NORMAL COMPLETIONS OF SMALL CATEGORIES

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1. Introduction. In (**3**), Isbell proposed a stronger definition for the term "complete category" and obtained many nice theorems for the resulting notion of a completion. In particular, he showed (**3**, Theorem 3.20) that completions of small categories satisfy a strong normality condition.

In this paper we shall always use the term "complete" in the weaker sense of Freyd (**1**). (In (**3**), Isbell used the term "small-complete" for this weaker notion.) We shall prove that the completions, in the sense of Freyd, of small categories also enjoy the same normality condition, provided they admit at least one bicategory structure. (The complete categories in the sense of Isbell always admit bicategory structures; see the remark following Proposition 2.4.)

In what follows, we let $\mathscr{A} \subseteq \mathscr{B}$ mean that $\mathscr{A}$ is a full subcategory of $\mathscr{B}$. Moreover, if $\mathscr{A} \subseteq \mathscr{B}$, then $\mathscr{A} = \mathscr{B}$ means that each object of $\mathscr{B}$ is equivalent to an object in $\mathscr{A}$. The terms "inf-complete" and "sup-complete" shall be used in place of Freyd's terms "left-complete" and "right-complete". We prefer the suggestive "inf" and "sup" terminology given by Lambek in (**6**), because of the useful analogy with lattices. Following (**6**), an *infimum* (or inf) is a generalized inverse limit (i.e., a left root in Freyd's nomenclature). *Supremum* (or sup) is defined dually.

Definition. A complete category $\mathscr{C}$ is a *completion* of the small category $\mathscr{A}$ if $\mathscr{A} \subseteq \mathscr{C}$ and there exists no proper intermediate complete category (i.e., $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{C}$ and $\mathscr{B}$ complete imply $\mathscr{B} = \mathscr{C}$).

$\mathscr{C}$ is a *normal completion* of the small category $\mathscr{A}$ if it is a completion which has no proper intermediate inf-complete or sup-complete categories.

(In this paper we shall not consider completions of large categories.)

We note that "bicategory structure" is defined in (**2**, p. 573). Our main result is the following theorem.

THEOREM 1.1. *Let $\mathscr{A}$ be a small full subcategory of the complete category $\mathscr{C}$. If $\mathscr{C}$ admits a bicategory structure (I, P), then there exists a normal completion $\mathscr{N}$ of $\mathscr{A}$ such that $\mathscr{A} \subseteq \mathscr{N} \subseteq \mathscr{C}$.*

The embedding of $\mathscr{A}$ into $\mathscr{N}$ preserves all infimums and supremums (of small diagrams) that might exist in $\mathscr{A}$. The embedding of $\mathscr{N}$ into $\mathscr{C}$ is retractable via a functor from $\mathscr{C}$ onto $\mathscr{N}$.

Finally, every normal completion of a small category is well-powered and co-well-powered, and therefore is complete in the sense of Isbell.

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We should mention that the above theorem was proved in (**3**) for a less general case.

We feel that some of our preliminary results are of individual interest. In particular, Corollary 2.2 resolves a question raised in (**3**; **4**). Proposition 2.6 yields useful smallness conditions which generalize a smallness result used in (**5**).

2. Preliminaries.

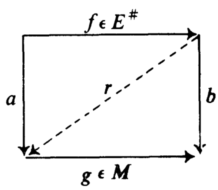
Notation. We shall let $M_{\mathscr{C}}$, or simply M, denote the class of all monomorphisms of $\mathscr{C}$. The class of epimorphisms shall be denoted by $E_{\mathscr{C}}$, or simply E. The extremal monomorphisms shall be denoted by $M_{\mathscr{C}}^{\#}$ or $M^{\#}$. By definition, $f \in M^{\#}$ if and only if $f \in M$ and $f = gh$ with $g \in E$ imply g is an equivalence. $E_{\mathscr{C}}^{\#}$ or $E^{\#}$ shall denote the dually defined class of extremal epimorphisms.

If I is any class of monomorphisms, then we define an *I-Subobject* to be a subobject which is representable by a monomorphism in I. The category $\mathscr{C}$ is *I-well-powered* if every object has no more than a set of I-subobjects. The dual terms *P-Quotient* and *P-co-well-powered*, for $P \subseteq E$, have the obvious definitions.

