

SUBGROUPS OF THE ADJOINT GROUP OF A RADICAL RING

B. AMBERG, O. DICKENSCHIED AND YA. P. SYSAK

ABSTRACT. It is shown that the adjoint group R° of an arbitrary radical ring R has a series with abelian factors and that its finite subgroups are nilpotent. Moreover, some criteria for subgroups of R° to be locally nilpotent are given.

1. Introduction. Let R be an associative ring, not necessarily with an identity element. The set of all elements of R is a semigroup with identity element $0 \in R$ under the operation $a \circ b = a + b + ab$ for all a and b in R . The group of all invertible elements of this semigroup is called the *adjoint group* of R and denoted by R° . Following Jacobson [5], a ring R is *radical* if $R = R^\circ$, which means that R coincides with its Jacobson radical. An important subclass of the class of radical rings is the class of *nil rings*, i.e., rings R such that for every element a of R there exists a positive integer $n = n(a)$ with $a^n = 0$. The relation between a radical ring and its adjoint group R° has been investigated for instance in [11], [8] and [1]. Here we will study how the nilpotency structure of a radical ring R is influenced when finiteness and nilpotency conditions are imposed on certain subgroups of R° .

The first theorem shows that the adjoint group of an arbitrary radical ring satisfies some solubility condition. Recall that a group G is an *SN-group* if it has a series with abelian factors (see [9], Vol. 1, pp. 9f and 25).

THEOREM A. *The adjoint group R° of every radical ring R is an SN-group in which every finite subgroup is nilpotent.*

Using Zelmanov's theorem on the restricted Burnside problem (see [14] and [15]), we can deduce the following from Theorem A.

COROLLARY 1. *Let G be a subgroup of the adjoint group of a radical ring and suppose that one of the following conditions holds:*

- (a) *G is locally finite,*
- (b) *G has finite exponent,*
- (c) *G is an n -Engel group for some $n \geq 1$,*
- (d) *G is locally artinian.*

Then the group G is locally nilpotent.

Received by the editors August 15, 1996.

The second author thanks the Federal State of Rheinland-Pfalz for a scholarship (LGFG). The third author is grateful to the Deutsche Forschungsgemeinschaft for financial support and the Department of Mathematics of the University of Mainz for its excellent hospitality during the preparation of this paper.

AMS subject classification: Primary: 16N20; secondary: 20F19.

©Canadian Mathematical Society 1998.

Note that a locally noetherian subgroup of the adjoint group of a radical ring need not be locally nilpotent. This can be seen from an example of Neroslavskii [8] of a radical algebra R over the field with p elements ($p > 2$) whose adjoint group R° contains a polycyclic subgroup G isomorphic with the semi-direct product $\langle a \rangle \rtimes \langle b \rangle$ of an infinite cyclic subgroup $\langle a \rangle$ and a cyclic normal subgroup $\langle b \rangle$ of order p with $a^{-1}ba = b^2$.

However, if a polycyclic subgroup G of the adjoint group of a radical ring R generates R as a ring, then G must be nilpotent. This follows from a theorem of Roseblade (see [10], Corollary 8.4.14) and is also contained in our next result. A group G is *strongly restrained* if there exists an integer n such that for every two elements $x, y \in G$, the subgroup $\langle x^{(y)} \rangle$ of G can be generated by n elements; see [6]. Obviously every polycyclic-by-finite group and every group with finite exponent is strongly restrained.

THEOREM B. *Let G be a strongly restrained subgroup of the adjoint group R° of a finitely generated radical ring R . If G generates R as a ring, then the ring R and so also the group G are nilpotent.*

Let x and y be elements of a ring R and define $[x, {}_k y]$ by $[x, {}_0 y] = x$ and $[x, {}_{k+1} y] = [[x, {}_k y], y]$, where $[x, y] = xy - yx$ denotes the ring commutator. A ring R is an *Engel ring* if for every two elements $x, y \in R$ there exists an integer $k \geq 0$ such that $[x, {}_k y] = 0$. Moreover, R is an *n -Engel ring* if $[x, {}_n y] = 0$ for all $x, y \in R$.

COROLLARY 2. *The following conditions on the finitely generated radical ring R are equivalent.*

- (1) R is an n -Engel ring for some $n \geq 1$,
- (2) R is a nilpotent ring,
- (3) R° is an n -Engel group for some $n \geq 1$,
- (4) R° is a nilpotent group.

Theorem B and Corollary 2 do not hold in general when the ring R is not finitely generated, since even a commutative radical ring need not be nil. Also it was noted by Brown (see [4], remarks before Corollary 4.6), that a radical ring R is locally nilpotent if it is generated as a ring by a locally polycyclic subgroup of R° with finite torsion-free rank. Note also that in statements (1) and (3) of Corollary 2, the n -Engel condition cannot be replaced by the Engel condition, as a well-known example of Golod shows (see for instance [10], Theorem 6.2.9).

Recall that the class \mathcal{E} of *elementary amenable* groups (see [7]) is the smallest class of groups which

- (1) contains all abelian and all finite groups,
- (2) is extension closed,
- (3) is closed under direct unions.

It is easy to see that \mathcal{E} contains every group whose finitely generated subgroups have an ascending series with locally finite or locally nilpotent factors.

