

BOUNDING BASIC CHARACTERISTICS OF SPATIAL EPIDEMICS WITH A NEW PERCOLATION MODEL

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Abstract

We introduce a new 1-dependent percolation model to describe and analyze the spread of an epidemic on a general directed and locally finite graph. We assign a two-dimensional random weight vector to each vertex of the graph in such a way that the weights of different vertices are independent and identically distributed, but the two entries of the vector assigned to a vertex need not be independent. The probability for an edge to be open depends on the weights of its end vertices, but, conditionally on the weights, the states of the edges are independent of each other. In an epidemiological setting, the vertices of a graph represent the individuals in a (social) network and the edges represent the connections in the network. The weights assigned to an individual denote its (random) infectivity and susceptibility, respectively. We show that one can bound the percolation probability and the expected size of the cluster of vertices that can be reached by an open path starting at a given vertex from above by the corresponding quantities for independent bond percolation with a certain density; this generalizes a result of Kuulasmaa (1982). Many models in the literature are special cases of our general model.

Keywords: Dependent percolation; epidemics

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1. Introduction, background, and main results

We consider an extension of the standard SIR (susceptible \rightarrow infectious \rightarrow removed) epidemic [1], [2], [11] on a directed graph $G = (V, E)$. Here V is the (countable) vertex set of the graph, and E consists of directed edges between vertices in V . An edge from u to v is denoted by uv , and we say that v is a (directed) neighbor of u . We assume that the graph $G = (V, E)$ is simple, that is, for any $u, v \in V$, there is at most one edge from u to v . This simplicity assumption can easily be dropped. Furthermore, we assume that the graph is locally finite, in the sense that both the in-degree and out-degree of every vertex are finite.

In a standard SIR epidemic, a vertex is identified with an individual which makes (asymmetric) contacts with each of its neighbors at rate τ . If an infectious individual u contacts a susceptible individual v , then v becomes infectious itself. If an individual becomes infectious, it will stay infectious for a random time; the infectious periods of different individuals are independent and identically distributed (i.i.d.). After the infectious period an individual is removed, which can either mean that it is recovered or that the individual has died. A removed

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individual never becomes susceptible or infectious again. Usually, we assume that there is one initially infectious individual v_0 , and that all other individuals in the network are initially susceptible. Furthermore, we assume that demography plays no role, in the sense that we ignore births, deaths not caused by the infectious disease, and migration. This is a reasonable assumption if we consider emerging infectious diseases for which the timescale of the spread is much smaller than the timescale of demography.

In the model just described, we implicitly assume that all individuals in the network are the same (at least with respect to the epidemic) apart from their position in the network. In particular, all individuals will have the same joint distribution of the total infectivity and susceptibility. In real life, however, infectivity and susceptibility show individual variation, notably because of immunological polymorphism, or due to polymorphic reactions to vaccination [4], [5]. The infectivity and susceptibility of one individual are in general not independent. Dependencies may arise because of confounding factors, such as age, general health status, or in the case of sexually transmitted diseases, promiscuity and levels of condom use, which affect the susceptibility and infectivity in the same direction.

In this paper we model heterogeneity of the population by assigning a random infectivity W_v and susceptibility \bar{W}_v to each vertex v in V , where the vectors (W_v, \bar{W}_v) are assumed to be i.i.d. and distributed as (W, \bar{W}) , taking values in a convex subset S of $\mathbb{R}_+^2 := [0, \infty)^2$. We do not assume that W_v and \bar{W}_v are independent. Conditionally on the weights, if u becomes infected and $uv \in E$, then v becomes infected (if it was not already) with probability $\kappa(W_u, \bar{W}_v)$, where κ is some connection function, specified in the model. In percolation terms, this means that the directed edge uv is open with (conditional) probability $\kappa(W_u, \bar{W}_v)$. Conditioned on the weights, the states of the edges are independent. However, without this conditioning, states of edges sharing an end vertex are dependent through the weights assigned to this common end vertex.

We are mainly interested in (i) the probability of a large outbreak, (ii) the probability that a disease spreads from one given individual to another, and (iii) the expected final size of an epidemic. Percolation models have served before as useful tools to analyze these quantities; see, for instance, [10], [14], [15], [16], [17], and [19] for related material.

