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# SUCCESSIVE RADII OF FAMILIES OF CONVEX BODIES

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**Abstract**

We study properties of the so-called inner and outer successive radii of special families of convex bodies. First we consider the balls of the $p$-norms, for which we show that the precise value of the outer (inner) radii when $p \geq 2$ $(1 \leq p \leq 2)$, as well as bounds in the contrary case $1 \leq p \leq 2$ $(p \geq 2)$, can be obtained as consequences of known results on Gelfand and Kolmogorov numbers of identity operators between finite-dimensional normed spaces. We also prove properties that successive radii satisfy when we restrict to the families of the constant width sets and the $p$-tangential bodies.

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## 1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, that is, compact convex sets, in the $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|_2$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively, and denote by $\mathrm{e}_i$ the $i$th canonical unit vector. For a 0-symmetric convex body $K \in \mathcal{K}^n$, which by definition satisfies $K = -K$, the well-known Minkowski functional $\min\{\lambda \geq 0 : x \in \lambda K\}$ defines a norm, denoted by $|\cdot|_K$, which has $K$ as its unit ball.

The set of all $i$-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}^n_i$, and for $L \in \mathcal{L}^n_i$, $L^\perp$ denotes its orthogonal complement. For $K \in \mathcal{K}^n$ and $L \in \mathcal{L}^n_i$, the orthogonal projection of $K$ onto $L$ is denoted by $K|L$. By $\mathrm{lin}\{u_1,\dots,u_m\}$ we denote the linear hull of the vectors $u_1,\dots,u_m$, and by $[u_1,u_2]$ the line segment with end-points $u_1,u_2$. Finally, for $S \subset \mathbb{R}^n$ we denote by $\mathrm{conv}\, S$ the convex hull of $S$ and by $\mathrm{int}\, S$ and $\mathrm{bd}\, S$ its interior and boundary. We write $\mathrm{relbd}\, S$ to represent the relative boundary, that is, the boundary of $S$ relative to its affine hull, $\mathrm{aff}\, S$, and $\dim S$ to represent the dimension of the set, that is, the dimension of $\mathrm{aff}\, S$.

The diameter, minimal width, circumradius and inradius of a convex body $K$ are denoted by $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For more information on these

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functionals and their properties we refer to [5, pages 56–59]. If $f$ is a functional on $\mathcal{K}^n$ depending on the dimension in which a convex body $K$ is embedded and if $K$ is contained in an affine space $A$, then we write $f(K; A)$ to stress that $f$ has to be evaluated with respect to the space $A$. Successive outer and inner radii are defined in the following way.

Definition 1.1. For $K \in \mathcal{K}^n$ and $i = 1, \dots, n$, let

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \quad \text{and} \quad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}(K \cap (x + L); x + L).$$

It is clear that the outer radii are increasing in $i$, whereas the inner radii are decreasing in $i$. Moreover, $\mathrm{R}_i(K)$ is the smallest radius of a solid cylinder with $i$-dimensional spherical cross section containing $K$, and $\mathrm{r}_i(K)$ is the radius of the greatest $i$-dimensional ball contained in $K$. We obviously have $\mathrm{R}_n(K) = \mathrm{R}(K)$, $\mathrm{R}_1(K) = \omega(K)/2$, $\mathrm{r}_n(K) = \mathrm{r}(K)$ and $\mathrm{r}_1(K) = \mathrm{D}(K)/2$.

The first systematic study of these and other families of successive radii was developed in [2]. For more information on these radii and their relation with other measures, we refer, for example, to [3, 16, 18, 19, 27] and the references therein. The successive radii of Definition 1.1 are closely related to some notions in approximation theory, namely, they are particular cases of the so-called Gelfand and Kolmogorov numbers of identity operators between finite-dimensional normed spaces (see, for example, [22, 25]), which will be defined in Section 2.

Here we are interested in computing and studying properties of the radii of special families of convex bodies. So far, only orthogonal boxes, orthogonal cross-polytopes [7, 13], simplices [1, 4, 7, 9] and ellipsoids [17] have been studied and their radii explicitly given. In this paper we mainly consider two families of convex bodies.

(i) Unit $p$-balls. For $p \geq 1$ we denote by $B_p^n$ the unit $p$-ball associated to the $p$-norm $|\cdot|_p$, that is,

$$B_p^n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq 1 \right\},$$

with $|x|_\infty = \max\{|x_i| : i = 1, \dots, n\}$. The normed space with unit ball $B_p^n$ is as usual denoted by $\ell_p^n$. For the sake of brevity, we will write $B_n = B_2^n$ to denote the $n$-dimensional Euclidean unit ball. Moreover, for $L \in \mathcal{L}_i^n$, we will write $B_{i,L} = B_n \cap L$. Notice that when $p = 1$ and $p = \infty$ the (unit) $p$-balls are, respectively, the regular cross-polytope $B_1^n = \mathrm{conv}\{\pm \mathrm{e}_1, \dots, \pm \mathrm{e}_n\}$ and the regular cube $B_\infty^n = \sum_{j=1}^n [-\mathrm{e}_j, \mathrm{e}_j]$ with edge length 2; here + denotes the usual Minkowski (vectorial) addition. The values of the successive radii of these particular $p$-balls are (see, for example, [7])

$$\mathrm{R}_i(B_\infty^n) = \sqrt{i}, \quad \mathrm{r}_i(B_\infty^n) = \sqrt{\frac{n}{i}}, \quad \mathrm{R}_i(B_1^n) = \sqrt{\frac{i}{n}}, \quad \mathrm{r}_i(B_1^n) = \sqrt{\frac{1}{i}}. \tag{1.1}$$