THEOREM C. *Every elementary amenable subgroup of the adjoint group of a nil ring is locally nilpotent.*

Golod's example shows that for each integer $r \geq 1$, there exists a finitely generated non-nilpotent subgroup of the adjoint group of a nil ring in which every r -generator subgroup is nilpotent. The last theorem strengthens this result slightly. Indeed, for each $r > 0$, every finitely generated non-nilpotent subgroup of the adjoint group of a nil ring contains a non-nilpotent subgroup, which is finitely generated but not r -generated.

THEOREM D. *Let G be a subgroup of the adjoint group of a nil ring. If there exists an integer $r > 0$ such that every finitely generated subgroup of G is r -generated or nilpotent, then the group G is locally nilpotent.*

Since every group with finite Prüfer rank satisfies the hypothesis of Theorem D, we obtain the following corollary, which can also be proved using a result of Wilson (see [13], Theorem 1).

COROLLARY 3. *Let G be a subgroup of the adjoint group of a nil ring.*

- (a) *If every finitely generated subgroup of G has finite Prüfer rank, then G is locally nilpotent.*
- (b) *If there is an integer $r > 0$ such that every two-generator subgroup of G has finite Prüfer rank bounded by r , then G is locally nilpotent.*

The notation is standard and can for instance be found in [9], [5] and [10].

2. General properties of the adjoint group of a radical ring. The proof of Theorem A requires the following two lemmas, the first of which is due to Neroslavskii.

LEMMA 1 ([8], PROPOSITION 6). *Let R be a radical algebra over the field F .*

- (a) *If F has characteristic zero, then the adjoint group R° is torsion-free.*
- (b) *If F has prime characteristic p , then every element of finite order in R° is a p -element.*

LEMMA 2. *Let R be a radical ring such that $\bigcap_n p^n R = 0$ for some prime p . Then every periodic subgroup of the adjoint group R° of R is a p -group.*

PROOF. Let g be an element of R° with prime order q . There exists a minimal integer $n \geq 0$ such that $g \notin p^n R$. Hence $n > 0$ and $g + p^n R$ has order q in the adjoint group of the ring $p^{n-1}R/p^n R$, which is obviously a radical algebra over the field with p elements. Thus $q = p$ by Lemma 1, and the lemma is proved. ■

The following simple observation will be used without further reference. Let G be a subgroup of the adjoint group of the radical ring R . If S is a radical subring of R and T an ideal of S , then $(G \cap S)/(G \cap T)$ is isomorphic with a subgroup of the adjoint group $(S/T)^\circ$ of the ring S/T .

PROOF OF THEOREM A. To prove the first assertion, consider R as a left R -module. We will use the notion of a series of left R -submodules defined in the same way as in [9], Vol. 1, pp. 9f for groups. It follows from Zorn's Lemma that the module R has a composition series, *i.e.*, a series with no proper refinement. Let T/S be a factor of this series. As a radical ring, R has no irreducible left R -modules; see [5], p. 9, Theorem 2. Thus $R(T/S) = 0$, and so $RT \subseteq S$. In particular TS as well as ST are contained in S . Hence S is a (two-sided) ideal of T . The left ideals S and T are both radical subrings of R . Since $TT \subseteq S$, the factor ring T/S has trivial multiplication and the factor group $T^\circ/S^\circ \cong (T/S)^\circ$ is abelian. Thus every composition series of the module R forms a series of R° with abelian factors, and R° is an SN -group.

Assume there exist finite non-nilpotent subgroups of the adjoint group of a radical ring R and let G be such a group with minimal order. We may suppose that R is the radical join $\langle G \rangle_{\text{rad}}$ of G , *i.e.*, the smallest radical subring of R which contains G . By Zorn's Lemma there exists an ideal M of R , which is maximal with respect to $M \cap G = 0$. The subgroup $\bar{G} = \{g + M \mid g \in G\}$ of $(R/M)^\circ$ is isomorphic with G . Passing to the factor ring $R/M = \langle \bar{G} \rangle_{\text{rad}}$, we may suppose that each non-trivial ideal of R intersects G non-trivially.

Assume that $pR \neq R \neq qR$ for two different primes p and q . Then $G \cap pR \neq G$ and $G \cap qR \neq G$. The choice of G implies that the groups $G \cap pR$ and $G \cap qR$ are both nilpotent. The factor group $G/(G \cap pR)$ is isomorphic with a subgroup of the adjoint group of the radical algebra R/pR over the field with p elements. By Lemma 1, $G/(G \cap pR)$ is a p -group. Similarly $G/(G \cap qR)$ is a q -group and hence $G = (G \cap pR)(G \cap qR)$. By Fitting's Theorem (see [9], part 1, p. 49, Theorem 2.18), G is nilpotent. This contradiction shows that there exists a prime p such that $qR = R$ for all primes $q \neq p$.