If $\mathscr{A}$ is a full subcategory of $\mathscr{C}$, then Prod $\mathscr{A}$ shall denote the full subcategory of all products $\prod A_i$ for which $A_i \in \mathscr{A}$ for all i. We use Sub $\mathscr{A}$ for the full subcategory of all objects which are (representatives of) subobjects of objects in $\mathscr{A}$. The terms I-Sub$(\mathscr{A})$ (for $I \subseteq M$), Coprod$(\mathscr{A})$, P-Quot$(\mathscr{A})$ (for $P \subseteq E$) and Quot $\mathscr{A}$ are defined analogously.

The category $\mathscr{C}$ is *factorable* if each morphism f of $\mathscr{C}$ can be factored as $f = me$ for $m \in M$ and $e \in E$.

PROPOSITION 2.1. *Let $\mathscr{C}$ have pullbacks* (see **1**, p. 40, for definition). *Consider the following diagram, where $bf = ga$, $f \in E^{\#}$, and $g \in M$ are given:*



Then there exists a morphism r such that $rf = a$ and $gr = b$.

Proof. Let $\bar{g}$ and $\bar{b}$ be morphisms such that $b\bar{g} = g\bar{b}$ is a pullback diagram. Then there exists h such that $\bar{g}h = f$ and $\bar{b}h = a$. However, it can be shown that $\bar{g} \in M$ as $g \in M$ and $b\bar{g} = g\bar{b}$ form a pullback diagram. Thus, $\bar{g}$ is an equivalence as $\bar{g}h = f \in E^{\#}$. Choose $r = \bar{b}(\bar{g})^{-1}$.

COROLLARY 2.2. *If $\mathscr{C}$ has pullbacks, then $E^{\#}$ is composition-closed.*

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Proof. Let bf be a given composition with $b \in E^{\#}$ and $f \in E^{\#}$. Assume that $bf = ga$ with $g \in M$. Then there exists r such that $b = gr$ which implies that g is an equivalence.

COROLLARY 2.3. *Let $\mathscr{C}$ have pullbacks. Let $f = me$ and $f = \bar{m}\bar{e}$ be two factorizations of f with $m, \bar{m} \in M$ and $e, \bar{e} \in E^{\#}$. Then there exists an equivalence h such that $e = h\bar{e}$ and $\bar{m} = mn$.*

The proof is straightforward.

The following type of proposition is well known and can, in effect, be found in (**4**).

PROPOSITION 2.4. *Let $\mathscr{C}$ be complete. Then $(M, E^{\#})$ is a bicategory structure on $\mathscr{C}$ if $\mathscr{C}$ is either $E^{\#}$-co-well-powered or well-powered.*

Proof (sketch). In view of the previous results, it suffices to show that each morphism f of $\mathscr{C}$ factors as $f = me$, where $m \in M$ and $e \in E^{\#}$. (For then $\mathscr{C}$ is factorable and the corollaries to Proposition 2.1 apply.) If $\mathscr{C}$ is $E^{\#}$-co-well-powered and f is given, it suffices to choose e to be the largest extremal epimorphism through which f factors (e turns out to be a supremum). Thus, f factors as $f = me$ and m must be a monic, for if $mg = mh$, then f factors through ce, where c is the coequalizer (or difference cokernel) of g and h. A similar argument applies if $\mathscr{C}$ is well-powered.

Remark. The same argument shows that if $\mathscr{C}$ is complete in the sense of Isbell, then $(M, E^{\#})$ and $(M^{\#}, E)$ are bicategory structures on $\mathscr{C}$.

PROPOSITION 2.5. *Let $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{C}$ be given, where $\mathscr{C}$ has arbitrary products. Assume that $\mathscr{B} = I$-*Sub(Prod $\mathscr{A}$)*, where I is a class of monomorphisms such that $hg \in I$ implies $g \in I$. Then each $B \in \mathscr{B}$ is an I-subobject of a product, $\prod A_\alpha$, via $g\colon B \to \prod A_\alpha$, where $g \in I$, $A_\alpha \in \mathscr{A}$ for all α, and where the projections of B onto the A_α's are distinct (i.e., $p_\alpha g = p_\beta g$ implies $\alpha = \beta$).*

Proof. Let $B \in \mathscr{B}$ be given. By our hypothesis, there is a map $f\colon B \to \prod A_i$, where $f \in I$ and $A_i \in \mathscr{A}$ for all i. Define an equivalence relation by $i \sim j$ if and only if $p_i f = p_j f$ and let α vary over the resulting set of equivalence classes. Define A_α so that $A_\alpha = A_i$ for $i \in \alpha$ and $g\colon B \to \prod A_\alpha$ so that $p_\alpha g = p_i f$ for $i \in \alpha$. Let $h\colon \prod A_\alpha \to \prod A_i$ be defined so that $p_i h = p_\alpha$, where α is the equivalence class containing i. Then $g \in I$ since $hg = f \in I$ and g has the required properties.