(ii) Constant width sets. A convex body $K \in \mathcal{K}^n$ has constant width if it has the same width b in all directions, that is, if its diameter and minimal width have the same value, $\mathrm{D}(K) = \omega(K) = \mathrm{b}$. The class of convex bodies of constant width will be denoted by $\mathcal{W}^n$. For a nice and thorough survey on convex bodies of constant width, see [11].

In Section 2 we introduce the Gelfand and Kolmogorov numbers, stating the necessary notation from Banach space theory and approximation theory. We also present some known properties of these numbers and show the relation between them and the successive radii of Definition 1.1.

Section 3 is devoted to studying the inner and outer radii of the (unit) $p$-balls, obtaining the following results.

Theorem 1.2. *Let $p \geq 2$ and $1 \leq q \leq 2$. For all $i = 1, \ldots, n$,*

$$\mathrm{R}_i(B_p^n) = i^{1/2-1/p} \quad \text{and} \quad \mathrm{r}_i(B_q^n) = i^{1/2-1/q}.$$

In the case $1 \leq p \leq 2$ (respectively, $q \geq 2$) we give matching upper and lower bounds for the outer (inner) radii which are collected in the next theorem. Here and later we use the notation $a_{n,i} \asymp b_{n,i}$ for some double sequences $a_{n,i}, b_{n,i}$ of nonnegative real numbers to mean that there exist absolute constants $c, C > 0$ such that $c\, a_{n,i} \leq b_{n,i} \leq C\, a_{n,i}$.

Theorem 1.3. *Let $1 \leq p \leq 2$ and $q \geq 2$. For all $i = 1, \ldots, n$,*

$$\mathrm{R}_i(B_p^n) \asymp \begin{cases} \left(\dfrac{i}{n}\right)^{1/2} & \text{for } i \geq n^{2(1-1/p)}, \\ n^{1/2-1/p} & \text{for } i \leq n^{2(1-1/p)}, \end{cases}$$

*and*

$$\mathrm{r}_i(B_q^n) \asymp \begin{cases} \left(\dfrac{n}{i}\right)^{1/2} & \text{for } i \geq n^{2/q}, \\ n^{1/2-1/q} & \text{for } i \leq n^{2/q}. \end{cases}$$

In Section 4 we study the relation between the radii and the constant width sets. It is well known (see, for example, [12, page 125]) that if $K \in \mathcal{W}^n$ with width b, then the inball and the circumball of $K$ are concentric and

$$\mathrm{R}(K) + \mathrm{r}(K) = \mathrm{b} \quad \text{and} \quad \mathrm{D}(K) + \omega(K) = 2\mathrm{b}. \tag{1.2}$$

So the natural question arises if an analogous relation holds for the more general inner and outer radii, namely,

$$\mathrm{R}_i(K) + \mathrm{r}_i(K) = \mathrm{b}, \quad i = 1, \ldots, n. \tag{1.3}$$

The next theorem shows that this relation is, in general, not true except, of course, when $i = 1, n$.

Theorem 1.4. *Let $K \in \mathcal{W}^n$ with width* b*. Then $\mathrm{R}_i(K) + \mathrm{r}_i(K) \leq \mathrm{b}$, and the inequality can be strict, as the Meissner body shows.*

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It can be easily seen (see Proposition 4.1) that for a different definition of inner radii in which projections are involved, it is possible to get an equality relation of the type (1.3). Some additional properties of the radii of constant width sets are also studied.

Finally, in Section 5 we consider an additional family of convex bodies, the so-called $p$-tangential bodies, for which a nice relation between their inner radii can be proved.

## 2. Gelfand numbers, Kolmogorov numbers and successive radii of symmetric convex bodies

The authors of [16] have already mentioned the close relation of successive radii to notions of width studied in approximation theory (see, for example, [10, 25, 26]). Nevertheless, it seems that up to now this intimate connection has not been highlighted in its full generality. Some results proved for successive radii in recent years can be translated from corresponding results about Gelfand numbers and Kolmogorov numbers of identity operators between finite-dimensional normed spaces. Our aim in this section is to point out the formal connection between successive radii and Gelfand and Kolmogorov numbers and to translate results from approximation theory to the geometric setting of successive radii.