We define a descending chain of ideals of R as follows:

$$\begin{aligned} R_0 &= R, \\ R_{\alpha+1} &= \bigcap_{n=1}^{\infty} p^n R_\alpha \quad \text{for every ordinal } \alpha \text{ and} \\ R_\lambda &= \bigcap_{\alpha < \lambda} R_\alpha \quad \text{for every limit ordinal } \lambda. \end{aligned}$$

Assume that there exists a prime $q \neq p$ such that $qR_\alpha \neq R_\alpha$ for some ordinal α and let α be minimal with this property. Clearly $\alpha = \beta + 1$ for some ordinal β , and so

$$R_\alpha = \bigcap_{n=1}^{\infty} p^n R_\beta.$$

If $a \in R_\alpha$, then $a = qb$ for some $b \in R_\beta$. For every $n \in \mathbb{N}$, there exist integers u and v such that $uq + vp^n = 1$. Hence

$$b = uqb + vp^n b = ua + vp^n b \in p^n R_\beta$$

for every $n \in \mathbb{N}$. Thus $b \in R_\alpha$ and so $qR_\alpha = R_\alpha$. This contradiction shows that $qR_\alpha = R_\alpha$ for every prime $q \neq p$ and every ordinal α .

Let $S = \bigcap_{\alpha} R_{\alpha}$. Then $pS = S$. Since also $qS = S$ for every prime $q \neq p$, the additive group S^+ of S is divisible. The torsion subgroup of S^+ forms an ideal T of R , which is the ring-theoretical direct sum of its primary components. Assume that more than one of them is non-trivial. Then T can be written as the direct sum of two non-trivial ideals T_1 and T_2 of R . The intersection $G \cap T_1$ is a non-trivial normal subgroup of G . Thus $|G/(G \cap T_1)| < |G|$ and so the factor group $G/(G \cap T_1)$ is nilpotent. Similarly $G/(G \cap T_2)$ is nilpotent. As $(G \cap T_1) \cap (G \cap T_2) = 0$, the group G is isomorphic with a subgroup of the direct product of $G/(G \cap T_1)$ and $G/(G \cap T_2)$ and hence is also nilpotent. This contradiction shows that T^+ is a q -group for some prime q .

Assume $q \neq p$ and $T \neq 0$. If $t \in T$, then $q^n t = 0$ for some $n \in \mathbb{N}$. As $qR = R$, every $r \in R$ is of the form $r = q^n s$ with $s \in R$. Therefore $rt = 0$ and $tr = 0$, which shows that T is contained in the two-sided annihilator of R . Thus the non-trivial subgroup $G \cap T$ is central in G . By the minimality of G , the factor group $G/(G \cap T)$ and so G are nilpotent, a contradiction. Hence T^+ is a p -group.

As $pS = S$, it follows as above that T is contained in the two-sided annihilator of S . Thus $G \cap T$ is a p -group. The group $(S/T)^+ = S^+/T^+$ is torsion-free and divisible, since S^+ is divisible. Hence S/T can be considered as an algebra over the field of rationals, and so $(S/T)^{\circ}$ is torsion-free by Lemma 1. This implies that $G \cap S = G \cap T$. Since every factor group $(G \cap R_{\alpha})/(G \cap S)$ is a p -group by Lemma 2, the group $G/(G \cap S)$ and so also G are p -groups, a contradiction. The theorem is proved. ■

PROOF OF COROLLARY 1. If (a) holds, the statement is trivial. Condition (b) implies (a), since every SN -group of finite exponent is locally finite; this can be seen in the same way as Theorem 7.16 of [9] by replacing Kostrikin's theorem by Zelmanov's theorem (see [14] and [15]). To deal with condition (c) note that every non-trivial finitely generated SN -group has a non-trivial finite epimorphic image. This implies that every n -Engel subgroup of the adjoint group of a radical ring is locally nilpotent by [6], Corollary 6 and Theorem A. Finally, condition (d) implies (a), since every artinian SN -group is soluble; see [9], part 1, p. 71, Corollary. ■

3. Finitely generated radical rings.

PROOF OF THEOREM B. Since each of the finitely many generators of the ring R can be written in terms of finitely many elements of G , it follows that R is generated as a ring by a finitely generated subgroup H of G . The subgroup H is an SN -group by Theorem A. This implies that each non-trivial finitely generated subgroup of H has a non-trivial finite epimorphic image. Hence H is polycyclic-by-finite by [6], Theorem A. Let $\Delta_{\mathbb{Z}}(H)$ be the augmentation ideal of the group ring $\mathbb{Z}H$ and let the \mathbb{Z} -linear map $\alpha: \Delta_{\mathbb{Z}}(H) \rightarrow R$ be defined by $(h - 1)\alpha = h$. Then α is a ring epimorphism whose kernel I is an ideal of $\mathbb{Z}H$. Thus $\Delta_{\mathbb{Z}}(H)/I$ is a radical ring, since it is isomorphic with R . Therefore $\Delta_{\mathbb{Z}}(H)/I$ is the Jacobson radical of $\mathbb{Z}H/I$. By Roseblade's theorem, the Jacobson radical of every epimorphic image of $\mathbb{Z}H$ is a nil ideal. Since $\mathbb{Z}H$ is also noetherian (see [9], Vol. 1, p. 163, Corollary from Lemma 5.35) it follows from [10], Theorem 2.6.23, that $\Delta_{\mathbb{Z}}(H)/I$

is a nilpotent ring. Thus the ring R and so also the group G are nilpotent. The theorem is proved. ■

If x and y are elements of a ring R and $k \geq 0$, then it is easy to see that

$$(*) \quad [x, {}_k y] = \sum_{i=0}^k (-1)^{i+1} \binom{k}{i} y^i x y^{k-i}.$$

Recall that a ring R is called a *PI-ring* if it satisfies a polynomial identity with integer coefficients (see [10], Definition 6.1.2).