We denote by P the probability measure governing the full process of assigning weights, and making the edges open or closed. (We do not really need to formally define the full sample space of the process.) One necessary property is that

$$P(\kappa(W_u, \bar{W}_v) \in [0, 1]) = 1, \quad (1.1)$$

since $\kappa(\cdot, \cdot)$ represents a probability. Sometimes it is useful to discuss the induced measure of P on the space $\Omega := \{\text{open, closed}\}^E$, that is, the induced measure on configurations of open and closed edges.

We consider connection functions $\kappa(x, y)$, which can be written as $\kappa(xy)$, where $\kappa(z)$ is nondecreasing and concave. Examples of functions satisfying these conditions are

$$\begin{aligned} \kappa_a(x, y) &= xy \quad \text{with } S = [0, 1]^2, \\ \kappa_b(x, y) &= 1 - e^{-\alpha xy} \quad \text{with } \alpha > 0 \text{ and } S = \mathbb{R}_+^2, \\ \kappa_c(x, y) &= xy(\beta + xy)^{-1} \quad \text{with } \beta > 0 \text{ and } S = \mathbb{R}_+^2. \end{aligned} \quad (1.2)$$

We note that if $\kappa(x, y)$ is *factorizable*, i.e. if there exist functions $\kappa_1(x)$ and $\kappa_2(y)$ such that $\kappa(x, y) = \kappa_1(x)\kappa_2(y)$, then we can without loss of generality assume that $\kappa(x, y) = xy$ and $S = [0, 1]^2$. This can easily be seen by first replacing W by $\kappa_1(W)$ and \bar{W} by $\kappa_2(\bar{W})$, and then

scaling W and \bar{W} such that they both take values in $[0, 1]$ with probability 1. This is possible since, by (1.1) and the independence of W_u and \bar{W}_v for $u \neq v$, there exists a constant $c > 0$ such that $P(\kappa_1(W) \leq c) = P(\kappa_2(\bar{W}) \leq 1/c) = 1$.

The connection functions κ_a , κ_b , and κ_c have important epidemiological interpretations. A factorizable connection function is appropriate in situations in which there is at most one contact from an infectious individual to a given neighbor during its infectious period, or when only at the first contact of an infectious individual with a given neighbor the infection may be transmitted. This last assumption is proposed in some models for the spread of HIV [20], where the number of sexual contacts per couple can be ignored and only the number of partners is of importance. In those models the probability of an infectious contact from u to v is given by $\kappa(W_u, \bar{W}_v) = W_u \bar{W}_v$, with $W_u = 1 - \exp[-\int_0^{\Lambda_u} \tau_u(x) dx]$, where Λ_u is the length of the infectious period of individual u and $\tau_u(x)$ is the (possibly inhomogeneous) rate at which individual u makes infectious contacts at time x after its infection, and where \bar{W}_v is the probability that individual v (if still susceptible) becomes infected at an infectious contact.

The choice in (1.2) can be found in [18], and (1.3) is discussed in [7], both in the context of complete graphs. In neither of these two papers are epidemiological interpretations of the connection functions given. To shed some light on a possible epidemiological interpretation of κ_b and κ_c , we write

$$\kappa_b(x, y) = 1 - e^{-\alpha xy} = 1 - \sum_{k=0}^{\infty} \frac{(\alpha x)^k}{k!} e^{-\alpha x} (1 - y)^k$$

and

$$\kappa_c(x, y) = \frac{xy}{\beta + xy} = 1 - \sum_{k=0}^{\infty} \frac{\beta}{x + \beta} \left(\frac{x}{x + \beta}\right)^k (1 - y)^k.$$

When $0 < y < 1$, we see that we can interpret these connection functions as follows. For κ_c , an infectious individual has a geometric- $\beta/(x + \beta)$ number of contacts with a neighbor, and each time, the probability that the infection is accepted is equal to y . For κ_b , a similar interpretation is possible, replacing the geometric number of attempts by a Poisson- αx number. In both cases, the number of attempts stochastically increases when x grows, in accordance to our interpretation of W as the infectivity.

It should be noted that (1.3) arises when the infectious period of every individual is exponentially distributed with parameter β and the per neighbor infection rate of individual u is W_u , while the probability that an infectious contact with susceptible individual v leads to an infection is given by \bar{W}_v . If the infectious period of individual u is Λ_u , and during its infectious period it makes infectious contacts with every neighbor at rate $\alpha \tau_u$, then the number of attempts will have a Poisson- $\alpha \tau_u \Lambda_u$ distribution; hence, both κ_b and κ_c arise naturally. Note that replacing the combination κ_c and (W, \bar{W}) by κ_b and $(\Lambda W/\alpha, \bar{W})$, where Λ is exponentially distributed with parameter β and independent of W , does not change the induced measure on Ω .