We start by introducing the necessary notation from Banach space theory and approximation theory. The letters $X, Y$ always stand for Banach spaces. The dual space of all bounded linear functionals on $X$ will be denoted by $X'$. In this particular setting, we will also represent the action of $a \in X'$ on $x \in X$ by $\langle x, a\rangle$. The Banach space $\mathcal{L}(X, Y)$ is the space of all linear bounded operators from $X$ to $Y$ with the usual operator norm, denoted by $\|\cdot\|$. Then the dual operator $T' \in \mathcal{L}(Y', X')$ of $T \in \mathcal{L}(X, Y)$ is given by $\langle x, T'b\rangle = \langle Tx, b\rangle$ for $x \in X$ and $b \in Y'$. It satisfies $\|T'\| = \|T\|$.

For $T \in \mathcal{L}(X, Y)$, we define the *kth approximation number* as

$$a_k(T) := \inf\{\|T - R\| : R \in \mathcal{L}(X, Y),\ \operatorname{rank} R < k\},$$

the *kth Gelfand number* as

$$c_k(T) := \inf\{\|T_{|_M}\| : M \text{ a linear subspace of } X,\ \operatorname{codim} M < k\},$$

and the *kth Kolmogorov number* as

$$d_k(T) := \inf\{\|q_N T\| : N \text{ a linear subspace of } Y,\ \dim N < k\};$$

here $T_{|_M}$ is the restriction of $T$ to the subspace $M$ and $q_N$ denotes the quotient mapping $Y \longrightarrow Y/N$.

More explicit descriptions of the Gelfand and Kolmogorov numbers are

$$c_k(T) = \inf_{\substack{M\subset X\\ \operatorname{codim} M<k}} \sup_{x\in M, \|x\|\le 1} \|Tx\|$$

and

$$d_k(T) = \inf_{\substack{N\subset Y\\ \dim N<k}} \sup_{x\in X, \|x\|\le 1} \inf_{y\in N} \|Tx - y\|.$$

In the following lemma we collect some basic known facts about these quantities. For this and more information on $s$-numbers of operators in the normed case we refer to [22, 25].

Lemma 2.1. *Let $s \in \{a, c, d\}$, $k \in \{1, \dots, n\}$ and $T \in \mathcal{L}(X, Y)$. Then:*

(i) $\|T\| \geq s_1(T) \geq s_2(T) \geq s_3(T) \geq \cdots \geq 0$;
(ii) $s_k(STR) \leq \|S\| s_k(T) \|R\|$, *for all operators $R, S$ for which the product $STR$ is defined;*
(iii) $c_k(T) \leq a_k(T)$ *and* $d_k(T) \leq a_k(T)$;
(iv) $c_k(T) = a_k(T)$ *whenever $X$ is a Hilbert space and* $d_k(T) = a_k(T)$ *whenever $Y$ is a Hilbert space;*
(v) $a_k(T') = a_k(T)$, *and* $d_k(T') = c_k(T)$ *whenever $T$ is a compact operator between Banach spaces.*

In order to state the connection of the above numbers with the successive radii, we need the well-known correspondence between a 0-symmetric convex body $K \in \mathcal{K}^n$ and the $n$-dimensional normed space $X_K = (\mathbb{R}^n, |\cdot|_K)$ with unit ball $K$. For two such bodies $K$ and $E$, let $I_K^E$ denote the identity operator of $\mathbb{R}^n$ considered as an operator between the corresponding normed spaces, $X_K \longrightarrow X_E$. If $K = B_p^n$, then we use the abbreviation $I_p^E$ for $I_K^E$. Similarly, if $E = B_q^n$, we write $I_K^q$ for $I_K^E$. The notation $I_p^q$ is self-explanatory.

Let $K^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \text{ for all } x \in K\}$ denote, as usual, the polar body of $K$. Recall that $K^*$ is the unit ball of the dual space of $X_K$, that is, $X_K' = X_{K^*}$. Moreover,

$$(I_K^E)' = I_{E^*}^{K^*}. \tag{2.1}$$

The following theorem gives the formal connection between the Gelfand and Kolmogorov numbers, and the successive radii.

Theorem 2.2. *Let $K \in \mathcal{K}^n$ be 0-symmetric. For all $i = 1, \dots, n$,*

$$\mathrm{r}_i(K) = c_{n-i+1}(I_2^K)^{-1} = d_{n-i+1}(I_{K^*}^2)^{-1} = a_{n-i+1}(I_2^K)^{-1}$$

*and*

$$\mathrm{R}_i(K) = d_{n-i+1}(I_K^2) = c_{n-i+1}(I_2^{K^*}) = a_{n-i+1}(I_K^2).$$

Proof. The last two equalities between the Gelfand, Kolmogorov and approximation numbers follow immediately from the properties of these numbers stated above (see Lemma 2.1 and (2.1)).