PROOF OF COROLLARY 2. If the ring R satisfies condition (1), then it is a *PI-ring* by the above formula (*). Hence R is nilpotent by theorems of Amitsur-Procesi and Braun; see [10], Theorems 6.3.3 and 6.3.39. Thus (1) implies (2), and it follows immediately from (*) that (2) also implies (1). Moreover, it is obvious that (2) implies (4) and that (4) implies (3). It remains to show that (3) implies (2). Indeed, if R° is an n -Engel group, it is strongly restrained (see [6], Lemma 1) and hence the ring R is nilpotent by Theorem B. ■

The ring of all rationals with even numerators and odd denominators is a radical ring in which no non-trivial finitely generated subring is radical. Hence Theorem B does not give information about arbitrary finitely generated subrings of a radical ring. However, more can be said for nil rings, because every subring of a nil ring is again nil. Moreover, it is easy to see that a radical ring is nil if each of its finitely generated subrings is radical. The following proposition gives another sufficient condition for a radical ring to be nil. We will need some definitions. If R is a radical ring, then its adjoint group R° operates on its additive group R^+ via the rule $r^s = r + rs$ for all $r \in R^+$ and $s \in R^\circ$. The semi-direct product $G(R) = R^\circ \ltimes R^+$ is called the *associated group* of R . The “diagonal subgroup” S of $G(R)$ gives rise to a so-called triple factorization

$$G(R) = R^\circ \ltimes R^+ = S \ltimes R^+ = R^\circ S$$

of $G(R)$. This allows to use results on radical rings in order to study factorized groups, and vice versa; see [1] or Chapter 6.1 of [2].

PROPOSITION. *The radical ring R is nil if and only if every element of $G(R)$ which is contained in R^+ is a right Engel element. In particular, if $G(R)$ is an Engel group, then R is nil.*

PROOF. If r is an element of R , let r^+ and r° denote the corresponding elements of $G(R)$ which are contained in R^+ and R° , respectively. By (x, y) we denote the commutator of x and y in the group $G(R)$, and write $(x, {}_0 y) = x$ and $(x, {}_{k+1} y) = ((x, {}_k y), y)$ for all $x, y \in G(R)$ and each $k \geq 1$. By induction on k we have $(r^+, {}_k s^\circ) = (r s^k)^+$ for all $r, s \in R$ and $k \geq 1$. For $s = r$, the proposition follows. ■

It seems unknown, whether the adjoint group of every nil ring is an Engel group.

4. Elementary amenable groups. It is clear that a maximal abelian subgroup of the adjoint group of a radical ring forms a radical subring. The next lemma implies a similar statement for every maximal locally nilpotent subgroup of the adjoint group of a nil ring.

LEMMA 3. *Let R be a nil ring. If G is a locally nilpotent subgroup of R° , the subring $\langle G \rangle_{\text{rg}}$ of R generated by G is locally nilpotent.*

PROOF. If H is a finitely generated subgroup of G , then H is nilpotent and the subring $\langle H \rangle_{\text{rg}}$ of R is finitely generated. Therefore by Theorem B the ring $\langle H \rangle_{\text{rg}}$ is nilpotent. Thus $\langle G \rangle_{\text{rg}}$ is a locally nilpotent ring. ■

Lemma 3 can also be proved without using Theorem B by induction on the nilpotency class of H , because the centre of H generates a nilpotent ideal in $\langle H \rangle_{\text{rg}}$.

Combining Corollary 1 and Lemma 3 we obtain the following.

COROLLARY 4. *If the adjoint group of a nil ring R satisfies one of the conditions of Corollary 1, then the ring R is locally nilpotent.*

LEMMA 4. *Let the ring R be generated by the subgroup G of R° . If N is a normal subgroup of G and S is the subring of R generated by N , then $I = SR + S$ is an ideal of R . If in addition S is nilpotent, then I is a nilpotent ideal of R .*

PROOF. It is easy to see that G normalizes the adjoint group S° of the subring S generated by N . Hence for every $g \in G$ and every $s \in S$, there exists an element $t \in S$ such that $s \circ g = g \circ t$. This implies that $sg = gt + t - s \in gS + S$. Thus $Sg \subseteq gS + S$ for every $g \in G$, from which it follows that $SR + S \subseteq RS + S$, since R is generated by G . Similarly $RS + S \subseteq SR + S$. Hence

$$I = SR + S = RS + S$$

is an ideal of R .

By induction on k we obtain $I^k = S^k R + S^k$ for every integer $k \geq 1$. Indeed

$$\begin{aligned} I^{k+1} &= (S^k R + S^k)(SR + S) \\ &= S^k(RS + S)R + S^k(RS + S) \\ &= S^k(SR + S)R + S^k(SR + S) \\ &= S^{k+1}R + S^{k+1}. \end{aligned}$$

Therefore, if $S^n = 0$, then also $I^n = S^n R + S^n = 0$. ■

LEMMA 5. *Let x and y be elements of a ring R with identity element 1. If $y \in R^\circ$, then*

$$\langle (1+y)^{-m}x(1+y)^m \mid m \geq 0 \rangle_{\text{rg}} = \langle (1+y)^{-m}[x, m, y] \mid m \geq 0 \rangle_{\text{rg}}.$$

In particular, if $y^n = 0$ for some $n \geq 1$, then the subring $\langle (1+y)^{-m}x(1+y)^m \mid m \geq 0 \rangle_{\text{rg}}$ of R is generated by at most $2n$ elements.

