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CONGRUENCES ON COMPLETELY REGULAR SEMIGROUPS

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1. Introduction and summary. There are two subjects in the literature on semigroups which have recently attracted great attention: the class of completely regular semigroups (that is semigroups which are unions of their subgroups) and congruences on regular semigroups. In completely regular semigroups, the most popular subject is that of varieties, even though other aspects of them, such as structure, congruences, amalgamation, received their due attention. On the other hand, the treatment of congruences on regular semigroups became especially interesting with the emergence of the kernel-trace approach. This method proved quite successful in the case of inverse semigroups, see [**6**], whereas the analysis for the general regular semigroups encounters considerable difficulties, see [**4**].

The kernel-trace approach consists of studying a congruence ρ on a regular semigroup S by means of its kernel (elements of S ρ-related to idempotents) and its trace (the restriction of ρ to the set $E(S)$ of idempotents of S). We offer here an alternative method for studying congruences on a completely regular semigroup S not based on its set of idempotents but on the greatest semilattice decomposition of S, which, fortunately, turns out to be the usual Green relation $\mathcal{D}$. For a basic fact about S is that it is a semilattice Y of completely simple semigroups S_α. If we now observe that every congruence ρ on S induces in a natural way a congruence ξ on Y and on completely simple components S_α by restriction, we thus immediately arrive at an aggregate of the form $(\xi; \eta_\alpha)$ where $\eta_\alpha = \rho|_{S_\alpha}$ for every $\alpha \in Y$.

Our main thrust is to describe, what we call, a congruence aggregate of the form $(\xi; \eta_\alpha)$ with $\xi \in \mathcal{C}(Y)$ and $\eta_\alpha \in \mathcal{C}(S_\alpha)$, where $\mathcal{C}(\ )$ stands for the congruence lattice, with conditions governing these parameters in order that they produce a congruence on S in a natural way. A part of these conditions provides that

$$\eta = \bigcup_{\alpha \in Y} \eta_\alpha$$

be a congruence on S. Clearly $\eta \subseteq \mathcal{D}$ and, on the other hand, ξ induces a congruence τ on S with the property that $\mathcal{D} \subseteq \tau$. We thus arrive at a pair of congruences (τ, η) with the property that $\rho \vee \mathcal{D} = \tau$ and $\rho \wedge \mathcal{D} = \eta$. Therefore, the problem of finding the appropriate definition of a congruence aggregate reduces to the determination of necessary and sufficient conditions on a pair (τ, η) of congruences on S in order that there exists a congruence ρ on S for which $\rho \vee \mathcal{D} = \tau$ and $\rho \wedge \mathcal{D} = \eta$.

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Section 2 contains the necessary preliminaries. One of the principal results of the paper, the solution of the problem evoked above, is proved in Section 3. Properties of the mappings $\rho \longrightarrow \rho \vee \mathcal{D}$ and $\rho \longrightarrow \rho \wedge \mathcal{D}$ are discussed in Section 4. Section 5 contains the main construction of congruences on a completely regular semigroup in terms of congruence aggregates. Some of the conditions arising in this study are transcribed in Section 6 into the language of a construction of an arbitrary completely regular semigroup.

2. Preliminaries. We generally follow the notation and terminology of [**1**] and [**5**]. In particular, we recall the following concepts.

A *completely regular semigroup* S is a union of its (maximal) subgroups. It is also a semilattice of its completely simple components; we will indicate this by writing $S = (Y; S_\alpha)$. In the entire paper, S stands for an arbitrary completely regular semigroup with $S = (Y; S_\alpha)$ unless specified otherwise.

A semigroup in which $ac = bc$ and $ca = cb$ imply that $a = b$ is said to be *weakly cancellative*. We will often make use of the fact that a completely simple semigroup is weakly cancellative. On any regular semigroup there is defined the *natural partial order* by: $a = b \Leftrightarrow a = eb = bf$ for some idempotents e and f. The set of idempotents of S will be denoted by $E(S)$. In S, we have the operation $x \longrightarrow x^{-1}$ of inversion; we will write $x^0 = xx^{-1} = x^{-1}x$.

The lattice of congruences on S will be denoted by $\mathcal{C}(S)$. Let $\rho \in \mathcal{C}(S)$. The *kernel* of ρ is defined as

$$\ker \rho = \{a \in S \mid \quad a\, \rho\, e \text{ for some } e \in E(S)\}.$$

Since S is completely regular, we actually have

$$\ker \rho = \{a \in S \mid \quad a\, \rho\, a^0\}.$$

The *trace* of ρ is defined as

$$\operatorname{tr} \rho = \rho|_{E(S)}.$$

It is easy to see that for any $a, b \in S$, we have

$$a\, \rho\, b \;\Leftrightarrow\; a^0 \operatorname{tr} \rho\, b^0, \quad ab^{-1} \in \ker \rho,$$

a fact which will be used often and without special reference. For more information on this subject, consult [**4**].

Symbols ϵ and ω denote the equality and the universal relations on any set X; if necessary we will write ϵ_X and ω_X, respectively.

We first collect several statements concerning congruences in the following lemma. Part (i) of it can be found in ([**3**], Lemma 2.2) whereas part (v) represents a strengthening of ([**2**], Theorem 3.3(iii) and [**3**], Theorem 2.1).

LEMMA 2.1. *Let* $\rho \in \mathcal{C}(S)$.

(i) *Let* $\alpha \geqq \beta$, *let* $a, c \in S_\alpha, b \in S_\beta$ *and* $a\ \rho\ b$. *Then* $c\ \rho\ d$ *for some* $d \in S_\beta$.

(ii) *If* $\alpha \geqq \beta$ *and* $a \in S_\alpha$, *then* $a \geqq b$ *for some* $b \in S_\beta$.

(iii) *Let* $\alpha \geqq \beta \geqq \gamma$, *let* $a \in S_\alpha$, *let* $c \in S_\gamma$ *and* $a\ \rho\ c$. *If* $x \in S_\alpha$, *then* $x\ \rho\ y$ *for some* $y \in S_\beta$. *If* $w \in S_\beta$, *then* $w\ \rho\ z$ *for some* $z \in S_\gamma$.

(iv) *Let* $a, x \in S_\alpha, b, y \in S_\beta, a\ \rho\ b$ *and* $x \geqq y$. *Then* $x\ \rho\ y$.

(v) *Let* $a \in S_\alpha, b \in S_\beta$ *and* $a\ \rho \vee \mathcal{D} b$. *Then there exist* $x, y \in S_{\alpha\beta}$ *such that* $a \geqq x, b \geqq y, a\ \rho\ x, b\ \rho\ y$. *Consequently* $\rho \vee \mathcal{D} = \rho \mathcal{D} \rho$.

(vi) *Let* $a, b, u \in S_\alpha, c, d, v \in S_\beta$, $\alpha \geqq \beta$, $ac\ \rho\ bc$, $da\ \rho\ db$, $u\ \rho\ v$. *Then* $a\ \rho\ b$.

Proof. (i) Let

$$I = \{x \in S_\alpha |\quad x\ \rho\ y \text{ for some } y \in S_\beta\}.$$

Then $a \in I$ so $I \neq \emptyset$. Since $\beta \leqq \alpha$, it follows easily that I is an ideal of S_α. But S_α is simple and thus $I = S_\alpha$.

(ii) Let $c \in S_\beta$ and $f = (aca)^0$. Then

$$fa = a^0 fa = a(a^{-1}fa)$$

where $a^{-1}fa \in E(S)$ so that $a \geqq b$ with $b = fa \in S_\beta$.

(iii) Let $e = a^0$. As in the proof of part (ii), there exists $f \in E(S_\beta)$ such that $e \geqq f$. Letting $g = c^0$, we obtain $e\ \rho\ g$ and $f = ef\ \rho\ gf$, $f = fe\ \rho\ fg$ so that

$$f\ \rho\ gfg\ \rho\ (gfg)^0 = g.$$

Therefore $e\ \rho\ f$ and part (i) implies that for $x \in S_\alpha$, there exists $y \in S_\beta$ such that $x\ \rho\ y$. In particular, $a\ \rho\ d$ for some $d \in S_\beta$ and hence $d\ \rho\ c$. If now $w \in S_\beta$, we get $w\ \rho\ z$ for $z \in S_\beta$ again by part (i).

(iv) Let $x \in S_\alpha$, $y \in S_\beta$ and $x \geqq y$. By part (i), we have $x\ \rho\ c$ for some $c \in S_\gamma$. Further, $y = ex = xf$ for some $e, f \in E(S)$. Hence

$$y\ \rho\ ec\ \rho\ cf$$

so that

$$c^0 = ((cf)(ec))^0\ \rho\ (y^2)^0\ \rho\ y^0$$

and thus $y^0\ \rho\ c^0$. Also

$$yc^{-1} = exc^{-1}\ \rho\ ec^0\ \rho\ ey^0\ \rho\ y^0$$

and $yc^{-1} \in \ker \rho$. But then $y\ \rho\ c$. Consequently $x\ \rho\ y$, as required.

