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Hyperbolic geometry and reflection groups

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The n-dimensional pseudospheres are the surfaces in $\mathbf{R}^{n+1}$ given by the equations ${x_1}^2 + {x_2}^2 + \ldots + {x_k}^2 - {x_{k+1}}^2 - \ldots - {x_{n+1}}^2 = 1$ $(1 \leqslant k \leqslant n+1)$. The cases $k = 1, n+1$ give, respectively a pair of hyperboloids, and the ordinary n-sphere.

In the first chapter we consider the pseudospheres as sufaces in $E_{n+1,k}$, where $E_{m,k} = \mathbf{R}^k \times (i\mathbf{R})^{m-k}$, and investigate their geometry in terms of the linear algebra of these spaces.

The main objects of investigation are finite sequences of hyperplanes in a pseudosphere. To each such sequence we associate a square symmetric matrix, the Gram matrix, which gives information about angle and incidence properties of the hyperplanes. We find when a given matrix is the Gram matrix of some sequence of hyperplanes, and when a sequence is determined up to isometry by its Gram matrix.

We also consider subspaces of pseudospheres and projections onto them. This leads to an n-dimensional cosine rule for spherical and hyperbolic simplices.

In the second chapter we derive integral formulae for the volume of an n-dimensional spherical or hyperbolic simplex, both in terms of its dihedral angles and its edge lengths. For the regular simplex with common edge length γ we then derive power series for the volume, both in $u = \sin(\gamma/2)$, and in γ itself, and discuss some of the properties of the coefficients. In obtaining these series we encounter an interesting family of entire functions, $R_n(p)$ (n a nonegative integer and $p \in \mathbf{C}$). We derive a functional equation relating $R_n(p)$ and $R_{n-1}(p)$.

Finally we classify, up to isometry, all tetrahedra with one or more vertices truncated, for which the dihedral angles along the edges formed by the truncations are all $\pi/2$, and the remaining dihedral angles are all submultiples of π. We show how to find the volumes of these polyhedra, and find presentations and small generating sets for the orientation-preserving subgroups of their reflection groups.

For particular families of these groups, we find low index torsion free subgroups, and construct associated manifolds and manifolds with boundary. In particular, we

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find a sequence of manifolds with totally geodesic boundary of genus $g \geqslant 2$, which we conjecture to be of least volume among such manifolds.

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