PROOF. Put $a_m = (1+y)^{-m}x(1+y)^m$ and $b_m = (1+y)^{-m}[x, {}_m y]$ for every $m \geq 0$. It is easy to see that

$$(1+a)^{-1}b(1+a) = b + (1+a)^{-1}[b, a]$$

for all $a, b \in R$. By induction,

$$a_i = \sum_{j=0}^i \binom{i}{j} b_j$$

for $i \geq 0$. Hence for $m \geq 0$, we have

$$\langle a_0, \dots, a_m \rangle_{\text{rg}} = \langle b_0, \dots, b_m \rangle_{\text{rg}},$$

which implies the first assertion. The second statement follows from (*). ■

LEMMA 6. *Let N be a locally nilpotent subgroup of the adjoint group R° of a nil ring R . If $x \in R^\circ$ normalizes N , then the subgroup $\langle N, x \rangle_{\circ}$ of R° generated by $N \cup \{x\}$ is locally nilpotent.*

PROOF. We will use a formal identity 1 for R . Clearly

$$x' \circ y \circ x = (1+x)^{-1}y(1+x)$$

for every $y \in R$. Hence the adjoint conjugation by x is an automorphism of the ring R . Thus x normalizes the adjoint group M° of the subring $M = \langle N \rangle_{\text{rg}}$ of R . Moreover, M is locally nilpotent by Lemma 3.

We will prove that $\langle N, x \rangle_{\text{rg}}$ is a locally nilpotent ring. If S is a finitely generated subring of $\langle N, x \rangle_{\text{rg}}$, then S is contained in $\langle K, x \rangle_{\text{rg}}$ for some finitely generated subring K of M . Hence it is sufficient to show that $\langle K, x \rangle_{\text{rg}}$ is nilpotent. The subring

$$L = \langle (1+x)^{-n}K(1+x)^n \mid n \geq 0 \rangle_{\text{rg}}$$

of R is finitely generated by Lemma 5. As x normalizes M° and K is contained in M , it follows that L is a subring of the locally nilpotent ring M and so is nilpotent. Clearly L° is normalized by x , which means that $L(1+x) = (1+x)L$. Hence $Lx \subseteq L + xL$ and it follows by induction that

$$Lx^i \subseteq L + xL + x^2L + \dots + x^iL$$

for every $i \geq 1$. Since $x^{n+1} = 0$ for some $n \geq 1$, we have that

$$\langle L, x \rangle_{\text{rg}} = \langle x \rangle_{\text{rg}} + L + xL + x^2L + \dots + x^nL,$$

where the additive group of $\langle x \rangle_{\text{rg}}$ is finitely generated. As the additive group L^+ of the finitely generated nilpotent ring L is also finitely generated, it follows that the additive group of the ring $\langle L, x \rangle_{\text{rg}}$ is finitely generated. Thus $\langle L, x \rangle_{\text{rg}}$ is nilpotent by [11]. In particular, $\langle K, x \rangle_{\text{rg}}$ is nilpotent. This proves the lemma. ■

To prove Theorem C we need the following characterization of the class E of elementary amenable groups. Let \mathfrak{X}_1 denote the class of all finitely generated abelian-by-finite groups and let

$$\begin{aligned} \mathfrak{X}_0 &= \{1\}, \\ \mathfrak{X}_{\alpha+1} &= (L\mathfrak{X}_\alpha)\mathfrak{X}_1 \quad \text{for each ordinal } \alpha \text{ and} \\ \mathfrak{X}_\lambda &= \bigcup_{\alpha < \lambda} \mathfrak{X}_\alpha \quad \text{for each limit ordinal } \lambda. \end{aligned}$$

If $\mathfrak{X} = \bigcup_\alpha \mathfrak{X}_\alpha$, then we have the following.

LEMMA 7 ([7], LEMMA 3.1).

- (a) $\mathfrak{X} = E$.
- (b) Each \mathfrak{X}_α is subgroup closed.

This lemma associates with every group $G \in E$ the least ordinal α such that $G \in \mathfrak{X}_\alpha$.

PROOF OF THEOREM C. By Lemma 7 it suffices to show that for every ordinal α , each finitely generated \mathfrak{X}_α -subgroup of the adjoint group of the nil ring R is nilpotent. Assume that this is false, and let α be the least ordinal for which there exists a counterexample $G \in \mathfrak{X}_\alpha$. Then $\alpha = \beta + 1$ for some ordinal β . Thus $G \in \mathfrak{X}_\alpha = (L\mathfrak{X}_\beta)\mathfrak{X}_1$, which means that G contains a normal subgroup $H \in L\mathfrak{X}_\beta$ such that the factor group G/H is abelian-by-finite. By the choice of α , it follows that the group H is locally nilpotent, so that we may suppose that the ring R is generated by G and that H is the Hirsch-Plotkin radical of G .