For a 0-symmetric convex body $K$, the definition of $\mathrm{r}_i(K)$ reduces to

$$\mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K \cap L; L).$$

Let $L \in \mathcal{L}_i^n$ be any $i$-dimensional linear subspace of $\mathbb{R}^n$. Observe that

$$\|I_2^K{}_{|_L}\| = \min\{R > 0 : |x|_K \leq R|x|_2 \text{ for all } x \in L\}$$

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and

$$\begin{aligned}\mathrm{r}(K\cap L;L) &= \max\{r>0 : rB_{i,L}\subset K\cap L\}\\ &= \max\Big\{r>0 : |x|_K \le \frac{1}{r}|x|_2 \text{ for all } x\in L\Big\}.\end{aligned}$$

Thus it follows that

$$\mathrm{r}(K\cap L;L) = \|I_2^K{}_{|_L}\|^{-1},$$

and taking the maximum over $L\in\mathcal{L}_i^n$, which is the same as taking the maximum over all $L$ with $\operatorname{codim} L < n-i+1$, we get $\mathrm{r}_i(K) = c_{n-i+1}(I_2^K)^{-1}$.

The equality for the outer radii is now deduced from the duality relation $d_{n-i+1}(I_K^2) = c_{n-i+1}(I_2^{K^*})$ stated above, the previously proved identity $\mathrm{r}_i(K^*) = c_{n-i+1}(I_2^{K^*})^{-1}$ and the known relation (see [16, (1.2)])

$$\mathrm{R}_i(K)\,\mathrm{r}_i(K^*) = 1. \tag{2.2}$$

We would like to emphasize that (2.2) can be also seen as a special case of the duality relation $c_k(T') = d_k(T)$ between Gelfand and Kolmogorov numbers. For completeness we give a self-contained short argument.

To this end, observe that

$$\begin{aligned}\mathrm{R}(K|L) &= \min\{R>0 : K|L\subset RB_n\}\\ &= \min\{R>0 : |P_Lx|_2 \le R|x|_K \ \text{ for all } x\in\mathbb{R}^n\}\\ &= \|P_LI_K^2\|,\end{aligned}$$

where $P_L$ denotes the orthogonal projection onto $L$ in the Euclidean space $\ell_2^n$. Then it follows from [24, Proposition 11.6.2] that

$$\begin{aligned}\mathrm{R}_i(K) &= \min_{L\in\mathcal{L}_i^n}\mathrm{R}(K|L) = \min_{L\in\mathcal{L}_i^n}\|P_LI_K^2\| = d_{n-i+1}(I_K^2)\\ &= c_{n-i+1}(I_2^{K^*}) = \frac{1}{\mathrm{r}_i(K^*)}.\end{aligned}$$ □

## 3. Successive radii of $p$-balls

In this section we use the general characterisation of inner and outer successive radii by approximation quantities given in Theorem 2.2 to deduce exact values and sharp asymptotic estimates for successive radii of $p$-balls. This shows that the results for $p=1$ and $p=\infty$ referred to in (1.1) can also be derived from known results about Gelfand and Kolmogorov numbers.

We start collecting the known results for Gelfand and Kolmogorov numbers $c_k(I_2^p)$ and $d_k(I_p^2)$ for $1\le p\le\infty$. It was proved by Steckin [30] and Pietsch [23] that for all $k=1,\ldots,n$,

$$d_k(I_1^2) = c_k(I_2^\infty) = \sqrt{\frac{n-k+1}{n}} \quad\text{and}\quad c_k(I_1^2) = d_k(I_2^\infty) = \sqrt{n-k+1}.$$

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By Theorem 2.2 this immediately implies (1.1). Pietsch actually computed all $s$-numbers

$$a_k(I_p^q) = c_k(I_p^q) = d_k(I_p^q) = (n-k+1)^{1/q-1/p}$$

when $1 \leq q \leq p \leq \infty$. In particular,

$$d_k(I_p^2) = (n-k+1)^{1/2-1/p} \quad \text{and} \quad c_k(I_2^q) = (n-k+1)^{1/q-1/2}$$

for $2 \leq p \leq \infty$ and $1 \leq q \leq 2$. Then, using Theorem 2.2, we get as a direct consequence the evaluations of the successive radii in Theorem 1.2.

The computation of the remaining Kolmogorov and Gelfand numbers of identity operators $I_p^q$ turned out to be more complicated. In the cases relevant for us, the exact values seem to be very difficult to determine. Nevertheless, matching lower and upper bounds up to multiplicative constants are known. The result we need is due to Gluskin [15], who proved that, for $q \geq 2$,

$$c_k(I_2^q) \asymp \begin{cases} \left(\dfrac{n-k+1}{n}\right)^{1/2} & \text{for } 1 \leq k \leq n+1-n^{2/q}, \\ n^{1/q-1/2} & \text{for } n+1-n^{2/q} \leq k \leq n, \end{cases}$$

and, by duality, for $1 < p \leq 2$,

$$d_k(I_p^2) \asymp \begin{cases} \left(\dfrac{n-k+1}{n}\right)^{1/2} & \text{for } 1 \leq k \leq n+1-n^{2(1-1/p)}, \\ n^{1/2-1/p} & \text{for } n+1-n^{2(1-1/p)} \leq k \leq n. \end{cases}$$

By Theorem 2.2, the direct consequence for successive radii is Theorem 1.3.