(v) Let $a\ \mathcal{D} \rho \mathcal{D}\ b$. Then $a\ \mathcal{D} x\ \rho\ y \mathcal{D}\ b$ for some $x, y \in S$ and thus $a\ \mathcal{D} e\ \rho\ f \mathcal{D}\ b$ where $e = x^0$ and $f = y^0$. It follows that $e\ \rho\ ef\ \rho\ f$ where $D_{ef} \leqq D_e$

and $D_{ef} \leqq D_f$. By part (i), there exist $u, v \in \mathcal{D}_{ef}$ such that $a \; \rho \; u \, \mathcal{D} \, v \; \rho \; b$. This proves that $\mathcal{D} \rho \mathcal{D} \subseteq \rho \mathcal{D} \rho$. As a consequence, we then have that $\rho \vee \mathcal{D} = \rho \mathcal{D} \rho$.

Let $a \; \rho \vee \mathcal{D} \; b$ so that $a \; \rho \; u \mathcal{D} v \; \rho \; b$ for some $u, v \in S$; say, $u, v \in S_\gamma$. Hence $a^2 \; \rho \; au$ and $av \; \rho \; ab$ where $a^2 \in S_\alpha$, $au, av \in S_{\alpha\gamma}$ and $ab \in S_{\alpha\beta}$. Similarly $(av)^2 \; \rho \; (ab)(av)$, where $(av)^2 \in S_{\alpha\beta}$ and $(ab)(av) \in S_{\alpha\beta\gamma}$. Applying part (i) to the components $S_\alpha, S_{\alpha\gamma}$ and to $S_{\alpha\gamma}, S_{\alpha\beta\gamma}$, we deduce that $a \; \rho \; s$ for some $s \in S_{\alpha\beta\gamma}$. But then part (iii) yields the existence of $t \in S_{\alpha\beta}$ for which $a \; \rho \; t$. By part (ii), $a \geqq x$ for some $x \in S_{\alpha\beta}$ so by part (iv) we obtain $a \; \rho \; x$. Symmetrically, there exists $y \in S_{\alpha\beta}$ for which $y \leqq b$ and $y \; \rho \; b$.

(vi) By part (i), there exist $a', b' \in S_\beta$ such that $a \; \rho \; a'$ and $b \; \rho \; b'$ and thus

$$a'(cd) \; \rho \; b'(cd) \quad \text{and} \quad (cd)a' \; \rho \; (cd)b'.$$

Letting $\theta = \rho|_{S_\beta}$, we get in $\bar{S}_\beta = S_\beta/\theta$ that

$$\overline{a'cd} = \overline{b'cd}, \quad \overline{cda'} = \overline{cdb'}$$

which by weak cancellation gives $\bar{a}' = \bar{b}'$. Hence

$$a \; \rho \; a' \; \rho \; b' \; \rho \; b,$$

as required.

We will also need the following simple statement.

LEMMA 2.2. *If elements a and b of S commute, so do $a, b, a^{-1}, b^0, a^0, b^0$.*

Proof. Indeed,

$$\begin{aligned} ab^{-1} &= (ab)b^{-2} = bab^{-2} = b^0(ba)b^{-2} = b^0ab^{-1} = b^{-1}(ba)b^{-1} \\ &= b^{-1}ab^0 = b^{-2}(ba)b^0 = b^{-2}ab = b^{-1}a. \end{aligned}$$

For $a^{-1}b^{-1} = b^{-1}a^{-1}$ apply the above to a^{-1}, b. Further, using this, we get

$$ab^0 = (ab^{-1})b = b^{-1}ab = b^0a.$$

Now apply this to a, b^0 to get $a^0b^0 = b^0a^0$. Finally, $a^0b^{-1} = b^{-1}a^0$ follows from the statements already proved.

3. The characterization. For the entire discussion in this section, the following lemma is of fundamental importance.

LEMMA 3.1. *For any $\rho \in C(S)$ and $a, b \in S$, we have*

$$a \; \rho \; b \;\Leftrightarrow\; a \; \rho \vee \mathcal{D} \, b, \quad ab \; \rho \wedge \mathcal{D} \, ba, \quad ab^{-1} \in \ker(\rho \wedge \mathcal{D}).$$

Proof. The direct part is trivial.

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Converse. By Lemma 2.1(v), we get $a \,\rho\, x \,\mathcal{D}\, y \,\rho\, b$ for some $x, y \in S$. Hence $ab \,\rho\, xy$ and $ba \,\rho\, yx$ so that $xy \,\rho\, yx$. In the quotient $\bar{S} = S/\rho$, we then have $\bar{x} \,\mathcal{D}\, \bar{y}$ and $\bar{x}\bar{y} = \bar{y}\bar{x}$ so that $\bar{a} \,\mathcal{H}\, \bar{b}$. Now $ab^{-1} \in \ker \rho$ implies $\bar{a}\bar{b}^{-1} \in E(S/\rho)$ which together with $\bar{a} \,\mathcal{H}\, \bar{b}$ implies that $\bar{a} = \bar{b}$. Consequently $a \,\rho\, b$, as required.

The following uniqueness result, to be used later, is the content of ([**3**], Corollary 4.4).

COROLLARY 3.2. *If $\lambda, \rho \in C(S)$, then*

$$\lambda \vee \mathcal{D} = \rho \vee \mathcal{D}, \quad \lambda \wedge \mathcal{D} = \rho \wedge \mathcal{D} \Rightarrow \lambda = \rho.$$

As additional information, we prove the following simple statement.

LEMMA 3.3. *For $\rho \in C(S)$ and $a, b \in S$, we have*

$$a^0 \,\rho\, b^0 \;\Leftrightarrow\; a^0 \,\rho \,\mathcal{D}\, b^0, \quad (ab)^0 \,\rho \wedge \mathcal{D}\, (ba)^0.$$

Proof. The direct part is trivial. We prove the converse. Then $a^0 \,\rho\, e \,\mathcal{D}\, f \,\rho\, b^0$ where we may take $e, f \in E(S_{\alpha\beta})$ if $a \in S_\alpha$, $b \in S_\beta$ by Lemma 2.1(v). Hence

$$a \,\rho\, ea \,\rho\, ae \quad \text{and} \quad b \,\rho\, fb \,\rho\, bf$$

whence $eabf \,\rho\, ab$ and $fbae \,\rho\, ba$, so that

$$(eabf)^0 \,\rho\, (ab)^0 \,\rho\, (ba)^0 \,\rho\, (fbae)^0.$$

But $(eabf)^0 = (ef)^0$ and $(fbae)^0 = (fe)^0$ which implies $(ef)^0 \,\rho\, (fe)^0$. Since $e \,\mathcal{D} f$, this yields $e \,\rho\, \mathcal{H} f \rho$ so $e\rho = f\rho$. But then $a^0 \,\rho\, e = f \,\rho\, b^0$, as required.

COROLLARY 3.4. *For $\rho \in C(S)$ and $a, b \in S$ we have*

$$a \,\rho\, b \;\Leftrightarrow\; a^0 \operatorname{tr}(\rho \vee \mathcal{D})\, b^0, \quad (ab)^0 \operatorname{tr}(\rho \wedge \mathcal{D})\,(ba)^0, \quad ab^{-1} \in \ker(\rho \wedge \mathcal{D}).$$

LEMMA 3.5. *For any $\lambda, \rho \in C(S)$, we have*

$$\operatorname{tr} \lambda = \operatorname{tr} \rho \;\Leftrightarrow\; \lambda \vee \mathcal{D} = \rho \vee \mathcal{D}, \quad \operatorname{tr}(\lambda \wedge \mathcal{D}) = \operatorname{tr}(\rho \wedge \mathcal{D}).$$

Proof. Direct part. Using ([**4**], Theorem 4.20), we get

$$\operatorname{tr}(\lambda \vee \mathcal{D}) = \operatorname{tr}\lambda \vee \operatorname{tr}\mathcal{D}\ \operatorname{tr}\rho \vee \operatorname{tr}\mathcal{D} = \operatorname{tr}(\rho \vee \mathcal{D}),$$

and analogously $\operatorname{tr}(\lambda \wedge \mathcal{D}) = \operatorname{tr}(\rho \wedge \mathcal{D})$. In addition,

$$\ker(\lambda \vee \mathcal{D}) = S = \ker(\rho \vee \mathcal{D})$$

and thus $\lambda \vee \mathcal{D} = \rho \vee \mathcal{D}$.

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Converse. Let $e,f \in E(S)$ be such that $e\ \lambda\ f$. Then

$$ef\ \lambda \wedge \mathcal{D}\ fe$$

and hence

$$(ef)^0\ \lambda \wedge \mathcal{D}\ (fe)^0.$$

The hypothesis implies that

$$(ef)^0\ \rho \wedge \mathcal{D}\ (fe)^0.$$

Further, $e\ \lambda\ f$ implies that $e\ \lambda \vee \mathcal{D} f$ which by hypothesis yields $e\ \rho \vee \mathcal{D} f$. Now Lemma 2.1(v) implies the existence of $x, y \in S$ such that

$$e\ \rho\ x\ \mathcal{D}\ y\ \rho\ f.$$

It follows that $(xy)^0\ \rho \wedge \mathcal{D}\ (yx)^0$. But then

$$x^0 = (xy)^0 x^0\ \rho\ (yx)^0 x^0 = (yx)^0\ \rho\ (yx)^0 (xy)^0 = y^0$$

which implies that $e\ \rho\ x^0\ \rho\ y^0\ \rho f$. Therefore $\operatorname{tr}\lambda \subseteq \operatorname{tr}\rho$ and equality follows by symmetry.