Let F/H be the Hirsch-Plotkin radical of G/H . As G/H is finitely generated and abelian-by-finite, the subgroup F/H is nilpotent and has finite index in G/H . We claim that $F = H$. Indeed, for each element x of F , the subgroup $V = \langle H, x \rangle_\circ$ of G generated by $H \cup \{x\}$ is locally nilpotent by Lemma 6. On the other hand, the cyclic subgroup V/H is subnormal in G/H . It follows that the locally nilpotent subgroup V is subnormal in G and hence is contained in H ; see [9], Theorem 2.31. Thus $H = F$ has finite index in the finitely generated group G , and so is likewise finitely generated. As H is locally nilpotent, this implies that H is even nilpotent.

Let $S = \langle H \rangle_{\text{rg}}$ be the subring of R generated by H . Then S is a finitely generated ring which is locally nilpotent by Lemma 3. It follows that S is nilpotent, and so $I = SR + S$ is a nilpotent ideal of R by Lemma 4. Consider the factor ring R/I , which is generated by the subgroup $\tilde{G} = \{g + I \mid g \in G\}$ of $(R/I)^\circ$. As $H \subseteq I$, it follows that \tilde{G} is an epimorphic image of the finite group G/H . Thus \tilde{G} is a finite nilpotent group by Theorem A. As R/I is generated by \tilde{G} , Lemma 3 implies that R/I is a nilpotent ring. Hence the ring R and so also the group G are nilpotent. This contradiction proves the theorem. ■

5. Groups with finite Prüfer rank and generalizations. The following result will play an essential role in the proof of Theorem D. Recall that the standard polynomial of degree n is given by

$$S_n(x_1, \dots, x_n) = \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi)x_{1\pi} \dots x_{n\pi}.$$

PROPOSITION ([3], PROPOSITION). *Let A be an abelian group of finite Prüfer rank at most r . Then the endomorphism ring $\text{End}(A)$ of A satisfies the standard polynomial identity of degree $2r$.*

LEMMA 8. *Let the group G be the cartesian product of nilpotent groups of finite Prüfer rank bounded by some positive integer r . Then G has an abelian normal subgroup A such that the factor group G/A is embedded in the group of units of a ring satisfying the standard polynomial identity of degree $2r$.*

PROOF. Let H be a nilpotent group with Prüfer rank $r(H) \leq r$. If A is a maximal abelian normal subgroup of H , then $C_H(A) = A$; see [9], Vol. 1, Lemma 2.19.1. Hence the factor group H/A can be embedded into the group of units $\text{End}(A)^*$ of $\text{End}(A)$. By the preceding proposition $\text{End}(A)$ satisfies the standard polynomial identity of degree $2r$, as $r(A) \leq r$.

Now let $G = \text{Cr}_{i \in I} H_i$ be the cartesian product of the nilpotent groups H_i with $r(H_i) \leq r$. If A_i is a maximal abelian normal subgroup of H_i for each i , then $A = \text{Cr}_{i \in I} A_i$ has the desired properties. ■

The next lemma is probably known.

LEMMA 9. *Let R be a ring with an identity element which satisfies a polynomial identity and G a finitely generated subgroup of the group R^* of units of R . If G has no non-abelian free subgroups, then G is hyperabelian-by-finite.*

PROOF. Note first that a ring S contains a non-trivial nilpotent ideal if and only if $N(S) \neq 0$, where $N(S)$ is the set of all $a \in S$ such that there exists some $n = n(a) \in \mathbb{N}$ with $(sa)^n = 0$ for every $s \in S$; see Theorem 2.6.17 and Proposition 2.6.26 of [10]. It follows by transfinite induction that R has an ascending series of ideals J_α with $J_0 = 0$ such that the factor ring $J_{\alpha+1}/J_\alpha$ is nilpotent for each ordinal α and non-trivial whenever $N(R/J_\alpha) \neq 0$. There exists an ordinal τ such that $J = J_\tau = J_{\tau+1}$, which implies that $N(R/J) = 0$. Thus by Theorem 6.1.26 of [10], the ring R/J is embedded in the ring of $n \times n$ matrices over some commutative ring K . If $H_\alpha = G \cap (1 + J_\alpha)$ for every $\alpha < \tau$, we obtain that the H_α form an ascending series of normal subgroups of G which finally reaches $H = G \cap (1 + J)$ such that its factors $H_{\alpha+1}/H_\alpha$ are nilpotent. Moreover, G/H is isomorphic with a subgroup of $R^*/(1 + J)$. As the latter is embedded in $\text{GL}(n, K)$, the group G/H is linear over the commutative ring K .

Assume that G/H contains a non-abelian free subgroup F/H . Then the extension F of H by F/H splits and thus G contains a subgroup isomorphic with F/H , a contradiction. Therefore G/H contains no non-abelian free subgroups. As G/H is finitely generated, we may suppose that the ring K is likewise finitely generated. It follows from [12], Theorem 13.31, that the group G/H is soluble-by-finite. Hence G is a hyperabelian-by-finite group. ■

The following two lemmas are perhaps of independent interest. The Hirsch number of the polycyclic group G will be denoted by $r_0(G)$. Moreover, for each $n \geq 1$, let G^n be the subgroup of G generated by all elements g^n with $g \in G$ and G' the derived subgroup of G .