In connection with the approximation of embeddings between function spaces, considerable work has been done to compute the Gelfand and Kolmogorov numbers of diagonal operators. We will now translate some of this work into results for successive radii. Let $D_t$ be the diagonal matrix with diagonal $t = (t_1, \ldots, t_n)$, considered as a map on $\mathbb{R}^n$. We will always assume that $t_1 \geq t_2 \geq \cdots \geq t_n > 0$. The following result is a special case of [24, Theorem 11.11.4].

**Proposition 3.1.** *Let $1 \leq q \leq 2$ and $p \geq 2$ and define positive numbers $r, s$ by $1/r = 1/q - 1/2$ and $1/s = 1/2 - 1/p$. Then*

$$c_k(D_t : \ell_2^n \longrightarrow \ell_q^n) = \left(\sum_{j=k}^n t_j^r\right)^{1/r} \quad \text{and} \quad d_k(D_t : \ell_p^n \longrightarrow \ell_2^n) = \left(\sum_{j=k}^n t_j^s\right)^{1/s}.$$

Let $K_p = D_t(B_p^n)$, $p \geq 2$, and $K^q = D_t^{-1}(B_q^n)$, $1 \leq q \leq 2$. That is, $K_p$ and $K^q$ are orthogonally dilated images of the balls $B_p^n$ and $B_q^n$, respectively, $t_i$ and $t_i^{-1}$ being the respective lengths of the half-axes in the direction $e_i$. Thus from the properties of the Gelfand and Kolmogorov numbers, we directly obtain from Proposition 3.1 that

$$c_k(I_2^{K^q}) = \left(\sum_{j=k}^n t_j^r\right)^{1/r} \quad \text{and} \quad d_k(I_{K_p}^2) = \left(\sum_{j=k}^n t_j^s\right)^{1/s}.$$

Finally, Theorem 2.2 leads to the following result.

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Theorem 3.2. *Let $1 \leq q \leq 2$ and $p \geq 2$, and define positive numbers $r, s$ by $1/r = 1/q - 1/2$ and $1/s = 1/2 - 1/p$. Let $t = (t_1, \dots, t_n)$ be such that $t_1 \geq t_2 \geq \cdots \geq t_n > 0$, and let $K_p = D_t(B_p^n)$ and $K^q = D_t^{-1}(B_q^n)$. Then*

$$\mathrm{r}_i(K^q) = \Big(\sum_{j=n-i+1}^{n} t_j^r\Big)^{-1/r} \quad \textit{and} \quad \mathrm{R}_i(K_p) = \Big(\sum_{j=n-i+1}^{n} t_j^s\Big)^{1/s}.$$

For $q = 1$ and $p = \infty$, the values of the inner radii of orthogonal cross-polytopes and the outer radii of orthogonal boxes are obtained (see [7, Theorem 4.4]); for $p = q = 2$ the successive radii of the ellipsoids can be deduced (see [17, page 18]), namely, $\mathrm{R}_i(K_2) = t_{n-i+1}$, $\mathrm{r}_i(K^2) = t_i$.

We also remark that the values of the outer radii of orthogonal cross-polytopes (and so the inner radii of orthogonal boxes) can be derived from [24, Theorem 11.11.7] via Theorem 2.2 (see [7, Proposition 4.3] and [13, Theorem 1]). Finally, we mention that the results from [20, 21] can be used to compute (or to estimate, up to multiplicative constants) the successive radii of unit balls of symmetric $n$-dimensional normed spaces; in particular, this applies to unit balls of Lorentz and Orlicz sequence spaces.

**3.1. Appendix: a geometrical proof of Theorem 1.2.** In this appendix we sketch a geometrical proof of Theorem 1.2. We point out that it partly follows the idea of the proof of [24, Theorem 11.11.4], from a geometric point of view.

In order to prove the theorem, we need the following two facts. On the one hand, it is an easy computation to check that:

$$\begin{aligned} &\text{if } 1 \leq p \leq 2 \quad \text{then } \mathrm{R}(B_p^n) = 1;\\ &\text{if } p \geq 2 \quad \text{then } \mathrm{R}(B_p^n) = n^{1/2-1/p}. \end{aligned} \tag{3.1}$$

On the other hand, we observe that if $P \subset \mathbb{R}^n$ is a polytope with $0 \in \operatorname{int} P$ then, for any $L \in \mathcal{L}_i^n$, $P_L = P \cap L$ is an $i$-dimensional polytope. Let $v$ be a vertex of $P_L$ and denote by $F$ the smallest (in the sense of dimension) face of $P$ containing $v$, which gives $F \cap L = \{v\}$. If we assume that $\dim F > n - i$, then $\dim(F + L) = i + \dim F > n$, which is not possible. Therefore $\dim F \leq n - i$, that, we have proved the following:

$$\begin{aligned} &\text{If } P \subset \mathbb{R}^n \text{ is a polytope with } 0 \in \operatorname{int} P, \text{ then any } L \in \mathcal{L}_i^n \text{ intersects}\\ &\quad P \text{ in one of its } (n-i)\text{-faces.} \end{aligned} \tag{3.2}$$