We define the usual independent bond percolation measure P_p^{bond} as the product measure on Ω in which edges are independently open with probability p [12]. If $\kappa(x, y) = xy$, $P(W = w^*, \bar{W} = \bar{w}^*) = 1$, and $w^* \bar{w}^* = p$, then the induced measure of P on Ω is just P_p^{bond} . If $\kappa(x, y) = xy$ and $P(W = \bar{W} = 1) = 1 - P(W = \bar{W} = 0) = p$, we denote the corresponding measure by P_p^{site} . Indeed, this measure corresponds to the edge representation of an independent site percolation model with parameter p [12], in which an edge is open if and only if both its starting and ending vertices are open. Note that although P_p^{bond} is defined on Ω , P_p^{site} is defined on the full space. This is perhaps slightly confusing, but it works best this way.

In order to state our results, we need a few definitions. An ordered set of edges in E , $\xi = (v_0v_1, v_1v_2, \dots, v_{n-1}v_n)$, is a (directed) path of length n from v_0 to v_n . If $v_i \neq v_j$ for all $0 \leq i < j \leq n$ then the path is *self-avoiding*. It is straightforward to extend this definition to self-avoiding paths of infinite length. If we assume that a path is self-avoiding, we will explicitly mention this. With some abuse of terminology we define a *trivial path* as a path without edges, but with a starting vertex and the same end vertex (i.e. a trivial path may be seen as a single vertex).

We say that a path is open if all edges in the path are open; a trivial path is always open. We use the notation $v_i \rightsquigarrow v_j$ if there is at least one open path from v_i to v_j . If the final vertex of a path ξ_1 is the first vertex of a path ξ_2 , we write (ξ_1, ξ_2) for the *conjunction* of ξ_1 and ξ_2 , that is, if $\xi_1 = (v_0v_1, v_1v_2, \dots, v_{n-1}v_n)$ and $\xi_2 = (v_nv_{n+1}, v_{n+1}v_{n+2}, \dots, v_{n+m-1}v_{n+m})$, then

$$(\xi_1, \xi_2) = (v_0v_1, v_1v_2, \dots, v_{n-1}v_n, v_nv_{n+1}, \dots, v_{n+m-1}v_{n+m}).$$

For a finite or infinite path $\xi = (v_0v_1, v_1v_2, \dots, v_{n-1}v_n, v_nv_{n+1}, \dots)$, we define the *truncation of ξ after n edges* as $\xi^s(n) := (v_0v_1, v_1v_2, \dots, v_{n-1}v_n)$. The *tail of ξ starting after n edges* is defined as $\xi^t(n) := (v_nv_{n+1}, \dots)$. Both the truncation after n edges and the tail starting after n edges may be trivial paths. We are now ready to specify the collections of paths we consider in this paper.

Definition 1.1. We say that a collection of paths Ξ is *weakly hoppable* if, for any $v \in V$, any $i, j \in \mathbb{N}$, and any two paths $\xi, \phi \in \Xi$ going through v , where v is the end vertex of the i th edge of ξ and the start vertex of the j th edge of ϕ , the conjunction $(\xi^s(i), \phi^t(j))$ is in Ξ as well.

In words, we need to be able to ‘hop’ from one path to another if they cross. We allow for $\xi = \phi$, which implies that if a path in a weakly hoppable collection of paths Ξ has a loop then this loop can be erased and the resulting path is still in Ξ .

Furthermore, let $E^{(n)}$ be the collection of the first n edges in E , according to some given enumeration of the edges for which $\bigcup_{n \in \mathbb{N}} E^{(n)} = E$. We ‘approximate’ Ξ by sets Ξ_n defined as follows: Ξ_n is the collection of all infinite paths which start in $E^{(n)}$ truncated at the first instance they leave $E^{(n)}$ together with all finite paths of which all edges are in $E^{(n)}$. If the first edge of an infinite path ξ is not in $E^{(n)}$ then the trivial path at the starting vertex of ξ is in Ξ_n .

Finally, for a collection of paths Ξ we denote by \mathcal{C}^Ξ the event that at least one path in Ξ is open. We note that, for a weakly hoppable collection of paths Ξ , \mathcal{C}^Ξ is equivalent to the event that at least one self-avoiding path in Ξ is open, because loop-erased paths from Ξ are also in Ξ .