The above lemma points to the fact that the trace of ρ is uniquely determined by $\rho \vee \mathcal{D}$ (or only by $\operatorname{tr}(\rho \vee \mathcal{D})$) and the trace of $\rho \wedge \mathcal{D}$. This, in effect, splits the trace of ρ into an "upper trace", equal to $\operatorname{tr}(\rho \vee \mathcal{D})$, and a "lower trace", equal to $\operatorname{tr}(\rho \vee \mathcal{D})$. The kernel of ρ is of course equal to the kernel of $\rho \wedge \mathcal{D}$.

The following theorem represents one of our principal results; the main construction theorem to be established in Section 5 is essentially its corollary.

THEOREM 3.6. *Let $\tau, \eta \in \mathcal{C}(S)$. Then there exists a congruence ρ on S such that $\rho \vee \mathcal{D} = \tau$ and $\rho \wedge \mathcal{D} = \eta$ if and only if*

(i) $\eta \subseteq \mathcal{D} \subseteq \tau$,
(ii) $D_a = D_b \geqq D_c,\ a\ \tau\ c,\ ac\ \eta\ bc,\ ca\ \eta\ cb\ \Rightarrow\ a\ \eta\ b$,
(iii) $a\ \tau\ b,\quad ab\ \eta\ ba,\quad ab^{-1} \in \ker\eta,\quad c \in S \Rightarrow acb\ \eta\ bca.$

In such a case, ρ defined by:

$$a\ \rho\ b\ \Leftrightarrow\ a\ \tau\ b,\quad ab\ \eta\ ba,\quad ab^{-1} \in \ker\eta\quad (a, b \in S)$$

is the unique congruence on S for which $\rho \vee \mathcal{D} = \tau$ and $\rho \wedge \mathcal{D} = \eta$.

Proof. Direct part. Let $\rho \in \mathcal{C}(S)$ and $\tau = \rho \vee \mathcal{D}, \eta = \rho \wedge \mathcal{D}$. Item (i) holds trivially. If the hypotheses of item (ii) are satisfied, then by Lemma 2.1(vi), we get $a\ \rho\ b$ and thus $a\ \eta\ b$ as well. Assume next that the hypotheses of item (iii) are satisfied. Then $a\ \rho \vee \mathcal{D}\ b$, $ab\ \rho\ ba$ and $ab^{-1} \in \ker \rho$ which by Lemma 3.1 gives that $a\ \rho\ b$. But then $acb\ \rho\ bca$ and hence also $acb\ \eta\ bca$.

Converse. Assume conditions (i)–(iii) and define ρ as above. Trivially ρ is reflexive. For $a \in S_\alpha, b \in S_\beta$ such that $a\ \rho\ b$, applying Lemma 2.2 to S/η and by the hypothesis that $ab^{-1} \in \ker \eta$, we get

$$\begin{aligned}(ba^{-1})(ab^{-1}) &= (ba^{-1})^0 ba^{-1}ab^{-1} = (ba^{-1})^0 ba^0 b^{-1} \\ &\eta\ (ba^{-1})^0 ba^0 b^{-1} ab^{-1}\ \eta\ (ba^{-1})^0 (ab^{-1})\end{aligned}$$

and analogoulsy

$$(ab^{-1})(ab^{-1})\ \eta\ (ab^{-1})(ba^{-1})^0.$$

Letting

$$\theta = \eta|_{S_{\alpha\beta}},$$

by weak cancellation in $S_{\alpha\beta}/\theta$, we conclude that

$$ba^{-1}\ \eta\ (ba^{-1})^0.$$

Therefore $ba^{-1} \in \ker \eta$ which proves symmetry of ρ.

Next let $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$ be such that $a\ \rho\ b$ and $b\ \rho\ c$. By condition (iii), we have $acb\ \eta\ bca$ and $bac\ \eta\ cab$. Also, by hypothesis, we have $ab\ \eta\ ba$ and $bc\ \eta\ cb$ which yield

(1) $\qquad acb\ \eta\ bca\ \eta\ cba\ \eta\ cab.$

We analogously obtain

(2) $\qquad bac\ \eta\ cab\ \eta\ cba\ \eta\ bca.$

Multiplying (1) on the right by acb and (2) on the left by bac, we get

(3) $\qquad (ac)(bacb)\ \eta\ (ca)(bacb), \quad (bacb)(ac)\ \eta\ (bacb)(ca),$

where $bacb \in S_{\alpha\beta\gamma}$. Since $a\ \tau\ b$ and $\tau \supseteq \mathcal{D}$, we have $ac\ \tau\ bacb$ which now by condition (ii) yields $ac\ \eta\ ca$.

We therefore have that a, b and c commute modulo η. Applying Lemma 2.2 to S/η gives

$$\begin{aligned}(ac^{-1})b^0\ \eta\ ab^0c^{-1} &= (ab^{-1})(bc^{-1})\ \eta\ (ab^{-1})(ab^{-1})(bc^{-1})(bc^{-1}) \\ &\eta\ (ab^{-1})(bc^{-1})(ab^{-1})(bc^{-1})\ \eta\ (ac^{-1})b^0(ac^{-1})b^0\ \eta\ (ac^{-1})^2 b^0\end{aligned}$$

and by commutativity modulo η, we deduce that

$$b^0(ac^{-1})\ \eta\ b^0(ac^{-1})^2.$$

By condition (ii) we conclude that

$$ac^{-1}\ \eta\ (ac^{-1})^2.$$

Consequently $ac^{-1} \in \ker\eta$ which completes the proof that $a\ \rho\ c$ and establishes transitivity of ρ.

In order to prove compatibility of ρ with the multiplication we again let $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$ with $a\ \rho\ b$. Then $a\ \tau\ b$ which gives $ac\ \tau\ bc$. Further, condition (iii) yields $acb\ \eta\ bca$ whence

$$(ac)(bc)\ \eta\ (bc)(ac).$$

Furthermore

$$\begin{aligned}
&(ac)(bc)^{-1}(ac)(bc)^{-1} && \\
&= (acb)b^{-1}(bc)^{-1}(acb)b^{-1}(bc)^{-1} && \\
&\eta\ (bc)ab^{-1}(bc)^{-1}(bc)ab^{-1}(bc)^{-1} && \text{by (iii)}\\
&= (bc)ab^{-1}(bc)^0ab^{-1}(bc)^{-1} && \\
&= (bc)a[b^{-1}(bc)^0a]bb^{-2}(bc)^{-1} && \\
&\eta\ bcb^0(bc)^0a^2b^{-2}(bc)^{-1} && \text{by (iii)}\\
&= (bca)ab^{-2}(bc)^{-1} && \\
&\eta\ (ac)bab^{-2}(bc)^{-1} && \text{by (iii)}\\
&\eta\ (ac)b^0ab^{-1}(bc)^{-1} && \text{by Lemma 2.2}\\
&= (acb)b^{-1}ab^{-1}(bc)^{-1} && \\
&\eta\ (bc)ab^{-1}ab^{-1}(bc)^{-1} && \text{by (iii)}\\
&\eta\ (bca)b^{-1}(bc)^{-1} && \text{since } ab^{-1} \in \ker\eta\\
&\eta\ (ac)b^0(bc)^{-1} && \text{by (iii)}\\
&= (ac)(bc)^{-1} &&
\end{aligned}$$

which proves that $(ac)(bc)^{-1} \in \ker\eta$. Therefore $ac\ \rho\ bc$.

Similarly, we have $ca\ \tau\ cb$ and $acb\ \eta\ bca$ whence $(ca)(cb)\ \eta\ (cb)(ca)$ and finally

$$
\begin{array}{ll}
(ca)(cb)^{-1}(ca)(cb)^{-1} & \\
= c[a(cb)^{-1}b]b^{-1}c[a(cb)^{-1}b]b^{-1} & \\
\eta\ (cb)(cb)^{-1}ab^{-1}(cb)(cb)^{-1}ab^{-1} & \text{by (iii)} \\
= (cb)^0ab^{-1}(cb)^0ab^{-1} & \\
= (cb)^0a[b^{-1}(cb)^0a]bb^{-2} & \\
\eta\ (cb)^0b^0(cb)^0a^2b^{-2} & \text{by (iii)} \\
= (cb)^0a^2b^{-2} & \\
\eta\ (cb)^0ab^{-1}ab^{-1} & \text{by Lemma 2.2} \\
\eta\ (cb)^0ab^{-1} & \text{since } ab^{-1} \in \ker\eta \\
= c[b(cb)^{-1}a]b^{-1} & \text{by (iii)} \\
\eta\ (ca)(cb)^{-1}b^0 & \\
= (ca)(cb)^{-1} &
\end{array}
$$

which proves that $ca\ \rho\ cb$. Therefore ρ is a congruence.