PROPOSITION 2.6. *Let $\mathscr{C}$ have pullbacks and arbitrary products and let $\mathscr{A} \subseteq \mathscr{C}$ be a small subcategory. Then:*

(1) *Let $\mathscr{B} =$* Sub(Prod $\mathscr{A}$)*. Each $X \in \mathscr{C}$ then has only a set of extremal quotients in $\mathscr{B}$;*

(2) *Let $\mathscr{C}$ also have pushouts* (see **1**, pp. 41–42 for definition) *and let $\mathscr{B} = M^{\#}$-*Sub(Prod $\mathscr{A}$)*. Each $X \in \mathscr{C}$ then has only a set of quotients in $\mathscr{B}$;*

(3) *Let (I, P) be a bicategory structure on $\mathscr{C}$ and let $\mathscr{B} = I$-*Sub(Prod $\mathscr{A}$).

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Then each $X \in \mathscr{C}$ has only a set of P-Quotients in $\mathscr{B}$, and $\mathscr{B}$ is a reflective subcategory of $\mathscr{C}$.

Proof. (1) Let $e: X \to B$ be an extremal epimorphism, where $B \in \mathscr{B}$. By using Proposition 2.5 (with $I = M$), we can find $m: B \to \prod A_\alpha$ with $m \in M$, $A_\alpha \in \mathscr{A}$ for all α and such that $p_\alpha m = p_\beta m$ implies $\alpha = \beta$. It follows that $p_\alpha me = p_\beta me$ implies $\alpha = \beta$. We claim that the quotient object of X represented by e is uniquely determined by the set $\{p_\alpha me\}$ of maps from X to members of $\mathscr{A}$. For, assume that $\bar{e}: X \to \bar{B}$ is another extremal epimorphism for which there exists a monomorphism $\bar{m}: \bar{B} \to \prod \bar{A}_i$ whose projections, $\bar{p}_i\bar{m}$, are distinct, and such that the sets $\{p_\alpha me\}$ and $\{\bar{p}_i\bar{m}\bar{e}\}$ coincide. Then $\{\bar{A}_i\}$ must be a one-to-one re-indexing of $\{A_\gamma\}$.

Thus, $\prod A_\gamma$ is equivalent to $\prod \bar{A}_i$, and there is no loss of generality in assuming that $\prod A_\gamma = \prod \bar{A}_i$ and $p_\alpha \bar{m}\bar{e} = p_\alpha me$. Then $\bar{m}\bar{e} = me$, and therefore e is equivalent to $\bar{e}$ by Corollary 2.3.

The proof of (2) is by the same type of argument, using the dual of Corollary 2.3.

As for (3), the above argument shows that each $X \in \mathscr{C}$ has only a set of P-Quotients in $\mathscr{B}$. (It is easily shown that $hg \in I$ implies that $g \in I$ so that Proposition 2.5 applies.) To show reflectivity, let $X \in \mathscr{C}$ be given and let $e_\alpha: X \to B_\alpha$ be a representative set of all P-Quotients of X for which $e_\alpha \in P$ and $B_\alpha \in \mathscr{B}$. Let $e: X \to \prod B_\alpha$ be determined by the equations $p_\alpha e = e_\alpha$ for all α. Factor $e = e_1 e_0$, where $e_1 \in I$ and $e_0 \in P$. A well-known type of argument shows that e_0 is a reflection map reflecting X into $\mathscr{B}$; cf. (**5**, 2.3 and 3.1).

