$, we obtain that $\lambda_n\in\sigma_p(T)$ for all $n\in\mathbb{N}$ and therefore $\lambda\in\sigma_{\mathrm{ap}}(T)$. In particular, it follows that $\mathrm{ex}(\pi_T(x))\subseteq\sigma(T)$ and therefore $\mathrm{ex}(\pi_T(x))\subseteq\mathrm{ex}(\tau_T(x))$. Conversely, given any $\lambda\in\mathrm{ex}(\tau_T(x))$, we have $\lambda\in\pi_T(x)$ so that λ can be written as a convex combination with non-zero coefficients of at most three extreme points of $\pi_T(x)$. Since we have shown that $\mathrm{ex}(\pi_T(x))\subseteq\sigma(T)$, these extreme points belong to $\tau_T(x)$ and hence have to be identical with λ. This shows that $\lambda\in\mathrm{ex}(\pi_T(x))$. The assertion follows.

If the operator $T\in L(X)$ does not have SVEP, then there exist non-zero $x\in X$ with empty local spectrum and hence with $\sigma_T(x)\neq\tau_T(x)$. But even for decomposable operators, the local spectrum can be strictly contained in the glocal spectrum, as witnessed by simple examples of multiplication operators on the space of continuous functions defined on an annulus in the plane. The reason is, of course, that the construction of $\tau_T(x)$ reflects only roughly the holes of $\sigma_T(x)$. For the applications to spectral inclusions in Section 2, these holes appear not to be a major issue. Actually, in some cases, it will be sufficient to replace the glocal spectrum by a suitable disc. The following theorem shows how to determine this disc in terms of the *local spectral radius* given by

$$r_T(x):=\limsup_{n\to\infty}\|T^n x\|^{1/n}\quad\text{for all}\quad T\in L(X)\text{ and }x\in X.$$

Recall from [**18**] that $r_T(x)=\max\{|\lambda|:\lambda\in\sigma_T(x)\}$ whenever T has SVEP and $0\neq x\in X$. The following result contains a glocal version of this formula which does not require SVEP.

THEOREM 1.9. *For every $T\in L(X)$ and non-zero $x\in X$, we have the following properties.*

(a) *The glocal spectral radius formula $r_T(x)=\max\{|\lambda|:\lambda\in\tau_T(x)\}$ holds.*

(b) *$\tau_T(x)\subseteq\nabla(\lambda,r(\lambda))$ for all $\lambda\in\mathbb{C}$, where $r(\lambda):=r_{T-\lambda}(x)$.*

(c) *There exists precisely one point $\lambda_0\in\mathbb{C}$ such that $r(\lambda_0)\le r(\lambda)$ for all $\lambda\in\mathbb{C}$.*

(d) *The disc $\nabla(\lambda_0,r(\lambda_0))$ has the smallest radius among all closed discs containing $\tau_T(x)$.*

Proof. The main ingredient of the proof is the identity $\mathfrak{X}_T(\nabla(0,r))=\{x\in X: r_T(x)\le r\}$ for all $r\ge 0$ from Proposition 2.1 of [**18**]. To show assertion (a), we first apply this identity with the choice $r:=\max\{|\lambda|:\lambda\in\tau_T(x)\}$. Then Theorem 1.8 implies that $x\in\mathfrak{X}_T(\tau_T(x))\subseteq\mathfrak{X}_T(\nabla(0,r))$ and therefore $r_T(x)\le r$. Conversely, we conclude from the same identity that