We show next that $\rho \vee \mathcal{D} = \tau$. Let $a\ \rho \vee \mathcal{D}\ b$. Then by Lemma 2.1(v), we have $a\ \rho\ x \mathcal{D} y\ \rho\ b$ for some $x, y \in S$. Hence $a\ \tau\ x$ and $y\ \tau\ b$ which together with $x\ \mathcal{D} y$ and $\mathcal{D} \subseteq \tau$ gives $a\ \tau\ b$. Therefore $\rho \vee \mathcal{D} \subseteq \tau$. Conversely, let $a\ \tau\ b$. Then

$$a^0\ \tau\ (aba)^0 \quad \text{and} \quad a^0(aba)^0 = (aba)^0a^0$$

which easily implies that $a^0\ \rho\ (aba)^0$. Analogously $b^0\ \rho\ (bab)^0$ whence

$$a\ \mathcal{D}\ a^0\ \rho\ (aba)^0\ \mathcal{D}\ (bab)^0\ \rho\ b^0\ \mathcal{D}\ b$$

and thus $a\ \rho \vee \mathcal{D}\ b$. Consequently $\tau \subseteq \rho \vee \mathcal{D}$ and equality prevails.

Next we prove that $\rho \wedge \mathcal{D} = \eta$. Let $a\ \rho \wedge \mathcal{D} b$. If $a \in S_\alpha$, then letting $\theta = \eta|_{S_\alpha}$, we get $ab\ \theta\ ba$ and $ab^{-1} \in \ker\theta$. Hence in S_α/θ, we have $\bar{a}\bar{b} = \bar{b}\bar{a}$ so that $\bar{a}\,\mathcal{H}\,\bar{b}$ since S_α/θ is completely simple. But then $\bar{a}^0 = \bar{b}^0$ whence $a^0\ \theta\ b^0$ which together with $ab^{-1} \in \ker\theta$ implies $a\ \theta\ b$. It follows that $a\ \eta\ b$ and thus $\rho \wedge \mathcal{D} \subseteq \eta$. Conversely, let $a\ \eta\ b$. By condition (i), we have $a\,\mathcal{D} b$, whence, again by (i), $a\ \tau\ b$. Since trivially $ab\ \eta\ ba$ and $ab^{-1} \in \ker\eta$, we conclude that $a\ \rho\ b$. Therefore $\eta \subseteq \rho \wedge \mathcal{D}$ and equality prevails.

Uniqueness of ρ follows directly from Corollary 3.2.

We can reformulate some of the above results by using the following concept.

Definition 3.7. A pair of congruences τ and η on S is *related by* $\mathcal{D}$ if it satisfies conditions (i), (ii) and (iii) of Theorem 3.6; the corresponding congruence ρ in Theorem 3.6 is then denoted by $\rho_{(\tau,\eta)}$.

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THEOREM 3.8. *If τ and η are congruences on S related by $\mathcal{D}$, then $\rho_{(\tau,\eta)}$ is the unique congruence ρ on S for which $\rho \vee \mathcal{D} = \tau$ and $\rho \wedge \mathcal{D} = \eta$. Conversely, if ρ is a congruence on S, then $\rho \vee \mathcal{D}$ and $\rho \wedge \mathcal{D}$ are related by $\mathcal{D}$ and $\rho = \rho_{(\rho\vee\mathcal{D},\rho\wedge\mathcal{D})}$.*

Proof. This follows directly from Theorem 3.6, Lemma 3.1 and Corollary 3.2.

4. Pairs of congruences with extremal values. For a congruence τ containing $\mathcal{D}$, we construct here the least congruence $\hat{\tau}$ with the property that $\hat{\tau} \vee \mathcal{D} = \tau$. Dually, for a congruence η contained in $\mathcal{D}$, we construct the greatest congruence $\bar{\eta}$ for which $\bar{\eta} \wedge \mathcal{D} = \eta$. We then summarize our findings in a result concerning the mappings $\rho \longrightarrow \rho \vee \mathcal{D}$ and $\rho \longrightarrow \rho \wedge \mathcal{D}$.

LEMMA 4.1. *Let τ be a congruence on S containing $\mathcal{D}$. Let $\hat{\tau}$ be the congruence on S generated by the set*

$$\gamma = \{(a,b) \in S \times S \mid \ a \geqq b,\ a\,\tau\,b\}.$$

Then $\hat{\tau}$ is the least element of the set $\{\rho \in \mathcal{C}(S) \mid \ \rho \vee \mathcal{D} = \tau\}$.

Proof. Since $\gamma \subseteq \tau$, we have $\hat{\tau} \subseteq \tau$ and hence $\hat{\tau} \vee \mathcal{D} \subseteq \tau$. Let $a\,\tau\,b$. Then $x = a(aba)^0$ has the properties: $a\,\tau\,x$ and

$$x = a(aba)^0 = [a(aba)^0a^{-1}]a$$

where $a(aba)^0a^{-1}$ is evidently an idempotent. Thus $a \geqq x$ so that $a\,\gamma\,x$. Similarly, letting $y = b(bab)^0$, we get $b\,\gamma\,y$. Now

$$a\,\hat{\tau}\,x\,\mathcal{D}\,y\,\hat{\tau}\,b$$

which gives $a\,\hat{\tau} \vee \mathcal{D}\,b$. Therefore $\tau \subseteq \hat{\tau} \vee \mathcal{D}$ and equality prevails.

Let $\rho \in \mathcal{C}(S)$ satisfy $\rho \vee \mathcal{D} = \tau$ and let $a\,\gamma\,b$. Then $a \geqq b$ and $a\,\tau\,b$ so that $a\,\rho \vee \mathcal{D}\,b$. Letting $a \in S_\alpha$, $b \in S_\beta$, by Lemma 2.1(v), there exists $x \in S_\beta$ such that $a\,\rho\,x$. But then Lemma 2.1(iv) yields that $a\,\rho\,b$. Therefore $\gamma \subseteq \rho$ and thus also $\hat{\tau} \subseteq \rho$.

We recall the following construction. If θ is an equivalence relation on a semigroup S, then the relation θ^0 defined on S by

$$a\,\theta^0\,b \iff xay\,\theta\,xby \quad \text{for all } x,y \in S^1 \quad (a,b \in S)$$

is the greatest congruence on S contained in θ.

LEMMA 4.2. *Let η be a congruence on S contained in $\mathcal{D}$. Define η' by*

$$a\,\eta'\,b \iff ab\,\eta\,ba,\ ab^{-1} \in \ker\eta \quad (a,b \in S).$$

Then η' is a reflexive and symmetric relation. Further define $\tilde{\eta}$ by

$$a\,\tilde{\eta}\,b \Leftrightarrow (a\,\eta'\,b \text{ and for any } x \in S, x\,\eta'\,a \Leftrightarrow x\,\eta'\,b) \quad (a, b \in S).$$

Then $\tilde{\eta}$ is an equivalence relation. Let $\bar{\eta} = \tilde{\eta}^0$. Then $\bar{\eta}$ is the greatest element of the set $\{\rho \in \mathcal{C}(S) | \rho \wedge \mathcal{D} = \eta\}$.

Proof. Clearly η' is reflexive. The argument for symmetry is the same as in the proof of Theorem 3.6. Trivially $\tilde{\eta}$ is reflexive and is symmetric since η' is. Let $a\,\tilde{\eta}\,b$ and $b\,\tilde{\eta}\,c$. Then for any $x \in S$, we have

(1) $\quad a\,\tilde{\eta}\,'b \quad$ and $\quad x\,\eta'\,a \Leftrightarrow x\,\eta\,b,$

(2) $\quad b\,\eta'\,c \quad$ and $\quad x\,\eta'\,b \Leftrightarrow x\,\eta'\,c.$

By the first part of (1), we have $a\,\eta'\,b$ and hence by the second part of (2), we get $a\,\eta'\,c$. For any $x \in S$, by the second parts of (1) and (2), we obtain $x\,\eta'\,a \Leftrightarrow x\,\eta'\,c$. Therefore $a\,\tilde{\eta}\,c$ and $\tilde{\eta}$ is also transitive.

Consequently $\bar{\eta} = \tilde{\eta}^0$ is defined and is the greatest congruence on S contained in $\tilde{\eta}$. Let $a\,\eta\,b$. Then $ab\,\eta\,ba$ and $ab^{-1} \in \ker\eta$ so $a\,\eta'\,b$. Let $x \in S$ and assume that $x\,\eta'\,a$. Hence $xa\,\eta\,ax$ and $xa^{-1} \in \ker\eta$, and thus $xa^{-1}\,\eta\,(xa^{-1})^0$. But then $a\,\eta\,b$ implies that $xb\,\eta\,bx$ and $xb^{-1}\,\eta\,(xb^{-1})^0$ whence $x\,\eta'\,b$. By symmetry, we conclude that $a\,\tilde{\eta}\,b$. Consequently $\eta \subseteq \tilde{\eta}$ and since η is a congruence, it follows that $\eta \subseteq \tilde{\eta}^0 = \bar{\eta}$. Therefore $\eta \subseteq \bar{\eta} \wedge \mathcal{D}$.