Proof of Theorem 1.2. Notice that in order to prove that $\mathrm{R}_i(B_p^n) = i^{1/2-1/p}$, $p \geq 2$, it suffices to show that

$$\mathrm{R}(B_p^n \cap L) \geq i^{1/2-1/p} \quad \text{for all } L \in \mathcal{L}_i^n; \tag{3.3}$$

then, using (3.1), since $\mathrm{R}(B_p^n|L) \geq \mathrm{R}(B_p^n \cap L)$ and

$$\mathrm{R}(B_p^n|\operatorname{lin}\{e_1, \dots, e_i\}) = \mathrm{R}(B_p^n \cap \operatorname{lin}\{e_1, \dots, e_i\}) = i^{1/2-1/p},$$

we get that $\mathrm{R}_i(B_p^n) = i^{1/2-1/p}$, as required.

Let $L \in \mathcal{L}_i^n$. By (3.2) there exists an $(n-i)$-face $F_{n-i}$ of the cube $B_\infty^n$ such that $L \cap F_{n-i} \neq \emptyset$. Let $x \in L \cap F_{n-i}$. Without loss of generality we assume that

$$F_{n-i} = \{(t_1, \dots, t_{n-i}, 1, \dots, 1) \in \mathbb{R}^n : |t_j| \leq 1,\ j = 1, \dots, n-i\},$$

that is, $x = (x_1, \dots, x_{n-i}, 1, \dots, 1)$ with $|x_j| \leq 1$, $j = 1, \dots, n-i$. Moreover, let $\lambda = (i + \sum_{j=1}^{n-i} |x_j|^p)^{-1/p} \in (0, 1]$. Then $z = \lambda x \in L \cap \operatorname{bd} B_p^n$, and since $p \geq 2$ and $|x_j| \leq 1$, we clearly get

$$|z|_2 = \frac{(i + \sum_{j=1}^{n-i} |x_j|^2)^{1/2}}{(i + \sum_{j=1}^{n-i} |x_j|^p)^{1/p}} \geq \frac{(i + \sum_{j=1}^{n-i} |x_j|^p)^{1/2}}{(i + \sum_{j=1}^{n-i} |x_j|^p)^{1/p}} = \Big(i + \sum_{j=1}^{n-i} |x_j|^p\Big)^{1/2-1/p} \geq i^{1/2-1/p}.$$

This proves (3.3). Finally, the value for the inner radii comes from (2.2) and the fact that $(B_p^n)^* = B_q^n$ with $1/p + 1/q = 1$. □

## 4. On constant width sets

Constant width sets have been intensively studied throughout the last century. In the plane they are well known, whereas the situation becomes much more complicated in dimension $n \geq 3$ (see, for example, [5, §15], [12, Ch. 7] and [11] for detailed surveys).

The best-known three-dimensional constant width sets are the revolution of planar convex bodies with constant width and the so-called Meissner bodies which are constructed, roughly speaking, in the following way. Let $T_3$ be a three-dimensional regular tetrahedron with edge length b, and consider the intersection $K$ of four balls of radius b having the vertices of $T_3$ as centers. Then $K$ is bounded by four pieces of sphere which meet in six circular arcs. However, $K$ is not a constant width set, because the distance between two of those opposite circular arcs is strictly greater than b. The Meissner bodies are then obtained by suitably rounding three of those arcs (see Figure 1). Notice that two Meissner bodies can be constructed, depending on whether the three rounded arcs converge to a vertex or form a triangle. For a more detailed construction of the Meissner bodies we refer to [5, page 144].

Proof of Theorem 1.4. For $K \in \mathcal{W}^n$ with width b, let $L' \in \mathcal{L}_i^n$ be such that

$$\mathrm{r}(K|L'; L') = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L). \tag{4.1}$$

It is well known (see, for example, [5, page 135]) that every orthogonal projection of a constant width set is also a body of constant width having the same width. Then, using (1.2) one can easily obtain that

$$\begin{aligned} \mathrm{b} &= \mathrm{R}(K|L') + \mathrm{r}(K|L'; L') \\ &\geq \mathrm{R}_i(K) + \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L) \\ &\geq \mathrm{R}_i(K) + \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}(K \cap (x + L); x + L) \\ &= \mathrm{R}_i(K) + \mathrm{r}_i(K). \end{aligned}$$

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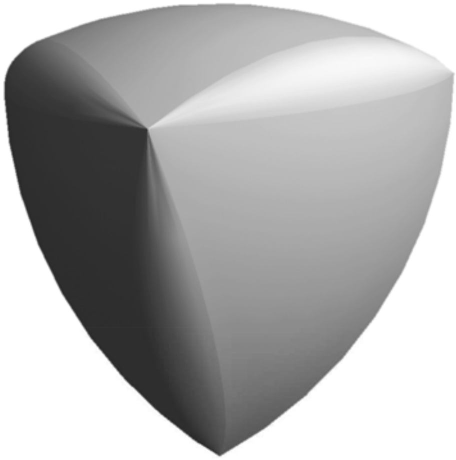

FIGURE 1. A Meissner body.