$x \in \mathfrak{X}_T(\nabla(0, r_T(x)))$; hence $\tau_T(x) \subseteq \nabla(0, r_T(x))$ by the definition of $\tau_T(x)$, and consequently $r \le r_T(x)$. This completes the proof of (a). Similarly, for every $\lambda \in \mathbb{C}$, we obtain that $x \in \mathfrak{X}_{T-\lambda}(\nabla(0, r(\lambda))) = \mathfrak{X}_T(\nabla(\lambda, r(\lambda)))$ and hence, by the definition of the glocal spectrum, that $\tau_T(x) \subseteq \nabla(\lambda, r(\lambda))$, which establishes (b). To prove assertion (c), let $r_0 := \inf\{r(\lambda)\colon \lambda \in \mathbb{C}\}$ and choose a sequence of points $\lambda_n \in \mathbb{C}$ such that $r(\lambda_n) \to r_0$ as $n \to \infty$. Because of $\emptyset \ne \tau_T(x) \subseteq \sigma(T) \cap \nabla(\lambda_n, r(\lambda_n))$, we have $|\lambda_n| \le r(\lambda_n) + v(T)$ for all $n \in \mathbb{N}$, where $v(T)$ denotes the spectral radius of T. Thus, without loss of generality, we may assume that (λ_n) converges to some $\lambda_0 \in \mathbb{C}$. Then clearly $r_0 \le r(\lambda_0)$. Now, if we suppose that $r_0 < r(\lambda_0)$, we can find an $n \in \mathbb{N}$ such that $\nabla(\lambda_n, r(\lambda_n))$ is strictly contained in $\nabla(\lambda_0, r(\lambda_0))$. Hence $\tau_T(x) \subseteq \nabla(\lambda_n, r(\lambda_n)) \subseteq \nabla(\lambda_0, r)$ for some $r < r(\lambda_0)$. We conclude that $x \in \mathfrak{X}_T(\tau_T(x)) \subseteq \mathfrak{X}_T(\nabla(\lambda_0, r)) = \mathfrak{X}_{T-\lambda_0}(\nabla(0, r))$ and therefore $r(\lambda_0) \le r$ by the identity noted at the beginning of the proof. This contradiction shows that $r(\lambda_0) = r_0$. Now suppose that $r(\lambda_0) = r_0 = r(\mu_0)$ holds for two different points $\lambda_0, \mu_0 \in \mathbb{C}$. From part (b) we obtain that $\tau_T(x) \subseteq \nabla(\lambda_0, r_0) \cap \nabla(\mu_0, r_0)$ and consequently $\tau_T(x) \subseteq \nabla(\lambda, r) \subseteq \nabla(\lambda_0, r_0) \cap \nabla(\mu_0, r_0)$ for a suitable disc $\nabla(\lambda, r)$ of radius $0 \le r < r_0$. As before, we conclude that $x \in \mathfrak{X}_T(\tau_T(x)) \subseteq \mathfrak{X}_T(\nabla(\lambda, r)) = \mathfrak{X}_{T-\lambda}(\nabla(0, r))$ and therefore $r_0 \le r(\lambda) \le r$, which is the desired contradiction. Thus $r_0 = r(\lambda_0)$ for exactly one $\lambda_0 \in \mathbb{C}$. Finally, the same kind of argument can be used to establish assertion (d). In fact, consider an arbitrary closed disc $\nabla(\lambda, r)$ which contains $\tau_T(x)$. It follows that $x \in \mathfrak{X}_T(\tau_T(x)) \subseteq \mathfrak{X}_T(\nabla(\lambda, r)) = \mathfrak{X}_{T-\lambda}(\nabla(0, r))$ and hence that $r_0 \le r(\lambda) \le r$, which completes the proof.

2. Commutators and spectral inclusions. Throughout this section, let $T \in L(X)$ and $S \in L(Y)$ be a given pair of operators on non-trivial complex Banach spaces X and Y, respectively, and let $C(S, T)\colon L(X, Y) \to L(X, Y)$ denote the corresponding commutator given by $C(S, T)(A) := SA - AT$ for all $A \in L(X, Y)$, where $L(X, Y)$ is the Banach space of all continuous linear mappings from X into Y. For brevity, we write $C := C(S, T)$ and $L := L(X, Y)$. One of our principal tools will be the following generalization of a result due to Foaiş and Vasilescu; cf. Theorem 2.4 of [**10**].

PROPOSITION 2.1. *If $F, K \subseteq \mathbb{C}$ are closed and $A \in \mathfrak{L}_C(K)$, then $A\mathfrak{X}_T(F) \subseteq \mathfrak{Y}_S(\overline{F + K})$.*