Next let $a\,\eta' \wedge \mathcal{D}\,b$. Then $ab\,\eta\,ba$, $ab^{-1} \in \ker\eta$ and $a\,\mathcal{D}\,b$. Let $\theta = \eta|_{D_a}$, $\bar{D}_a = D_a/\theta$ and $\bar{x} = x\theta$ for any $x \in D_a$. Then $\overline{ab} = \overline{ba}$ so $\bar{a}\,\mathcal{H}\,\bar{b}$ since $\bar{D}_a$ is completely simple. Hence $\bar{a}^0 = \bar{b}^0$ and thus $\overline{a^0} = \overline{b^0}$, that is, $a^0\,\theta\,b^0$. This together with $ab^{-1} \in \ker\theta$ implies that $a\,\theta\,b$. Therefore $a\,\eta\,b$ which proves that $\eta' \wedge \mathcal{D} \subseteq \eta$. Since $\bar{\eta} \subseteq \tilde{\eta} \subseteq \eta'$, it follows that $\bar{\eta} \wedge \mathcal{D} \subseteq \eta$. We have proved that $\eta = \bar{\eta} \wedge \mathcal{D}$.

Now let $\rho \in \mathcal{C}(S)$ be such that $\rho \wedge \mathcal{D} = \eta$ and let $a\,\rho\,b$. Then $ab\,\rho\,ba$ and $ab^{-1} \in \ker\rho$, where $ab\,\mathcal{D}\,ba$. It follows that $ab\,\eta\,ba$ and $ab^{-1} \in \ker\rho = \ker\eta$. Hence $a\,\eta'\,b$. Let $x \in S$ be such that $x\,\eta'\,a$. Thus $xa\,\eta\,ax$ and $xa^{-1} \in \ker\eta$. But then $xa\,\rho\,ax$ and $xa^{-1} \in \ker\rho$. Further, $a\,\rho\,b$ implies that $xb\,\rho\,bx$ and $xb^{-1} \in \ker\rho$ whence $xb\,\eta\,bx$ and $xb^{-1} \in \ker\eta$, that is $x\,\eta'\,b$. By symmetry, we conclude that $a\,\tilde{\eta}\,b$. This shows that $\rho \subseteq \tilde{\eta}$ and thus $\rho \subseteq \tilde{\eta}^0 = \bar{\eta}$ since ρ is a congruence. Therefore $\bar{\eta}$ is the greatest congruence with the given property.

That the map $\rho \longrightarrow \rho \vee \mathcal{J}$ is a homomorphism in an intraregular semigroup is stated in [**3**] by Jones as proved in another of his preprints. A special case of this will be proved in the next theorem. The above constructions will be used, with the notation introduced therein, in the following result.

THEOREM 4.3. (i) *The mapping*

$$\Gamma\colon \rho \longrightarrow \rho \vee \mathcal{D} \quad (\rho \in \mathcal{C}(S))$$

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is a complete homomorphism of $\mathcal{C}(S)$ *onto the interval* $[\mathcal{D}, \omega]$. *For every* $\tau \in [\mathcal{D}, \omega]$, *we have* $\tau\Gamma^{-1} = [\hat{\tau}, \tau]$.

(ii) *The mapping*

$$\Delta: \rho \longrightarrow \rho \wedge \mathcal{D} \quad (\rho \in \mathcal{C}(S))$$

is a complete $\wedge$*-homomorphism of* $\mathcal{C}(S)$ *onto the interval* $[\epsilon, \mathcal{D}]$. *For every* $\eta \in [\epsilon, \mathcal{D}]$, *we have* $\eta\Delta^{-1} = [\eta, \bar{\eta}]$.

Proof. (i) In order to establish the first statement, we must show that for every family $\{\rho_i\}_{i \in I}$ of congruences on S, we have

$$(1) \qquad \left(\bigwedge_{i \in I}\right) \vee \mathcal{D} = \bigwedge_{i \in I} (\rho_i \vee \mathcal{D})$$

Let

$$a \in S_\alpha, \quad b \in S_\beta, \quad a \bigwedge_{i \in I} (\rho_i \vee \mathcal{D})\, b.$$

Then for each $i \in I$, $a\ \rho_i \vee \mathcal{D}\ b$ so by Lemma 2.1(v), we have $a\ \rho_i\ x_i$, $y_i\ \rho_i\ b$ for some x_i, $y_i \in S_{\alpha\beta}$ for each $i \in I$. Let $e = a^0$. Lemma 2.1(ii), there exists $f \in E(S_{\alpha\beta})$ such that $e \geqq f$. Hence

$$f = ef\ \rho_i\ x_i^0 f, \quad f = fe\ \rho_i\ f x_i^0$$

and thus

$$f\ \rho_i\ x_i^0 f x_i^0\ \rho_i\ (x_i^0 f x_i^0)^0 = x_i^0.$$

Since $e\ \rho_i\ x_i^0$, it follows that $e\ \rho_i\, f$ for any $i \in I$. Thus

$$e \bigwedge_{i \in I} \rho_i\ f$$

so Lemma 2.1(i) implies that

$$a \bigwedge_{i \in I} \rho_i\ c \quad \text{for some } c \in S_{\alpha\beta}.$$

Analogously

$$d \bigwedge_{i \in I} \rho_i\ b \quad \text{for some } d \in S_{\alpha\beta}.$$

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Therefore

$$a \bigwedge_{i\in I} \rho_i\ c\ \mathcal{D}\ d \bigwedge_{i\in I} \rho_i\ b$$

and thus

$$a \left(\bigwedge_{i\in I} \rho_i \right) \vee \mathcal{D}\ b.$$

This proves the nontrivial inclusion in (1).

This establishes the first statement; the second follows easily from Lemma 4.1.

(ii) The first statement is trivial and the second follows easily from Lemma 4.2.

The mapping $\rho \longrightarrow \rho \wedge \mathcal{D}$ is not a $\vee$-homomorphism. Indeed, let S be a semilattice of nontrivial groups G_1 and G_0 determined by an injective homomorphism $\varphi\colon G_1 \longrightarrow G_0$. Let λ be the Rees congruence on S relative to the ideal G_0 and ρ be the congruence on S induced by the retraction $\psi = \varphi \cup \iota_{G_0}$, where

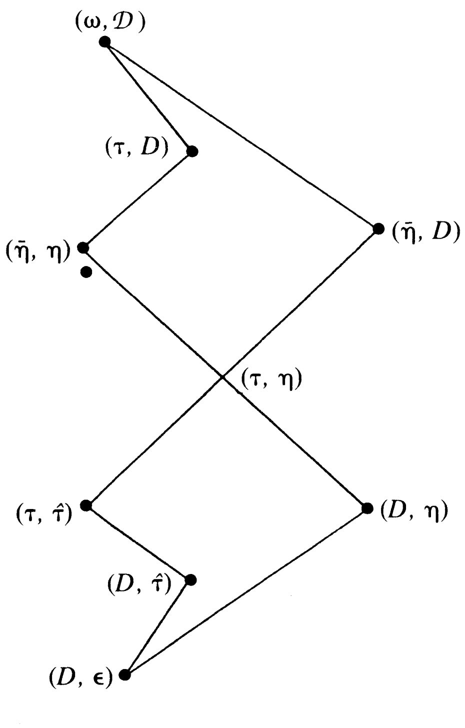

Diagram 1: The network resulting from the pair (τ, η).

ι_{G_0} is the identity map on G_0. Then $\lambda \vee \rho = \omega$ so that $(\lambda \vee \rho) \wedge \mathcal{D} = \mathcal{D}$, and

$$(\lambda \wedge \mathcal{D}) \vee (\rho \wedge \mathcal{D}) = \lambda \vee \epsilon = \lambda \neq \mathcal{D}$$

since G_1 is nontrivial. Therefore

$$(\lambda \wedge \mathcal{D}) \vee (\rho \wedge \mathcal{D}) \neq (\lambda \vee \rho) \wedge \mathcal{D}.$$

We can illustrate the intervals occurring in the above theorem by the following Diagram 1 of the resulting "network" of pairs. At the lower end $(\mathcal{D}, \epsilon)$ corresponds to the equality relation ϵ and at the upper $(\omega, \mathcal{D})$ to the universal relation ω.

5. The main theorem. Using the characterization of a congruence ρ in terms of the corresponding pairs τ and η of congruences, we can now deduce easily the desired construction. First note that there is a natural correspondence of congruences on S containing $\mathcal{D}$ and congruences on $S/\mathcal{D}$ given by $\rho \longrightarrow \rho/\mathcal{D}$. Since $S/\mathcal{D}$ and Y are isomorphic by $D_a \longrightarrow \alpha$ if $a \in S_\alpha$, congruences on $S/\mathcal{D}$ are in a natural one-to-one correpsondence with congruences on Y. We may thus essentially identify congruences on S containing $\mathcal{D}$ with congruences on Y. For congruences on S contained in $\mathcal{D}$, we have the following simple result.