LEMMA 10. *Let F be a finitely generated torsion-free nilpotent group.*

- (a) *If p is a prime then F has a subgroup V such that V/V^p is an elementary abelian p -group of rank $r_0(F)$.*
- (b) *If T is a finite nilpotent group, then $r(T \times F) = r(T) + r(F)$.*

PROOF. It is well known that every finitely generated nilpotent group is polycyclic. Thus all subgroups of F are finitely generated. For each such subgroup U of F let $d(U)$ denote the minimal number of generators of U . We will prove (a) by induction on $r = r_0(F)$, the result being clear if $r = 0$. Suppose now that $r \geq 1$. Then F has a normal subgroup F_1 with $r_0(F_1) = r - 1$ such that the factor group F/F_1 is infinite cyclic. Thus $F = \langle f \rangle \times F_1$ for some element f of F . By induction, F_1 has a subgroup W such that W/W^p has rank $r - 1$. It follows from a theorem of Glushkov (see [9], part 2, p. 140) that

$$r_0(F_1) \geq r_0(W) = r(W) \geq r(W/W^p) = r_0(F_1).$$

So $r_0(F_1) = r_0(W)$, which means that W and hence also W^p have finite index in F_1 . As the finitely generated group F_1 has only finitely many subgroups of index $|F_1 : W^p|$, the characteristic core $C = \bigcap \{(W^p)^\alpha \mid \alpha \in \text{Aut}(F_1)\}$ of W^p in F_1 is a characteristic subgroup of finite index in F_1 . Thus f induces an automorphism of the finite factor group F_1/C . Therefore some power $g = f^n$ of f centralizes F_1/C , which implies that $[W, g] \subseteq [F_1, g] \subseteq W^p$. Thus if V is the subgroup of F generated by W and g , then W^p is a normal subgroup of V and V/W^p is the direct product of the elementary abelian p -group W/W^p of rank $r - 1$ and the infinite cyclic group generated by $\bar{g} = gW^p$. It follows that V has a normal subgroup N such that V/N is an elementary abelian p -group of rank r . Hence the rank of V/V^p is at least r , while on the other hand Glushkov's theorem implies that the rank of V/V^p is at most $r(F) = r_0(F) = r$. Hence part (a) is proved.

To prove (b) let T_1 be a subgroup of T such that $r(T) = d(T_1)$. If $\bar{T}_1 = T_1/T_1'$, then it is easy to see that $d(T_1) = d(\bar{T}_1)$. As \bar{T}_1 is a finite abelian group, we have $r(T) = d(\bar{T}_1) = d((\bar{T}_1)_p)$ for some prime p , where $(\bar{T}_1)_p$ denotes the primary p -component of \bar{T}_1 . Thus T has an elementary abelian p -section of rank $r(T)$, while F has an elementary abelian p -section of rank $r_0(F) = r(F)$ by (a). Therefore $T \times F$ has an elementary abelian p -section of rank $r(T) + r(F)$, which shows that $r(T \times F) \geq r(T) + r(F)$. As clearly also $r(T \times F) \leq r(T) + r(F)$, the lemma is proved. ■

LEMMA 11. *Let G be a finitely generated nilpotent group and r a positive integer. If every subgroup of finite index in G is an r -generator group, then the Prüfer rank of G does not exceed r .*

PROOF. Clearly the maximal periodic normal subgroup T of G is finite and the factor group G/T is torsion-free. We will first show that $r(G) = r(T) + r_0(G)$.

As a polycyclic group, G has a torsion-free normal subgroup F of finite index. Since $T \cap F = 1$, we have $r(G) \geq r(T \times F) = r(T) + r(F)$ by Lemma 10. It follows from Glushkov's theorem and the finiteness of G/F that $r(F) = r_0(F) = r_0(G)$. Hence $r(G) \geq r(T) + r_0(G)$.

On the other hand, we have $r(G) \leq r(T) + r(G/T)$, where $r(G/T) = r_0(G/T) = r_0(G)$, since G/T is torsion-free. Thus $r(G) = r(T) + r_0(G)$.

Let U be a subgroup of G such that $d(U) = r(G)$. Then $T \cap U$ is the maximal periodic normal subgroup of U . By the above equation, we obtain $r(U) = r(T \cap U) + r_0(U)$ as well as $r(U) = d(U) = r(G) = r(T) + r_0(G)$. As $r(T \cap U) \leq r(T)$ and $r_0(U) \leq r_0(G)$, this implies $r_0(U) = r_0(G)$. Hence U must have finite index in G and thus $r(G) = d(U) \leq r$. The lemma is proved. ■

LEMMA 12. *Let R be a residually nilpotent ring and G a subgroup of R° , in which every finitely generated non-nilpotent subgroup is an r -generator group for some fixed $r > 0$. Then G is elementary amenable.*

PROOF. If $G_n = G \cap R^n$ for each n , then we have $\bigcap_n G_n = 0$, since R is a residually nilpotent ring. As each factor group G/G_n is isomorphic with a subgroup of the nilpotent group $(R/R^n)^\circ$, it follows that the group G is residually nilpotent.

In order to prove that $G \in \bar{E}$, it suffices to show that $U \in \bar{E}$ for every r -generator subgroup U of G , since every nilpotent group is elementary amenable. Therefore we may suppose that G is an r -generator group.

If there is an $n \in \mathbb{N}$ such that G/G_n contains a subgroup H/G_n of finite index which is not an r -generator group, then H is finitely generated but not an r -generator group. Thus H is nilpotent by hypothesis, which implies that G is nilpotent-by-finite and in particular $G \in \bar{E}$.