Proof. Because of $\mathfrak{X}_T(F) = \mathfrak{X}_T(\sigma(T) \cap T)$ and $\mathfrak{L}_C(K) = \mathfrak{L}_C(\sigma(C) \cap K)$, we may assume, without loss of generality, that both F and K are compact, in which case $F + K = (F + K)^-$. Let $x \in \mathfrak{X}_T(F)$, and consider an analytic function $f\colon \mathbb{C} \backslash F \to X$ such that $(T - \mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \backslash F$ and an analytic function $B\colon \mathbb{C} \backslash K \to L$ such that $(C - \xi)B(\xi) = A$ for all $\xi \in \mathbb{C} \backslash K$. Now, fix an arbitrary $\epsilon > 0$, and then put $U_\varepsilon := \mathbb{C} \backslash (F + K + \nabla(0, \epsilon))$. Choose an admissible contour Γ in $K + V(0, \epsilon)$ which surrounds K. Since $\lambda - \xi \in \mathbb{C} \backslash F$ for all $\lambda \in U_\varepsilon$ and $\xi \in \Gamma$, we may define

$$g_\varepsilon(\lambda) := \frac{1}{2\pi i} \int_\Gamma B(\xi) f(\lambda - \xi)\, d\xi \quad (\lambda \in U_\varepsilon).$$

From the analyticity of f and B it is easy to see that this definition does not depend on the choice of the admissible contour Γ and that the resulting function $g_\varepsilon\colon U_\varepsilon \to Y$ is analytic. It also follows that $g_\delta | U_\varepsilon = g_\varepsilon$ whenever $0 < \delta < \epsilon$. Hence the definition $g(\lambda) := g_\varepsilon(\lambda)$ for arbitrary $\epsilon > 0$ and $\lambda \in U_\varepsilon$ yields an analytic function $g\colon \mathbb{C} \backslash (F + K) \to Y$. To show that Ax

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belongs to $\mathfrak{Y}_S(F+K)$, it remains to verify that $(S-\lambda)g(\lambda)=Ax$ for all $\lambda \in \mathbb{C}\backslash(F+K)$. To this end, let $\epsilon>0$ and $\lambda \in U_\varepsilon$ be arbitrarily given. Then, for all $\xi \in \Gamma$, we obtain

$$(S-\lambda)B(\xi)f(\lambda-\xi)=(C-\xi)B(\xi)f(\lambda-\xi)+B(\xi)Tf(\lambda-\xi)+(\xi-\lambda)B(\xi)f(\lambda-\xi)$$
$$=Af(\lambda-\xi)+B(\xi)(T-(\lambda-\xi))f(\lambda-\xi)=Af(\lambda-\xi)+B(\xi)x$$

and consequently

$$(S-\lambda)g(\lambda)=\frac{1}{2\pi i}\int_\Gamma Af(\lambda-\xi)\,d\xi+\frac{1}{2\pi i}\int_\Gamma B(\xi)x\,d\xi.$$

Here, the first integral on the right hand side vanishes because of the analyticity of f on $\mathbb{C}\backslash F$. In the second integral, the analyticity of B on $\mathbb{C}\backslash K$ allows us to replace Γ by the circle Λ centered at 0 with radius $\|C\|+1$. Since $B(\xi)=(C-\xi)^{-1}A$ for all $\xi \in \Lambda$, we conclude from the Riesz functional calculus that

$$(S-\lambda)g(\lambda)=\frac{1}{2\pi i}\int_\Gamma B(\xi)x\,d\xi=\frac{1}{2\pi i}\int_\Lambda B(\xi)x\,d\xi=\frac{1}{2\pi i}\int_\Lambda (C-\xi)^{-1}Ax\,d\xi=Ax.$$

The assertion follows.

In the particular case that all of the operators S, T and C have SVEP, the preceding result applies directly to local spectra and shows that $\sigma_S(Ax)\subseteq\sigma_T(x)+\sigma_C(A)$ for all $x\in X$. This leads easily to certain spectral inclusions for S and T. In general, Proposition 2.1 will have to be applied to more subtle choices of the sets F and K, for instance to suitable glocal spectra. The following results contrast with the spectral inclusions $\sigma_{su}(C)\supseteq\sigma_{su}(S)-\sigma_{ap}(T)$ and $\sigma_{ap}(C)=\sigma_{ap}(S)-\sigma_{su}(T)$, which have been proved in [**5**]: while Davis and Rosenthal obtain information about the entire spectrum of the commutator $C=C(S,T)$, we aim to reach conclusions about the extent to which $\sigma(T)$ and $\sigma(S)$ are perturbed when passing from T to S, which will involve the local spectral theory for C at suitable A. The strongest results will be available in the intertwining case $SA=AT$ and the slightly more general case of asymptotic intertwining $r_C(A)=0$. By the glocal spectral radius formula of Theorem 1.9, the latter condition means precisely that $\tau_C(A)=\{0\}$, whenever A is non-zero.

