The four points therefore form a tetrahedron inscribed in the ellipsoid

$$x^2 / A + y^2 / B + z^2 / C = 3 / m,$$

and self-conjugate w.r.t. the homothetic and concentric virtual ellipsoid

$$x^2 / A + y^2 / B + z^2 / C = - 1 / m;$$

whence we deduce that they form a circumscribed tetrahedron of the real ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{3m}.$$

These ellipsoids are similar to Legendre's equimomental ellipsoid.

The Number of Lines that may lie upon a Surface of given Order.

By H. W. Richmond.

The greatest number of straight lines that can lie upon a surface of order n (not being a ruled surface) is unknown, except if n is three. Salmon and Clebsch have shown that the points of contact of lines which have a four-point contact with the surface lie upon a locus of order $n(11n - 24)$, the intersection of the surface of order n with another of order $11n - 24$. Since a straight line lying wholly on the former surface must form a part of this locus, the number $n(11n - 24)$ is an upper limit to the number of lines; if n is three, this gives 27, the correct number. But for values of $n > 3$, it is improbable that this limit[1] can be reached.

At the Colloquium, held at St Andrews in July 1930, the surfaces

$$\text{(i)}\ x^3 + y^3 + z^3 + t^3 = 0; \quad \text{(ii)}\ x^4 + y^4 = u^4 + v^4;$$

were considered, and a question was asked as to the number of lines lying on each. By a generalization of the result a theorem is obtained in regard to surfaces of any order on which lie an unexpectedly large number of lines.

"*There exist surfaces of order n, not ruled and without conical points, on which lie $3n^2$ straight lines; of these $n(n + 2)$ may be real.*"

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[1] See *Encyklopädie d. math. Wissenschaften*, Band III, Teil 2, p. 665.

The surface

$$ax^n + by^n + cz^n + dt^n = 0$$

proves the first part of the statement. The equations

$$ax^n + by^n = 0, \qquad cz^n + dt^n = 0,$$

separately represent n planes, passing respectively through the lines

$$x = y = 0, \text{ and } z = t = 0.$$

In combination they define n^2 lines of intersection of a plane of the first set with one from the second set; and all these lie on the surface. The surface therefore has upon it n^2 lines joining any one of n points on $x = y = 0$ to any one of n points on $z = t = 0$. By associating x with z and y with t, or x with t and y with z, we obtain two further sets of n^2 lines on the surface, $3n^2$ in all.

Few of these lines are real if x, y, z, t are real planes. If n is odd, three are real. If n is even, eight are real when two of the constants a, b, c, d are positive and two negative; otherwise none are real. A more interesting result may be derived. Let X, Y, Z, T be real linear functions of the coordinates, and consider the real surface

$$(X + iY)^n + (X - iY)^n = (Z + iT)^n + (Z - iT)^n.$$

The left hand member breaks into n real factors, and if equated to zero represents n real planes; similarly the right hand member. The n^2 lines of intersection of these planes are real lines lying on the surface: they join n real points of the line $X = Y = 0$ to n real points of $Z = T = 0$. The other lines of the surface belong to the two systems whose equations are

$$\text{(i)}\ (X + iY) = \alpha (Z + iT); \qquad (X - iY) = \beta (Z - iT):$$
$$\text{(ii)}\ (X + iY) = \alpha (Z - iT); \qquad (X - iY) = \beta (Z + iT):$$

where α and β are any n^{th} roots of unity: the lines are imaginary except when α and β are conjugate imaginary roots, *i.e.* except when $\alpha\beta = 1$. The surface therefore contains $3n^2$ straight lines of which $n(n+2)$ are real.

Among known quartic surfaces that of Weddle contains 25 straight lines all of which may be real; the surface has six conical points. From the foregoing work we see that the surface

$$X^4 - 6X^2Y^2 + Y^4 = Z^4 - 6Z^2T^2 + T^4$$

contains 48 straight lines, 24 of them real and 24 imaginary.