Hence we may suppose that for each $n \in \mathbb{N}$, every subgroup of finite index in the group G/G_n is an r -generator group. By Lemma 11, the Prüfer ranks of the groups G/G_n are bounded by r . As G can be embedded into the cartesian product of the factor groups G/G_n , Lemma 8 implies that G has an abelian normal subgroup A such that G/A is embedded into the group of units of a ring satisfying a polynomial identity. Let $H/A = \langle h_1A, \dots, h_kA \rangle$ be a finitely generated non-nilpotent subgroup of G/A . Then $H/A = XA/A$ for the finitely generated subgroup $X = \langle h_1, \dots, h_k \rangle$, which cannot be nilpotent, since it has H/A as an epimorphic image. Thus X and so also H/A are r -generator groups. It follows that the hypothesis of the lemma carries over from the group G to its epimorphic image G/A . In particular, G/A has no non-abelian free subgroups. It follows from Lemma 9 that G/A and hence also G are hyperabelian-by-finite. Therefore G is elementary amenable and the lemma is proved. ■

PROOF OF THEOREM D. Of course we may suppose that G is finitely generated and that it generates the ring R . We will show that the ring R is nilpotent. Consider the ideal $R^\omega = \bigcap_{n \in \mathbb{N}} R^n$ of R . Then R/R^ω is a residually nilpotent ring which is generated by the group $\bar{G} = \{g + R^\omega \mid g \in G\}$, in which every finitely generated non-nilpotent subgroup is an r -generator group. By Lemma 12, the group \bar{G} is elementary amenable and thus locally nilpotent by Theorem C. Hence R/R^ω is a locally nilpotent ring by Lemma 3. As R is generated by any finite set of generators of G , the ring R/R^ω is even nilpotent. Hence $R^n = R^{n+1}$ for some positive integer n . Since R^n is also a finitely generated ring, it follows from [5], p. 200, Proposition 2, that $R^n = 0$. The theorem is proved. ■

PROOF OF COROLLARY 3. Statement (a) follows immediately from Theorem D. To prove (b), let every two-generator subgroup U of G have finite Prüfer rank bounded by r . Then each such U is nilpotent by (a). Therefore G is strongly restrained and thus locally nilpotent by Theorem B. ■

Theorem D has the following consequence.

COROLLARY 5. *If G is a group in which every finitely generated non-nilpotent subgroup is an r -generator group for some fixed $r \in \mathbb{N}$ and if the augmentation ideal $\Delta_K(G)$ of the group ring KG over some commutative ring K with identity is a nil ideal, then G is a locally nilpotent group.*

PROOF. Let $R = \Delta_K(G)$ be the augmentation ideal of the group ring KG and $G^* = \{g - 1 \mid g \in G\}$. Then G^* is a subgroup of R° which is isomorphic with G . Hence $G \cong G^*$ is locally nilpotent by Theorem D. ■

REFERENCES

1. B. Amberg and O. Dickenschied, *On the adjoint group of a radical ring*. Canad. Math. Bull. (3) **38**(1995), 262–270.
2. B. Amberg, S. Franciosi, and F. de Giovanni, *Products of groups*. Oxford University Press, New York, 1992.
3. B. Amberg and Ya. P. Sysak, *Locally soluble products of two minimax subgroups*. Proceedings of ‘Groups-Korea 1994’, W. de Gruyter, Berlin, 1995, 8–14.
4. K. A. Brown, *The Nullstellensatz for certain group rings*. J. London Math. Soc. (2), **26**(1982), 425–434.
5. N. Jacobson, *Structure of Rings*. Amer. Math. Soc. Colloq. Publ. **37**, 1964.
6. Y. K. Kim and A. H. Rhemtulla, *Weak maximality condition and polycyclic groups*. Proc. Amer. Math. Soc. (3) **123**(1995), 711–714.
7. P. H. Kropholler, P. A. Linnell, and J. A. Moody, *Applications of a new K -theoretic theorem to soluble group rings*. Proc. Amer. Math. Soc. (3) **104**(1988), 675–684.
8. O. M. Neroslavskii, *Structures that are connected with radical rings (Russian)*. Vesci Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk (2) **134**(1973), 5–10.
9. D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*. Springer-Verlag, Berlin-Heidelberg-New York, 1972.
10. L. H. Rowen, *Ring Theory*. Academic Press, New York, 1988.
11. J. F. Watters, *On the adjoint group of a radical ring*. J. London Math. Soc. **43**(1968), 725–729.
12. B. A. F. Wehrfritz, *Infinite Linear Groups*. Springer-Verlag, Berlin, 1973.
13. J. S. Wilson, *Two-generator conditions for residually finite groups*. Bull. London Math. Soc. **23**(1991), 239–248.
14. E. Zelmanov, *The solution of the restricted Burnside problem for groups of odd exponent*. Izv. Akad. Nauk SSSR, Ser. Mat. (1) **54**(1990), 42–59.
15. ———, *The solution of the restricted Burnside problem for 2-groups*. Mat. Sb. (4) **182**(1991), 568–592.

Fachbereich Mathematik der Universität Mainz
D-55099 Mainz
Germany

email: amberg@mat.mathematik.uni-mainz.de
email: dickenschied@mat.mathematik.uni-mainz.de

Institute of Mathematics
Ukrainian Academy of Science
252601 Kiev

Ukraine
email: mathkiev@imat.gluk.apc.org