So it remains to prove that the inequality can be strict. Let $K_M \in \mathcal{W}^3$ be a Meissner body with width b. It is known (see, for example, [6, page 37]) that the orthogonal projection of $K_M$ onto the plane $\Pi$ determined by two opposite edges of the generating tetrahedron is a two-dimensional ball with radius b/2. Then, since $\mathrm{R}_2(K_M) \geq \mathrm{R}_1(K_M) = \mathrm{b}/2$ and $\mathrm{R}(K_M|\Pi) = \mathrm{b}/2$, we get $\mathrm{R}_2(K_M) = \mathrm{b}/2$. So, we have to prove that $\mathrm{r}_2(K_M) < \mathrm{b}/2$. In order to show this, we assume that $\mathrm{r}_2(K_M) = \mathrm{b}/2$, and we will get a contradiction.

From the definition of $\mathrm{r}_2(K_M)$, there exist $L \in \mathcal{L}_2^3$ and $x \in L^\perp$ such that

$$\frac{\mathrm{b}}{2} = \mathrm{r}_2(K_M) = \mathrm{r}(K_M \cap (x + L); x + L),$$

and thus there exists a circle $C$ of radius b/2 contained in $K_M \cap (x + L)$. Moreover, $C = K_M \cap (x + L)$, otherwise there would exist a point $p \in (K_M \cap (x + L))\backslash C$, and then $\mathrm{D}(K_M) \geq \mathrm{D}(K_M \cap (x + L)) > \mathrm{b}$, which is not possible. Let $y \in \operatorname{int} K_M$ such that $C = y + (\mathrm{b}/2)B_{i,L}$, and let $v \in \operatorname{relbd} B_{i,L}$. The point $y + (\mathrm{b}/2)v \in y + (\mathrm{b}/2) \operatorname{relbd} B_{i,L} = \operatorname{relbd} C$, and thus $y + (\mathrm{b}/2)v$ cannot be a vertex of $K_M$.

On the one hand, if $y + (\mathrm{b}/2)v$ lies on one of the four pieces of sphere bounding $K_M$, by the construction of the Meissner body and, taking into account that the segment $[y - (\mathrm{b}/2)v, y + (\mathrm{b}/2)v] \subset C$ and that it has length b, then $y - (\mathrm{b}/2)v$ should be the opposite vertex, which is not possible. On the other hand, if $y + (\mathrm{b}/2)v$ lies on one of the (rounded) arcs, then $C$ should touch one of the opposite sphere pieces of $K_M$, which leads to the previous case and again to a contradiction. □

We notice that (4.1) indeed defines another sequence of successive inner radii, namely,

$$\widetilde{\mathrm{r}}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L);$$

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see [2] for a detailed study of these and other radii. If these new inner radii are involved, then an equality of type (1.2) is obtained. Moreover, it is well known that for any constant width set $K \in \mathcal{W}^n$ of width b,

$$\mathrm{b}\left(1 - \sqrt{\frac{n}{2(n+1)}}\right) \le \mathrm{r}(K) \le \mathrm{R}(K) \le \mathrm{b}\sqrt{\frac{n}{2(n+1)}} \tag{4.2}$$

(see, for example, [11, page 68] or [12, page 125]); the analogous result for these inner and the outer radii can be easily obtained.

Proposition 4.1. *For any $K \in \mathcal{W}^n$ of width* b *and all $i = 1, \dots, n$,*

$$\mathrm{R}_i(K) + \widetilde{\mathrm{r}}_i(K) = \mathrm{b}$$

*and*

$$\mathrm{b}\left(1 - \sqrt{\frac{i}{2(i+1)}}\right) < \widetilde{\mathrm{r}}_i(K) \le \mathrm{R}_i(K) < \mathrm{b}\sqrt{\frac{i}{2(i+1)}}. \tag{4.3}$$

Proof. Notice that for any $K \in \mathcal{W}^n$, say of width b, and for any $i = 1, \dots, n$, the $i$-plane $L' \in \mathcal{L}_i^n$ giving the value for $\mathrm{R}_i(K)$ also gives $\widetilde{\mathrm{r}}_i(K)$: indeed, if $\mathrm{R}_i(K) = \mathrm{R}(K|L')$, since $K|L$ is also a constant width set of width b satisfying $\mathrm{R}(K|L) + \mathrm{r}(K|L; L) = \mathrm{b}$ for all $L \in \mathcal{L}_i^n$ (see (1.2)), then

$$\mathrm{r}(K|L'; L') = \mathrm{b} - \mathrm{R}(K|L') \ge \mathrm{b} - \mathrm{R}(K|L) = \mathrm{r}(K|L; L)$$

for all $L \in \mathcal{L}_i^n$, and so $\widetilde{\mathrm{r}}_i(K) = \mathrm{r}(K|L'; L')$. Therefore,

$$\mathrm{R}_i(K) + \widetilde{\mathrm{r}}_i(K) = \mathrm{R}(K|L') + \mathrm{r}(K|L'; L') = \mathrm{b};$$

moreover, applying (4.2) to the $i$-dimensional set $K|L'$ gives the left and right inequalities in (4.3). In order to conclude the proof of (4.3) notice that, since $K$ is a constant width set and $\widetilde{\mathrm{r}}_1(K) = \mathrm{D}(K)/2$ (see [2]), it follows that $\widetilde{\mathrm{r}}_i(K) \le \widetilde{\mathrm{r}}_1(K) = \mathrm{D}(K)/2 = \omega(K)/2 = \mathrm{R}_1(K) \le \mathrm{R}_i(K)$.