LEMMA 5.1. *Let $S = (Y; S_\alpha)$. For each $\alpha \in Y$, let $\eta_\alpha \in \mathcal{C}(S_\alpha)$ and assume:*

$$a, b \in S_\alpha, \quad c \in S_\beta, \quad \alpha \geqq \beta \quad or$$

$$\beta \geqq \alpha,\ a \ \eta_\alpha \ b \ \Rightarrow \ ac \ \eta_{\alpha\beta} \ bc, \quad ca \ \eta_{\alpha\beta} \ cb.$$

Then $\eta = \bigcup_{\alpha \in Y} \eta_\alpha$ is a congruence on S contained in $\mathcal{D}$. Conversely, every congruence on S contained in $\mathcal{D}$ can be so constructed.

Proof. Direct part. Let $a \ \eta \ b$ so that $a, b \in S_\alpha$ for some $\alpha \in Y$ and let $c \in S_\beta$. Then $cb \in S_{\alpha\beta}$ and thus

$$a(cb) \ \eta_{\alpha\beta} \ b(cb).$$

Now $acb, bcb, cb \in S_{\alpha\beta}$ whence

$$(ac)(bcb) \ \eta_{\alpha\beta} \ (bc)(bcb).$$

Also $(bcb)a \ \eta_{\alpha\beta} \ (bcb)b$ so that

$$(bcb)(ac) \ \eta_{\alpha\beta} \ (bcb)(bc),$$

again by hypothesis. Now weak cancellation in $S_{\alpha\beta}/\eta_{\alpha\beta}$ yields $ac \ \eta_{\alpha\beta} \ bc$. Analogously $ca \ \eta_{\alpha\beta} \ cb$ so that $\rho \in \mathcal{C}(S)$. Trivially $\rho \subseteq \mathcal{D}$.

Converse. This is trivial.

The above statement can be summarized thus: vertical compatibility conditions (i.e., $\alpha \geqq \beta$ or $\beta \geqq \alpha$) imply the general compatibility (in view of weak cancellation in the homomorphic images of the components).

Definition 5.2. For $\rho \in \mathcal{C}(S)$, the congruence ξ induced on Y by the congruence $(\rho \vee \mathcal{D})/\mathcal{D}$ is the *reflection* of ρ, in notation $\xi = \operatorname{re}\rho$.

Explicitly, for $\alpha, \beta \in Y$,

$$\alpha\ \xi\ \beta \Leftrightarrow \text{there exists } a \in S_\alpha, b \in S_\beta, c, d \in S_\gamma \text{ such that}$$

$$a\ \rho\ c, d\ \rho\ b \text{ and } \gamma \leqq \alpha\beta.$$

In view of Lemma 2.1(v), we may assume that $\gamma = \alpha\beta$, $a \geqq c$ and $b \geqq d$.

We now arrive at the concept basic for our considerations.

Definition 5.3. Let $S = (Y; S_\alpha)$. Let $\xi \in \mathcal{C}(Y)$ and $\eta_\alpha \in \mathcal{C}(S_\alpha)$ for every $\alpha \in Y$. We call $(\xi; \eta_\alpha)$ a *congruence aggregate for* S if the following conditions are satisfied.

(i) $a, b \in S_\alpha,\ c \in S_\beta,\ \alpha \geqq \beta$ or
$\beta \geqq \alpha,\ a\ \eta_\alpha\ b\ \Rightarrow\ ac\ \eta_{\alpha\beta}\ bc,\ ca\ \eta_{\alpha\beta}\ cb.$

(ii) $a, b \in S_\alpha, c \in S_\beta,\ \alpha \geqq \beta, \alpha\ \ \xi\ \beta,\ ac\ \eta_\beta\ bc,\ ca\ \eta_\beta\ cb\ \Rightarrow\ a\ \eta_\alpha\ b.$

(iii) $a \in S_\alpha,\ b \in S_\beta,\ c \in S_\gamma,\ \alpha\ \xi\ \beta,\ ab\ \eta_{\alpha\beta}\ ba,$
$ab^{-1} \epsilon \ker \eta_{\alpha\beta} \Rightarrow acb\ \eta_{\alpha\beta\gamma}\ bca.$

In such a case, define a relation $\rho_{(\xi;\eta_\alpha)}$ on S by: for $a \in S_\alpha$, $b \in S_\beta$,

$$a\ \rho_{(\xi;\eta_\alpha)}\ b\ \Leftrightarrow\ \alpha\ \xi\ b, \quad ab\ \eta_{\alpha\beta}\ ba, \quad ab^{-1} \in \ker \eta_{\alpha\beta}.$$

Denote by $\mathcal{CA}(S)$ the set of all congruence aggregates for S with componentwise ordering.

We can now easily derive our main result from the statements already established.

THEOREM 5.4. *If* $(\xi; \eta_\alpha)$ *is a congruence aggregate for* S, *then* $\rho_{(\xi;\eta_\alpha)}$ *is the unique congruence* ρ *on* S *such that* $\operatorname{re}\rho = \xi$ *and*

$$\rho|_{S_\alpha} = \eta_\alpha \quad \textit{for all } \alpha \in Y.$$

Conversely, if ρ *is a congruence on* S, *then* $(\operatorname{re}\rho; \rho|_{S_\alpha})$ *is a congruence aggregate for* S *and*

$$\rho = \rho_{(\operatorname{re}\rho;\rho|_{S_\alpha})}.$$

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Proof. Let $(\xi;\eta_\alpha) \in \mathcal{CA}(S)$. Condition (i) in Definition 5.3 gives, in view of Lemma 5.1, that

$$\eta = \bigcup_{\alpha\in Y} \eta_\alpha$$

is a conguence on S, evidently contained in $\mathcal{D}$. Letting τ be the congruence on S induced by ξ, we obtain the pair (τ,η) satisfying conditions (i), (ii) and (iii) of Theorem 3.6 in view of conditions (ii) and (iii) here. It is now clear that the congruence ρ in Theorem 3.6 and $\rho_{(\xi;\eta_\alpha)}$ coincide. Therefore

$$\rho_{(\xi;\eta_\alpha)} \in \mathcal{C}(S);$$

its uniqueness follows easily from the uniqueness assertion in Theorem 3.6.

Conversely, let $\rho \in \mathcal{C}(S)$. By Lemma 3.1, we have for any $a, b \in S$,

$$a\ \rho\ b \ \Leftrightarrow\ a\ \rho\ \vee \mathcal{D}\ b, \quad ab\ \rho \wedge \mathcal{D}\ ba, \quad ab^{-1} \in \ker(\rho \wedge \mathcal{D}).$$

If $a \in S_\alpha$ and $b \in S_\beta$, then it follows that $a\ \rho\ \vee \mathcal{D}\ b$ is equivalent to $\alpha\ \xi\ \beta$ where $\xi = \operatorname{re}\rho$. Letting $\eta_\alpha = \rho|_{S_\alpha}$ for all $\alpha \in Y$, by the converse of Theorem 3.6, we have that $(\xi;\eta_\alpha) \in \mathcal{CA}(S)$ and therefore by the above, $\rho = \rho_{(\xi;\eta_\alpha)}$.

Of course, we could have proved the above theorem directly. However, it seems conceptually more transparent why this construction produces all congruences on S if we pass through the congruences τ and η and then describe these as explained above.

COROLLARY 5.5. (i) *The mapping*

$$\gamma\colon \rho_{(\xi;\eta_\alpha)} \longrightarrow \xi \quad ((\xi;\eta_\alpha) \in \mathcal{CA}(S))$$

is a complete homomorphism of $\mathcal{C}(S)$ onto $\mathcal{C}(Y)$. For each $\xi \in \mathcal{C}(Y)$, we have

$$\xi\gamma^{-1} = [\rho_{(\xi;\xi_\alpha)}, \rho_{(\xi;\omega_{S_\alpha})}]$$

where $\xi_\alpha = \hat{\tau}|_{S_\alpha}$ for each $\alpha \in Y$ and $\tau \supseteq \mathcal{D}$, $\operatorname{re}\tau = \xi$ (see Lemma 4.1 for $\hat{\tau}$).

(ii) *The mapping*

$$\delta\colon \rho_{(\xi;\rho_\alpha)} \longrightarrow (\eta_\alpha) \quad ((\xi;\eta_\alpha) \in \mathcal{CA}(S))$$

is a complete $\wedge$-homomorphism of $\mathcal{C}(S)$ onto the $\wedge$-semilattice

$$\left\{(\eta_\alpha) \in \prod_{\alpha\in Y} \mathcal{C}(S_\alpha) \,|\, \{\eta_\alpha\} \textit{ satisfy condition (i) of Definition 5.3}\right\}$$

of the direct product $\prod_{\alpha\in Y}\mathcal{C}(S_\alpha)$. For each element (η_α) of this set, we have

$$(\eta_\alpha)\delta^{-1} = [\rho_{(\epsilon;\eta_\alpha)}, \rho_{(\operatorname{re}\bar{\eta};\eta_\alpha)}]$$

(for $\bar{\eta}$ *se Lemma* 4.2*).*

Proof. This follows easily from Theorems 4.3 and 5.4.