PROPOSITION 2.7. *Let $\mathscr{A} \subseteq \mathscr{C}$ and assume that $\mathscr{C} = M^\#$-Sub(Prod $\mathscr{A}$). Then there is no proper reflective subcategory of $\mathscr{C}$ which contains $\mathscr{A}$.*

Proof. Assume that $\mathscr{B}$ is reflective and that $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{C}$. Let $X \in \mathscr{C}$ be given. There exists $m: X \to B$ such that $m \in M^\#$ and $B \in$ Prod($\mathscr{A}$). Let $e: X \to \bar{X}$ reflect X into $\mathscr{B}$. Then there exists $f: \bar{X} \to B$ with $fe = m$ since $B \in \mathscr{B}$ (as Prod $\mathscr{A} \subseteq \mathscr{B}$). It suffices to show that e is epic, and hence an equivalence (as $m \in M^\#$) which shows that $\mathscr{B} = \mathscr{C}$. To show that $e \in E$, let $ge = he$, where $g, h: \bar{X} \to Y$. Let $n: Y \to Z \in$ Prod $\mathscr{A} \subseteq \mathscr{B}$ be a monomorphism. Then $nge = nhe$. This implies that $ng = nh$ as e is a reflection map and $Z \in \mathscr{B}$. However, $n \in M$, and hence $g = h$.

PROPOSITION 2.8. *Let $\mathscr{C}$ be sup-complete and let $\mathscr{A} \subseteq \mathscr{C}$ be such that $\mathscr{C} = M^\#$-Sub Prod $\mathscr{A}$. Then the embedding of $\mathscr{A}$ into $\mathscr{C}$ preserves all supremums of small diagrams.*

Proof. Let I be a small category and let $\Gamma: I \to A$ be a diagram such that $A =$ Sup Γ exists (in $\mathscr{A}$). Thus, $A \in \mathscr{A}$ and there exists $a_i: \Gamma(i) \to A$ which form a natural transformation from Γ to the constant functor A such that the requirements for a supremum are met.

Regarding Γ as a diagram in $\mathscr{C}$, let $X \in \mathscr{C}$ together with $c_i: \Gamma(i) \to X$ be the supremum of Γ in $\mathscr{C}$. By the properties of supremums, there exists

$f: X \to A$ with $fc_i = a_i$ for all i. We claim that f is epic. To prove $f \in E$, assume that $gf = hf$, where $g, f: A \to \bar{A}$. Since $\mathscr{C} = \text{Sub}(\text{Prod } \mathscr{A})$, it suffices to consider only the case for which $\bar{A} \in \mathscr{A}$. Then $h = g$ since $ha_i = hfc_i = ga_i$ for all i and since A is a supremum of Γ for the category $\mathscr{A}$.

Since $X \in \mathscr{C} = M^{\#}\text{-Sub}(\text{Prod } \mathscr{A})$, there exists $m: X \to \Pi A_\alpha$ with $m \in M^{\#}$ and $A_\alpha \in \mathscr{A}$ for all α. For each fixed α, determine $g_\alpha: A \to A_\alpha$ so that $g_\alpha a_i = p_\alpha mc_i$ for all i. These maps determine a map g for which $p_\alpha g = g_\alpha$ for all α. Clearly, $m = gf$ which implies that f is an equivalence as $m \in M^{\#}$. The proposition now follows immediately.

3. Proof of Theorem 1.1. We now assume that $\mathscr{A}$ and $\mathscr{C}$ satisfy the hypotheses of Theorem 1.1. In what follows, we shall say that $\mathscr{B}$ is an *intermediate category* if $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{C}$. It is convenient to first consider a special case.

Special case. Consider the case in which

$$\mathscr{C} = \text{Sub}(\text{Prod } \mathscr{A}) = E^{\#}\text{-Quot}(\text{Coprod } \mathscr{A}).$$

In this case, e is an epimorphism of $\mathscr{C}$ if and only if $fe = ge$ implies $f = g$ for all f and g with codomain in $\mathscr{A}$. (The proof uses the fact that each $X \in \mathscr{C}$ can be represented as a subobject of a product of objects in $\mathscr{A}$.) It follows that if $\mathscr{B}$ is an intermediate category, then a morphism of $\mathscr{B}$ is epic in $\mathscr{B}$ if and only if it is epic in $\mathscr{C}$. A similar statement holds for monomorphisms. It then follows that an extremal epimorphism (or extremal monomorphism) of $\mathscr{C}$ is still extremal in any intermediate category. Moreover, by the dual of (2) of Proposition 2.6, $\mathscr{C}$ is well-powered. Hence, any intermediate category is well-powered, since there are no new monomorphisms. By Proposition 2.4, $(M^{\#}, E)$ and $(M, E^{\#})$ are both bicategory structures on $\mathscr{C}$.