The first result below makes no particular assumptions on S or T, but covers only certain subsets of the spectra. In the following remark we shall include some counterexamples to show that Proposition 2.2 cannot be improved in general, even if one insists on the strong intertwining condition $SA=AT$ which implies that $\tau_C(A)=\{0\}$. Note that part (d) of Proposition 2.2 generalizes a corresponding result of Grabiner [**12**] in the intertwining case.

PROPOSITION 2.2. *For arbitrary operators S and T, the following properties hold.*
(a) *If A is injective, then $\sigma_p(T)\subseteq\sigma_{ap}(S)-\tau_C(A)$.*
(b) *If A is surjective, then $\sigma_{su}(S)\subseteq\sigma_{su}(T)+\tau_C(A)$.*
(c) *If A has dense range, then $\sigma_{com}(S)\subseteq\sigma_{su}(T)+\tau_C(A)$.*
(d) *If A is injective and F is a component of $\sigma(T)$, then $F\cap[\sigma_{ap}(S)-\tau_C(A)]\neq\emptyset$.*

Proof. (a) By an obvious translation argument, it suffices to prove that 0 belongs to

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$\sigma_{ap}(S) - \tau_C(A)$ whenever 0 is an eigenvalue of T. Since we know from Theorem 1.8 that $A \in \mathfrak{L}_C(\tau_C(A))$, we conclude from Proposition 2.1 and basic facts on glocal subspaces that $A \ker T \subseteq A\mathfrak{X}_T(\{0\}) \subseteq \mathfrak{Y}_S(\tau_C(A))$. By the injectivity of A, we obtain that $\mathfrak{Y}_S(\tau_C(A)) \neq \{0\}$ and therefore $\tau_C(A) \cap \sigma_{ap}(S) \neq \emptyset$ by Theorem 1.5. Thus $0 \in \sigma_{ap}(S) - \tau_C(A)$, as desired.

(b) Now assume that A is surjective. Again from Proposition 1.3, Proposition 2.1 and Theorem 1.8, we conclude that

$$Y = AX = A\mathfrak{X}_T(\sigma_{su}(T)) \subseteq \mathfrak{Y}_S(\sigma_{su}(T) + \tau_C(A))$$

and thus $\sigma_{su}(S) \subseteq \sigma_{su}(T) + \tau_C(A)$.

(c) If both X and Y are reflexive, then clearly $-\tau_{C(T^*,S^*)}(A^*) = \tau_{C(S,T)}(A)$ so that the desired inclusion follows easily from part (a) by duality. However, the general case seems to require a somewhat different argument. With $F := \sigma_{su}(T) + \tau_C(A)$ we obtain exactly as before that $AX = A\mathfrak{X}_T(\sigma_{su}(T)) \subseteq \mathfrak{Y}_S(F)$. Since A has dense range, it follows that $\mathfrak{Y}_S(F)$ is dense in Y and hence has a trivial annihilator in Y^*. Now, if $\lambda \in \mathbb{C}\backslash F$, then $\ker(S^* - \lambda) \subseteq \mathfrak{Y}^*_{S^*}(\{\lambda\}) \subseteq \mathfrak{Y}_S(F)^{\perp} = \{0\}$ by the result of Frunză mentioned before; cf. Lemma 2 of [**11**]. It follows that λ is not an eigenvalue of S^* and therefore $\sigma_p(S^*) \subseteq F$. Since $\sigma_{com}(S) = \sigma_p(S^*)$, this yields the desired inclusion.