6. Congruences on the standard representation. By the standard representation of a completely regular semigroup S we mean the following construction taken from [**7**].

THEOREM 6.1. *Let* Y *be a semilattice. For every* $\alpha \in Y$*, let*

$$S_\alpha = \mathcal{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P_\alpha)$$

be a Rees matrix semigroup such that P_α *is normalized at an element also denoted by* α *and suppose that* $S_\alpha \cap S_\beta = \emptyset$ *if* $\alpha \neq \beta$*. Let*

$$\langle\ ,\ \rangle : S_\alpha \times I_\beta \longrightarrow I_\beta,$$

$$S_\alpha \longrightarrow G_\beta \text{ in notation } a \longrightarrow a_\beta,$$

$$[\ ,\ ] : \Lambda_\beta \times S_\alpha \longrightarrow \Lambda_\beta$$

be functions defined whenever $\alpha \geqq \beta$ *and satisfying the following conditions. Let* $a \in S_\alpha$ *and* $b \in S_\beta$*.*

(i) *If* $\alpha \geqq \beta, i \in I_\beta, \lambda \in \Lambda_\beta$*, then*

$$p_{\lambda\langle a,i\rangle} a_\beta p_{[\beta,\alpha]i} = p_{\lambda\langle\alpha,\beta\rangle} a_\beta p_{[\lambda,\alpha]}).$$

(ii) *If* $i \in I_\alpha$ *and* $\lambda \in \Lambda_\alpha$*, then*

$$a = (\langle a, i\rangle, a_\alpha, [\lambda, a]).$$

On $S = \bigcup_{a \in Y} S_\alpha$ *define a multiplication by*

$$a \circ b = (\langle a, \langle b, \alpha\beta\rangle\rangle, a_{\alpha\beta} p_{[\alpha\beta,a]\langle b,\alpha\beta\rangle} b_{\alpha\beta}, [[\alpha\beta, a], b]).$$

(iii) *If* $\gamma \leqq \alpha\beta, i \in I_\gamma, \lambda \in \Lambda_\gamma$*, then*

$$\begin{aligned}&(\langle a, \langle b, i\rangle\rangle, a_\gamma p_{[\gamma,a]\langle b,\lambda\rangle} b_\gamma, [[\lambda, a], b]) \\ &= (\langle a \circ b, i\rangle, (a \circ b)_\gamma, [\lambda, a \circ b]).\end{aligned}$$

Then S *is a completely regular semigroup such that* $S/\mathcal{D} \cong Y$ *and whose multiplication restricted to each* S_α *coincides with the given multiplication. Conversely, every completely regular semigroup is isomorphic to one so constructed.*

We can use the representation of congruences in Theorem 5.4 on this particular completely regular semigroup. We may further represent congruences on the completely simple components S_α in terms of admissible triples since S_α is already given as a Rees matrix semigroup with normalized sandwich matrix. For this purpose, recall the following construction.

Definition 6.2. Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a Rees matrix semigroup with normalized sandwich matrix P. Let r be an equivalence on I, N be a normal subgroup of G and π be an equivalence on Λ. If the condition

$$(1) \qquad i \; r \; j \quad \text{or} \quad \lambda \; \pi \; \mu \;\Rightarrow\; p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1} \in N$$

is satisfied, then (r, N, π) is an *admissible triple* for S. In such a case, define a relation $\rho_{(r,N,\pi)}$ on S by

$$(i, g, \lambda) \; \rho_{(r,N,\pi)} \; (j, h, \mu) \;\Leftrightarrow\; i \; r \; j, \quad gh^{-1} \in N, \quad \lambda \; \pi \; \mu.$$

For the proof of the following theorem, see ([**1**], III. 4).

THEOREM 6.3. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ with normalized P. If (r, N, π) is an admissible triple for S, then $\rho_{(r,N,\pi)}$ is a congruence on S. Conversely, every congruence on S can be so represented for a unique admissible triple.*

Combining the two theorems above with the description of congruences in Theorem 5.4, our task consists in transcribing conditions (i), (ii) and (iii) in Definition 5.3 in terms of parameters $Y, I_\alpha, G_\alpha, \Lambda_\alpha$ for all $\alpha \in Y$, as well as the defining expression for $\rho_{(\xi;\eta_\alpha)}$. For each $\alpha \in Y$, we let

$$\eta_\alpha = \rho_{(r_\alpha, N_\alpha, \pi_\alpha)}$$

and, as before, $\xi \in \mathcal{C}(Y)$.

The next result translates condition (i) in Definition 5.3 into the new notation. A similar analysis, albeit with more complex formulae, is possible for conditions (ii) and (iii), which we will, however, omit. The same goes for the expression for $\rho_{(\xi;\eta_\alpha)}$; indeed, an explicit form for this congruence in terms of the above parameters does not seem, unfortunately, to shed any further light on its nature.

LEMMA 6.4. *Condition* (i) *in Definition* 5.3 *is equivalent to the following set of conditions: for $a = (i, g, \lambda)$, $b = (j, h, \mu) \in S_\alpha$, $c \in S_\beta$,*

$$(\alpha) \quad a \; \eta_\alpha \; b, k \in I_\beta, \alpha \geqq \beta \;\Rightarrow\; \langle a, k \rangle \; r_\alpha \; \langle b, k \rangle,$$

$$(\beta) \quad a \; \eta_\alpha \; b, \alpha \geqq \beta \;\Rightarrow\; a_\beta (b_\beta)^{-1} \in N_\beta,$$

$$(\gamma) \quad a \; \eta_\alpha \; b, \nu \in \Lambda_\beta, \alpha \geqq \beta \;\Rightarrow\; [\nu, a] \; \pi_\alpha \; [\nu, b],$$

$$(\delta) \quad i \; r_\alpha \; j, \beta \geqq \alpha \;\Rightarrow\; \langle c, i \rangle \; r_\alpha \; \langle c, j \rangle,$$

$$(\epsilon) \quad \lambda \; \pi_\alpha \; \mu, \beta \geqq \alpha \;\Rightarrow\; [\lambda, c] \; \pi_\alpha \; [\mu, c].$$

Proof. First assume condition (i) in Definition 5.3. Let $a\ \eta_\alpha\ b$, $c = (k, e, \beta)$, where e is the identity of G_β, and $\alpha \geqq \beta$. Then $ac\ \eta_\beta\ bc$ and thus

$$(\langle a, k\rangle, a_\beta p_{[\beta,a]k} e, \beta)\ \eta_\beta\ (\langle b, k\rangle, b_\beta p_{[\beta,b]k} e, \beta)$$

which in terms of admissible triples gives

$$\langle a, k\rangle\ r_\beta\ \langle b, k\rangle, (a_\beta p_{[\beta,a]k})(b_\beta p_{[\beta,b]k})^{-1} \in N_\beta. \tag{2}$$

It follows that (α) holds and

$$a_\beta p_{[\beta,a]k} p^{-1}_{[\beta,b]k} (b_\beta)^{-1} \in N_\beta. \tag{3}$$

Analogously, $ca\ \rho\ cb$ and thus

$$(k, ep_{\beta\langle a,\beta\rangle} a_\beta, [\beta, a])\ \eta_\beta\ (k, ep_{\beta\langle b,\beta\rangle} b_{\beta,[\beta,b]})$$

which yields

$$[\beta, a]\ \pi_\alpha\ [\beta, b]. \tag{4}$$

Now (4) together with (1) in the definition of an admissible triple gives

$$p_{[\beta,a]k} p^{-1}_{[\beta,b]k} \in N_\beta$$

which in conjunction with (3) yields

$$a_\beta (b_\beta)^{-1} \in N_\beta$$

verifying (β). Condition (γ) follows by an analogous argument.

Next let

$$i\ r_\alpha\ j, \quad a = (i, e, \alpha), \quad b = (j, e, \alpha)$$

where e is the identity of G_α. Then $a\ \eta_\alpha\ b$ and hence $ca\ \eta_\alpha\ cb$ whence

$$(\langle c, i\rangle, \quad , \quad)\ \eta_\alpha\ (\langle c, j\rangle, \quad , \quad)$$

and thus $\langle c, i\rangle\ r_\alpha\ \langle c, j\rangle$. This verifies (δ); condition (ϵ) follows symmetrically.

Conversely, assume the validity of conditions (α)–(ϵ). Let $a, b \in S_\alpha$, $a\ \eta_\alpha\ b$ and $c \in S_\gamma$.