Definition 1.2. We say that a collection of paths Ξ is *hoppable* if it is weakly hoppable and if in addition, for some enumeration of the edges,

$$\mathcal{C}^\Xi = \lim_{n \rightarrow \infty} \mathcal{C}^{\Xi_n}.$$

Remark 1.1. 1. Most natural and useful collections of paths are hoppable. For instance, the collection of infinite paths starting at a given vertex (or, more generally, in a finite set) is hoppable, as is the collection of all paths from a given vertex u to a given vertex v (or, more generally, from a finite set to another finite set). To see the first claim, we note that if Ξ is the collection of all infinite paths starting at a finite set then $\mathcal{C}^{\Xi_{n+1}} \subset \mathcal{C}^{\Xi_n}$ for all n and $\mathcal{C}^\Xi = \bigcap_{n=1}^\infty \mathcal{C}^{\Xi_n}$. If Ξ is the collection of all paths from a finite set to another finite set then $\mathcal{C}^{\Xi_n} \subset \mathcal{C}^{\Xi_{n+1}}$ for all n and $\mathcal{C}^\Xi = \bigcup_{n=1}^\infty \mathcal{C}^{\Xi_n}$.

2. If Ξ_n had been defined as ‘the collection of all paths which start in $E^{(n)}$, truncated at the first instance they leave $E^{(n)}$ ’, then the collection of all paths from vertex u to vertex v would not have been hopplable. Indeed, on an infinite graph we have $\mathcal{C}^{\Xi_{n+1}} \subset \mathcal{C}^{\Xi_n}$ for all n , but $\mathcal{C}^{\Xi} \neq \bigcap_{n=1}^{\infty} \mathcal{C}^{\Xi_n}$, since $\bigcap_{n=1}^{\infty} \mathcal{C}^{\Xi_n}$ occurs if there is an open path from u to v or if there is an infinite open path starting at u .

The main results of this paper are the following, generalizing results in [15] (see also [16] for related results for the case in which W and \bar{W} are independent).

Theorem 1.1. *Let (W, \bar{W}) be a random vector taking values in S , and let $\kappa(x, y) = \kappa(xy)$ be such that $\kappa(z)$ is increasing and concave. Then, for*

$$p = \kappa(\max[E(W\bar{W}), E(W)E(\bar{W})])$$

and any hopplable collection of paths Ξ , we have

$$P(\mathcal{C}^{\Xi}) \leq P_p^{\text{bond}}(\mathcal{C}^{\Xi}).$$

Theorem 1.2. *Let $\kappa(x, y) = xy$ and $S = [0, 1]^2$. Then, for $p = E(W\bar{W})$ and any hopplable collection Ξ of paths in E , we have*

$$P(\mathcal{C}^{\Xi}) \geq P_p^{\text{site}}(\mathcal{C}^{\Xi}).$$

Theorems 1.1 and 1.2 are corollaries of Theorem 3.1 below.

We note that if E is symmetric, i.e. $uv \in E$ is equivalent to $vu \in E$, and if $P(W = \bar{W}) = 1$ and $\kappa(x, y) = \kappa(y, x)$, then the law of the cluster of vertices that can be reached by open paths from $v \in V$ on G is the same as the law of the open cluster containing v on the undirected counterpart of G (the graph obtained by replacing the two edges connecting the same vertices by one undirected edge) [10]. Hence, many questions on undirected graphs can be addressed as questions on directed graphs, and Theorems 1.1 and 1.2 may be applied to percolation on undirected graphs as well.

We will illustrate the results by a numerical example, and then by discussing the situation on a tree. In this latter example, we also shed some light on the reason why $E(W\bar{W})$ and $E(W)E(\bar{W})$ play an important role in our analysis.

Example 1.1. Suppose that $W = \bar{W}$ is uniformly distributed on $(a, 1)$ for some $a \in (0, 1)$, $G = \mathbb{L}^2$ is the square lattice with nearest neighbor (directed) edges and $\kappa(x, y) = xy$. If $a \leq \sqrt{\frac{3}{4}} - \frac{1}{2} \approx 0.37$ then $E(W\bar{W}) = E(W^2) \leq \frac{1}{2}$. Because the critical value for independent bond percolation is $\frac{1}{2}$ [12], in view of the remark above on the relation between percolation on directed and undirected graphs, the probability that the cluster of individuals that can be reached by an open path from the origin is infinite is 0 for this model. Let $p_c^{\text{site}}(\mathbb{L}^2) \approx 0.59$ be the critical value for site percolation on \mathbb{L}^2 . If $a > \frac{1}{2}(\sqrt{12p_c^{\text{site}}(\mathbb{L}^2)} - 3 - 1) \approx 0.51$ then the probability that the cluster of individuals that can be reached by an open path from the origin is infinite is positive for this model.