(d) Since $\sigma(T)$ is compact, we know from general topology that every component of $\sigma(T)$ is the intersection of a family of clopen subsets of $\sigma(T)$; cf. Theorem 6.1.23 of [**6**]. Hence, again by compactness, it suffices to show that every non-empty clopen subset F of $\sigma(T)$ meets $\sigma_{ap}(S) - \tau_C(A)$. This follows as in the proof of part (a). Indeed, again by Theorem 1.8 and Proposition 2.1 we have $A\mathfrak{X}_T(F) \subseteq \mathfrak{Y}_S(F + \tau_C(A))$. Since $\mathfrak{X}_T(F)$ contains the range of the non-zero projection associated with the spectral set F through the Riesz functional calculus, the injectivity of A implies that $\mathfrak{Y}_S(F + \tau_C(A)) \neq \{0\}$ and consequently $[F + \tau_C(A)] \cap \sigma_{ap}(S) \neq \emptyset$ by Theorem 1.5. Hence $F \cap [\sigma_{ap}(S) - \tau_C(A)] \neq \emptyset$, which completes the proof.

REMARK 2.3. Even in the Hilbert space setting, it is easy to construct an example of a bijective operator $T \in L(X)$ and a surjective operator $A \in L(X, Y)$ such that $T(\ker A)$ is strictly contained in $\ker A$. In this situation, let $Sy := ATx$ for all $y \in Y$ and $x \in X$ with $y = Ax$. This yields a well-defined operator $S \in L(Y)$, which satisfies $SA = AT$ and is surjective, but not injective. Thus 0 belongs both to $\rho(T)$ and to $\sigma_p(S)\backslash\sigma_{su}(S)$, which shows that $\sigma(S) \nsubseteq \sigma(T)$. By considering the dual operators in this situation, we obtain immediately an example of operators A, S, T where A is injective, $SA = AT$, and $\sigma(T) \nsubseteq \sigma(S)$. Finally observe that there are examples of bounded linear operators A, S, T on a Hilbert space such that A and A^* are injective, $SA = AT$, S is quasi-nilpotent, and the spectrum of T is the unit disc, see for instance [**9**] or [**13**]. Obviously, in this case, none of the sets $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma_{ap}(T^*)$, $\sigma_{su}(T^*)$ is contained in $\sigma(S) = \sigma(S^*)$. This shows that, even in the case $\tau_C(A) = \{0\}$, the spectral inclusions of Proposition 2.2 cannot be improved in general.

To obtain inclusions for the entire spectrum, we now impose assumptions from local spectral theory on the operators T and S. It should be noted that, by the characterization of the properties (β) and (δ) provided in [**1**], the following results apply to restrictions and quotients of decomposable operators.

THEOREM 2.4. *If A is injective and T has property (δ), then $\sigma(T) \subseteq \sigma_{ap}(S) - \tau_C(A)$. If A has dense range and S has property (C), then $\sigma(S) \subseteq \sigma_{su}(T) + \tau_C(A)$.*

Proof. First assume that A is injective and that T has property (δ). By Proposition 2.2, we know that $\sigma_p(T)$ is contained in $F := \sigma_{ap}(S) - \tau_C(A)$. Since $\sigma_p(T) \cup \sigma_{su}(T) = \sigma(T)$, it remains to show that $\sigma_{su}(T) \subseteq F$. To this end, fix an arbitrary open neighborhood U of F and choose an open set $V \subseteq \mathbb{C}$ such that $U \cup V = \mathbb{C}$ and $F \cap V^- = \emptyset$. Then $V^- + \tau_C(A)$ and $\sigma_{ap}(S)$ are disjoint. Hence we conclude from Theorem 1.5, Theorem 1.8, and Proposition 2.1 that $A\mathfrak{X}_T(V^-) \subseteq \mathfrak{Y}_S(V^- + \tau_C(A)) = \{0\}$ and therefore $\mathfrak{X}_T(V^-) = \{0\}$ by the injectivity of A. Since T has property (δ), we also have $X = \mathfrak{X}_T(U^-) + \mathfrak{X}_T(V^-)$. Consequently $X = \mathfrak{X}_T(U^-)$ and therefore $\sigma_{su}(T) \subseteq U^-$ for every open neighborhood U of F. Thus $\sigma_{su}(T) \subseteq F$, which completes the proof of the first assertion. Next assume that A has dense range and that S has property (C) and hence SVEP by Proposition 1.2 of [**16**]. Again, it follows from Lemma 3 of [**15**] that $\sigma(S) = \sigma_{su}(S)$. Hence we may proceed as in the proof of part (b) of Proposition 2.2. In fact, Theorem 1.8 and Proposition 2.1, together with some basic properties of spectral subspaces, imply that

$$Y = (AX)^- = (A\mathfrak{X}_T(\sigma_{su}(T)))^- \subseteq \mathfrak{Y}_S(\sigma_{su}(T) + \tau_C(A))$$

and therefore $\sigma(S) = \sigma_{su}(S) \subseteq \sigma_{su}(T) + \tau_C(A)$.