Let $\alpha \geqq \beta$ and $c = (k, t, \nu)$. Then

$$ac = (\langle a, k\rangle, a_\beta p_{[\beta,a]k} t, \nu), \quad bc = (\langle b, k\rangle, b_\beta p_{[\beta,b]k} t, \nu)$$

with

$$\langle a,k\rangle\ r_\beta\ \langle b,k\rangle,\quad a_\beta(b_\beta)^{-1}\in N_\beta,\quad [\beta,a]\ \pi_\beta\ [\beta,b]$$

in view of $(\alpha),(\beta),(\gamma)$. By (1), we have

$$p_{[\beta,a]k}p_{[\beta,b]k}^{-1}\in N_\beta$$

and thus

$$a_\beta p_{[\beta,a]k}t(b_\beta p_{[\beta,b]k}t)^{-1}=a_\beta p_{[\beta,a]k}p_{[\beta,b]k}^{-1}(b_\beta)^{-1}\in N_\beta$$

and therefore $ac\ \eta_\beta\ bc$. Symmetrically, one can show that $ca\ \eta_\beta\ cb$.

Next let $\beta\geqq\alpha$, $a=(i,g,\lambda)$ and $b=(j,h,\mu)$. Then $i\ r_\alpha\ j$, $gh^{-1}\in N_\alpha$, $\lambda\ \pi_\alpha\ \mu$, which by (δ) and (ϵ) gives

$$\langle c,i\rangle\ r_\alpha\ \langle c,j\rangle\quad\text{and}\quad[\lambda,c]\ \pi_\alpha\ [\mu,c].$$

Further

$$ac=(i,gp_{\lambda\langle c,\alpha\rangle}c_\alpha,[\lambda,c]),\quad bc=(j,hp_{\mu\langle c,\alpha\rangle}c_\alpha,[\mu,c])$$

where

$$p_{\lambda\langle c,\alpha\rangle}p_{\mu\langle c,\alpha\rangle}^{-1}\in N_\alpha$$

by (1), so that

$$(gp_{\lambda\langle c,\alpha\rangle}c_\alpha)(hp_{\mu\langle c,\alpha\rangle}c_\alpha)^{-1}\in N_\alpha.$$

It then follows that $ac\ \eta_\alpha\ bc$ and symmetrically also $ca\ \eta_\alpha\ cb$. This verifies condition (i) of Definition 5.3.

In view of Lemma 5.1, condition (i) of Definition 5.3 is necessary and sufficient for a construction of congruences contained in $\mathcal{D}$. We thus have the following consequence of the above results.

COROLLARY 6.5. *Let S be given the standard representation. For each $\alpha\in Y$, let $(r_\alpha,N_\alpha,\pi_\alpha)$ be an admissible triple for S_α and let*

$$\eta_\alpha=\rho_{(r_\alpha,N_\alpha,\pi_\alpha)}.$$

If conditions (α)–(ϵ) in Lemma 6.4 are fulfilled, then

$$\eta=\bigcup_{\alpha\in Y}\eta_\alpha$$

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is a congruence on S contained in $\mathcal{D}$. Conversely, every congruence on S contained in $\mathcal{D}$ can be so constructed.

Proof. This follows directly from Lemmas 5.1 and 6.4.

There is a special case when the conditions in a congruence aggregate simplify considerably and $\rho_{(\xi;\eta_\alpha)}$ takes on a particularly simple form. That is the case of a normal band of groups. These are precisely strong semilattices of completely simple semigroups. For definitions and assertions relevant to this subject, see ([**5**], IV.4). We denote such a semigroup by $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$.

THEOREM 6.6. *Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be completely regular. Condition* (i) *of Definition* 5.3 *is equivalent to:*

$$(\alpha) \quad a, b \in S_\alpha,\ a\ \eta_\alpha\ b,\ \alpha \geqq \beta\ \Rightarrow\ a\varphi_{\alpha,\beta}\ \eta_\beta\ b\varphi_{\alpha,\beta};$$

condition (ii) *is equivalent to:*

$$(\beta) \quad a, b \in S_\alpha,\ \alpha \geqq \beta,\ a\varphi_{\alpha,\beta}\ \eta_\beta\ b\varphi_{\alpha,\beta}, \quad a\ \xi\ \beta\ \Rightarrow\ a\ \eta_\alpha\ b;$$

condition (iii) *follows from* (α). *The expression for $\rho_{(\xi;\eta)}$ becomes: for $a \in S_\alpha$ and $b \in S_\beta$,*

$$a\ \rho_{(\xi;\eta_\alpha)}\ b\ \Leftrightarrow\ \alpha\ \xi\ \beta,\ a\varphi_{\alpha,\alpha\beta}\ \eta_{\alpha\beta}\ b\varphi_{\beta,\alpha\beta}.$$

Proof. First assume condition (i) and let the hypotheses of (α) hold. Then

$$(a\varphi_{\alpha,\beta})(a^0\varphi_{\alpha,\beta}) = a(a^0\varphi_{\alpha,\beta})\ \eta_\beta\ b(a^0\varphi_{\alpha,\beta}) = (b\varphi_{\alpha,\beta})(a^0\varphi_{\alpha,\beta})$$

and symmetrically

$$(a^0\varphi_{\alpha,\beta})(a\varphi_{\alpha,\beta})\ \eta_\beta\ (a^0\varphi_{\alpha,\beta})(b\varphi_{\alpha,\beta}),$$

which by weak cancellation in S_β/η_β yields $a\varphi_{\alpha,\beta}\ \eta_\beta\ b\varphi_{\alpha,\beta}$, that is, (α) holds. The proof that (α) implies condition (i) is straightforward and is omitted.

Next assume condition (ii) and let the hypotheses of (β) hold. Similarly as above, we get

$$a(a^0\varphi_{\alpha,\beta})\ \eta_\beta\ b(a^0\varphi_{\alpha,\beta}), \quad (a^0\varphi_{\alpha,\beta})a\ \eta_\beta\ (a^0\varphi_{\alpha,\beta})b$$

which by the hypothesis implies that $a\ \eta_\alpha\ b$. That conversely (β) implies condition (ii) is straightforward to prove and is omitted.

Now let $a \in S_\alpha$, $b \in S_\beta$, $ab\ \eta_{\alpha\beta}\ ba$ and $ab^{-1} \in \ker \eta_{\alpha\beta}$. Then

$$(a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta})\ \eta_\beta\ (b\varphi_{\beta,\alpha\beta})(a\varphi_{\alpha,\alpha\beta})$$

which in $\bar{S}_{\alpha\beta} = S_{\alpha\beta}/\eta_{\alpha\beta}$ implies that

$$\overline{a\varphi_{\alpha,\alpha\beta}}\ \mathcal{H}\ \overline{b\varphi_{\beta,\alpha\beta}}.$$

It then follows that

$$\overline{a\varphi_{\alpha,\alpha\beta}}^{0} = \overline{b\varphi_{\beta,\alpha\beta}}^{0}$$

and hence

$$(a\varphi_{\alpha,\alpha\beta})^{0}\ \eta_{\alpha\beta}\ (b\varphi_{\beta,\alpha\beta})^{0}.$$

The condition $ab^{-1} \in \ker \eta_{\alpha\beta}$ evidently implies that

$$(a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta})^{-1} \in \ker \eta_{\alpha\beta}.$$

But then

$$a\varphi_{\alpha,\alpha\beta}\ \eta_{\alpha\beta}\ b\varphi_{\beta,\alpha\beta}.$$

Next let $c \in S_{\gamma}$. Then by condition (α), we get

$$a\varphi_{\alpha,\alpha\beta\gamma}\ \eta_{\alpha\beta\gamma}\ b\varphi_{\beta,\alpha\beta\gamma}$$

and therefore

$$acb = (a\varphi_{\alpha,\alpha\beta\gamma})(c\varphi_{\gamma,\alpha\beta\gamma})(b\varphi_{\beta,\alpha\beta\gamma})$$

$$\eta_{\alpha\beta\gamma}\ (b\varphi_{\beta,\alpha\beta\gamma})(c\varphi_{\gamma,\alpha\beta\gamma})(a\varphi_{\alpha,\alpha\beta\gamma}) = bca$$

as required.

If $a\ \rho_{(\xi;\eta_\alpha)}\ b$, then $ab\ \eta_{\alpha\beta}\ ba$ and $ab^{-1} \in \ker \eta_{\alpha\beta}$; we have shown that

$$a\varphi_{\alpha,\alpha\beta}\ \eta_{\alpha\beta}\ b\varphi_{\beta,\alpha\beta}.$$

Conversely, if

$$\alpha\ \xi\ \beta \quad \text{and} \quad a\varphi_{\alpha,\alpha\beta}\ \eta_{\alpha\beta}\ b\varphi_{\beta,\alpha\beta},$$

then it follows at once that

$$ab\ \eta_{\alpha\beta}\ ba \quad \text{and} \quad ab^{-1} \in \ker \eta_{\alpha\beta}$$

so that

$$a\ \rho_{(\xi;\eta_\alpha)}\ b.$$

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The above theorem shows that the definitions of the congruence aggregate $(\xi;\eta_\alpha)$ as well as the congruence $\rho_{(\xi;\eta_\alpha)}$ here and in **[8]** coincide for the case of a strong semilattice of completely simple semigroups.

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