We claim that $\mathscr{N} = M^{\#}\text{-Sub}(\text{Prod } \mathscr{A})$ is the desired normal completion. $\mathscr{N}$ is well-powered and, by Proposition 2.6, $\mathscr{N}$ is also co-well-powered and reflective in $\mathscr{C}$. Therefore, $\mathscr{N}$ is complete.

If $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{N}$ and if $\mathscr{B}$ is inf-complete, then by the dual of Proposition 2.8, the embedding of $\mathscr{B}$ in $\mathscr{C}$ is inf-preserving. Since $\mathscr{B}$ is well-powered, inf-complete, and has a small cogenerating family (viz., $\mathscr{A}$), it follows by Freyd's special adjoint functor theorem as stated by Lambek (**6**, Proposition 7.1) that $\mathscr{B}$ is reflective in $\mathscr{C}$. Hence, $\mathscr{B}$ is reflective in $\mathscr{N}$; this implies that $\mathscr{B} = \mathscr{N}$ by Proposition 2.7 which shows that $\mathscr{N}$ has no proper reflective subcategory containing $\mathscr{A}$.

Finally, assume that $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{N}$ and that $\mathscr{B}$ is sup-complete. By the dual of the above argument, $\mathscr{B}$ is coreflective in $\mathscr{N}$; hence $\mathscr{B}$ is inf-complete. Thus, the above argument shows that $\mathscr{B} = \mathscr{N}$. Thus, $\mathscr{N}$ is a normal completion of $\mathscr{A}$.

General case. Let (I, P) be a bicategory structure on $\mathscr{C}$. By Proposition 2.6, $\mathscr{C}' = I\text{-Sub}(\text{Prod } \mathscr{A})$ is reflective in $\mathscr{C}$, hence is complete. Moreover,

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the restriction of (I, P) to $\mathscr{C}'$ is clearly a bicategory structure on $\mathscr{C}'$ so we may as well assume that $\mathscr{C} = \mathscr{C}'$. Then $\mathscr{C}$ is P-co-well-powered, and hence $E^{\#}$-co-well-powered as $E^{\#} \subseteq P$ (consider (I, P) factorizations of members of $E^{\#}$).

By the dual considerations, we may as well assume that

$$\mathscr{C} = P\text{-Quot}(\text{Coprod } \mathscr{A}),$$

and hence that $\mathscr{C}$ is I-well-powered, and thus $M^{\#}$-well-powered. By Proposition 2.4 we have that $(M^{\#}, E)$ and $(M, E^{\#})$ are bicategory structures on $\mathscr{C}$.

Moreover, as mentioned in the above case, we have that all monomorphisms of intermediate categories are monic in $\mathscr{C}$. Thus, an extremal epimorphism of $\mathscr{C}$ is still extremal in all intermediate categories. Hence, by repeating the above argument for the bicategory structure $(M, E^{\#})$ instead of (I, P), we may as well assume that $\mathscr{C} = M\text{-Sub}(\text{Prod } \mathscr{A})$ and that

$$\mathscr{C} = E^{\#}\text{-Quot}(\text{Coprod } \mathscr{A}).$$

This is precisely the special case.

The other parts of the theorem follow easily. For example, the embedding of $\mathscr{A}$ into $\mathscr{N}$ preserves supremums and infimums in view of Proposition 2.8 and its dual. The category $\mathscr{C}$ can be retracted onto $\mathscr{N}$ via a composition of several reflective and coreflective functors.

The category $\mathscr{N}$ that we constructed is clearly well-powered and co-well-powered. This is true whenever $\mathscr{C}$ is a normal completion of $\mathscr{A}$. For in this case, $\mathscr{C} = \text{Sub}(\text{Prod } \mathscr{A})$ since $\text{Sub}(\text{Prod } \mathscr{A})$ is inf-complete. Thus, $\mathscr{C}$ is $E^{\#}$-co-well-powered, and hence has a bicategory structure, $(M, E^{\#})$. It follows that $\mathscr{C} = \mathscr{N}$ is well-powered and co-well-powered.

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