THEOREM 2.5. *Assume that S has property (C) and T has property (δ). If A is injective, then $\sigma(T) \subseteq \sigma_{ap}(S) - \sigma_C(A)$. If A has dense range, then $\sigma(S) \subseteq \sigma_{su}(T) + \sigma_C(A)$.*

Indeed, since it has been shown in Theorem 2.4 of [**16**] that, under the present assumptions on S and T, their commutator C has SVEP, the proof of the preceding result carries over verbatim with $\tau_C(A)$ replaced by $\sigma_C(A)$. Here the effect of the added assumption on S and T is the replacement of the glocal spectrum of C at A by a smaller and perhaps more familiar object, the local spectrum. This principle of substituting for $\tau_C(A)$ other, possibly less nebulous, but not necessarily smaller, sets may be applied in general: thanks to Theorem 1.9 we may always replace $\tau_C(A)$ by suitable larger closed discs.

As indicated earlier, both Theorems 2.4 and 2.5 apply whenever T is similar to the quotient of a decomposable operator and S is similar to the restriction of a decomposable operator onto one of its closed invariant subspaces. In this case, the two assertions in each of these results are closely connected, since we know from [**1**] that the properties (β) and (δ) are dual to each other. In fact, if T has (δ) and if the assumption on S is strengthened to (β) instead of (C), then the second inclusion in Theorem 2.5 follows immediately from the first by a straightforward duality argument, since it is easily seen that $-\sigma_{C(T^*,S^*)}(A^*) \subseteq \sigma_{C(S,T)}(A)$. Moreover, the two assertions are equivalent, if both X and Y are reflexive. A similar remark holds for Theorem 2.4. In this connection, it should be noted that it remains an intriguing open problem whether Bishop's property (β) and Dunford's property (C) are actually equivalent.

We conclude with some remarks on special cases and applications of the preceding results. First, recall that the operators T and S are said to be *quasi-similar* if there exist injective operators $A \in L(X, Y)$ and $B \in L(Y, X)$ with dense range such that $SA = AT$ and $TB = BS$. Evidently, Theorem 2.4 implies, in particular, that quasi-similarity preserves the spectrum for quotients and restrictions of decomposable operators. This improves a number of previous results in this direction and covers, for instance, the case of hyponormal and cohyponormal operators, see [**9**], [**13**], [**16**] for further information. Moreover, it follows that the same kind of spectral invariance holds for the weaker notion of asymptotic quasi-similarity, where the intertwining condition is relaxed to $r_{C(S,T)}(A) = r_{C(T,S)}(B) = 0$. This shows, in particular, that quasi-nilpotent equivalent quotients and restrictions of decomposable operators have the same spectrum **[4]**. Again, see **[16]** for a discussion of these and related results and a list of suitable references.

Another natural example arises in harmonic analysis, where the Fourier transform acts as an injective intertwiner for a given convolution operator and the corresponding multiplication operator. Indeed, given any regular Borel measure μ on a locally compact abelian group G, Theorem 2.4 implies that $\sigma(\mu)$ equals the closure of the range $\hat{\mu}(\Gamma)$ of the Fourier-Stieltjes transform $\hat{\mu}$ on the dual group Γ provided that the operator of convolution by μ on the group algebra $L_1(G)$ has property (δ). We refer to **[16]** and **[18]** for details and applications to the spectral theory of convolution operators.

Finally, Theorem 1.9 and Proposition 2.1 immediately imply the following result from **[18]**: if $r := r_{C(S,T)}(A)$, then $A\mathfrak{X}_T(F) \subseteq \mathfrak{Y}_S(F + \nabla(0, r))$ for all closed $F \subseteq \mathbb{C}$. Inclusions of this type are very useful in the theory of spectral decompositions; cf. **[4]**, **[21]**. As shown in **[18]**, it follows, for instance, that the properties (β) and (δ) are preserved under limits in the spectral distance of operators **[21]** and that, for a wide class of Banach algebras, the multipliers with property (δ) form a closed subalgebra of the multiplier algebra.

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