We observe that the equality $\widetilde{\mathrm{r}}_i(K) = \mathrm{R}_i(K)$ holds for any constant width set $K \in \mathcal{W}^n$ such that $K|L' = (\mathrm{b}/2)B_{i,L'}$. □

## 5. A property on $p$-tangential bodies

We conclude this paper by stating a property for the so-called $p$-tangential bodies. A convex body $K \in \mathcal{K}^n$ containing the Euclidean ball $B_n$ is called a $p$-tangential body of $B_n$, $0 \le p \le n - 1$, if each support plane of $K$ that is not a support plane of $B_n$ contains only $(p - 1)$-singular points of $K$ [29, page 86]. Here $x \in \mathrm{bd}\, K$ is said to be an $r$-singular point of $K$ if the dimension of the normal cone in $x$ is at least $n - r$. For further characterisations and properties of $p$-tangential bodies, we refer to [29, Section 2.2].

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So a 0-tangential body of $B_n$ is just $B_n$ itself, and each $p$-tangential body of $B_n$ is also a $q$-tangential body for $p \leq q \leq n-1$. A 1-tangential body can be seen as the convex hull of $B_n$ and countably many points such that the line segment joining any pair of those points intersects the ball. A celebrated result of Favard [14] states a nice characterisation of $n$-dimensional $p$-tangential bodies in terms of the so-called quermassintegrals of $K$, namely, that the first $n-p+1$ ones coincide. We will not enter here into the definition of these measures, but refer the interested reader to [29, page 431].

Here we show a result in the spirit of Favard's theorem mentioned above, in the sense that now, for a $p$-tangential body, many inner radii also coincide.

Proposition 5.1. *Let $K \in \mathcal{K}^n$ be a $p$-tangential body of $B_n$, $0 \leq p \leq n-1$. Then*

$$\mathrm{r}_n(K) = \mathrm{r}_{n-1}(K) = \cdots = \mathrm{r}_{p+1}(K) = 1.$$

Proof. It is a direct consequence of the definition that any $p$-tangential body of $B_n$ has inradius 1. So if $p = n-1$ then $\mathrm{r}_n(K) = 1$ and the result follows. Thus, we assume that $1 \leq p \leq n-2$.

Since the inner radii form a decreasing sequence, it follows that

$$1 = \mathrm{r}_n(K) \leq \mathrm{r}_{n-1}(K) \leq \cdots \leq \mathrm{r}_{p+1}(K),$$

and it suffices to show that $\mathrm{r}_{p+1}(K) \leq 1$. So we assume that $\mathrm{r}_{p+1}(K) > 1$ and we will get a contradiction. On the one hand, by definition of inner radii, there exist $t \in \mathbb{R}^n$ and $L \in \mathcal{L}^n_{p+1}$ such that

$$t + \mathrm{r}_{p+1}(K) B_{p+1,L} \subseteq K. \tag{5.1}$$

On the other hand, in [28, Lemma 2.5] it is shown, in particular, that $K$ is a $p$-tangential body of $B_n$, $1 \leq p \leq n-2$, if and only if $K|u^\perp$ is a $p$-tangential body of $B_{n-1,u^\perp}$ for any unit vector $u \in \mathbb{R}^n$. From this result it can be easily obtained that the orthogonal projection $K|L$ is again a $p$-tangential body of the ball $B_n|L = B_{p+1,L}$, and then

$$\mathrm{r}(K|L; L) = \mathrm{r}(B_{p+1,L}; L) = 1. \tag{5.2}$$

Moreover, from (5.1) we get that $t|L + \mathrm{r}_{p+1}(K) B_{p+1,L} \subseteq K|L$, and then, together with (5.2), we obtain the desired contradiction:

$$1 = \mathrm{r}(K|L; L) \geq \mathrm{r}_{p+1}(K) > 1. \qquad \square$$

This result shows (see [8, Lemma 3.2]) that $p$-tangential bodies of the Euclidean ball $B_n$ are $\{\mathrm{r}_{p+1}, \ldots, \mathrm{r}_{n-1}\}$-isoradial. We recall that a convex body $K$ is called $\mathrm{r}_j$-isoradial if, for every $L \in \mathcal{L}^n_j$, there exists $t \in \mathbb{R}^n$ such that $(t + \mathrm{r}_j(K) B_n) \cap (t + L) \subset K$, and is said to be $\{\mathrm{r}_j : j \in I\}$-isoradial, for a subset $I \subset \{1, \ldots, n-1\}$, if it is $\mathrm{r}_j$-isoradial for all $j \in I$.

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