Example 1.2. Let $G = (V, E)$ be the rooted tree in which all vertices have out-degree d and in-degree 1, apart from the root v_0 which has in-degree 0. We say that the root is the generation-0 vertex; if there is a path of length n from the root to vertex v , then v is said to be a generation- n vertex.

Let $\kappa(x, y) = xy$, and let Ξ_k be the set of all paths of length k starting at the root. Furthermore, let Z_k be the number of open paths in Ξ_k , i.e. Z_k is the number of generation- k

vertices that can be reached by an open path from the root. For $k \geq 1$ and generation-1 vertex v_1 , we define $Y_k(v_1)$ as the number of generation- k vertices v_k , for which there is an open path from v_0 to v_k through v_1 , conditioned on $W_{v_0} = 1$ and $\bar{W}_u = 1$ for all generation- k vertices u .

We claim that $Y_k(v_1)$ is distributed as the size of the $(k - 1)$ th generation of a Galton–Watson process [13] starting with one individual, in which individuals have no offspring with probability $E(1 - \bar{W}) + E(\bar{W}(1 - W)^d)$ and offspring of size j with probability $E(\bar{W} \binom{d}{j} W^j (1 - W)^{d-j})$ for $0 < j \leq d$. Indeed, we can adopt the point of view that, for a vertex to have any offspring, it first has to accept the disease from its predecessor in the tree, and, if it does, in addition it needs to send the disease to the next generation. This leads to an offspring mean of $d E(\bar{W} W)$.

It now seems quite natural, given the computation above, to consider a class of probability measures for which $E(W \bar{W})$ is constant (compare [4] and [5]). However, in order to prove the inequality $P(\mathcal{C}^\Xi) \leq P_p^{\text{bond}}(\mathcal{C}^\Xi)$ for all hopable collections of paths in E , we need the additional assumption that $p \geq E(W) E(\bar{W})$. This can be seen by assuming that Ξ consists of a single edge. The marginal probability that this edge is open is given by $E(W) E(\bar{W})$. This explains, to some extent, the importance of the quantities $E(W \bar{W})$ and $E(W) E(\bar{W})$.

2. Discussion

Before we start proving the results, we collect in this section a number of remarks.

- Contrary to Kuulasmaa [15] and Miller [16], we deal with infectious diseases for which the susceptibility and infectivity of individuals might be dependent. This dependence complicates the proofs and means that we cannot apply the results from [15] and [16] immediately. If $\kappa(x, y)$ is not factorizable then Miller’s result for independent W and \bar{W} is slightly stronger than our result. This is because he provided a bond percolation upper bound with parameter $p = E(\kappa(W \bar{W}))$, which, for concave κ , is bounded above by our bond percolation parameter $\kappa(E(W \bar{W}))$.
- The class of possible collections Ξ for which the result is true is larger than the class of hopable collections. We have chosen this formulation because it contains most collections of interest, and also because of its elegance. One class of paths that can be seen to satisfy the results is the class of infinite *backwards* paths ending at a given vertex v . Indeed, by simply interchanging the role of W and \bar{W} (see also [16] for this trick), it is easy to see that our results are valid for this class as well.
- Suppose that $\kappa(x, y) = xy$. By choosing Ξ as the collection of paths from u to v , we find that among all measures with $E(W \bar{W}) \geq E(W) E(\bar{W})$, it is the case that $E[1(u \rightsquigarrow v)] = P(u \rightsquigarrow v)$ is at most $P_{E(W \bar{W})}^{\text{bond}}(u \rightsquigarrow v)$ and at least $P_{E(W \bar{W})}^{\text{site}}(u \rightsquigarrow v)$. Let C_u be the set of vertices that can be reached by an open path from vertex u . The observations above give

$$E(|C_u|) = \sum_{v \in V} P(u \rightsquigarrow v) \leq \sum_{v \in V} P_{E(W \bar{W})}^{\text{bond}}(u \rightsquigarrow v) = E_{E(W \bar{W})}^{\text{bond}}(|C_u|).$$

- It is not possible to use straightforward stochastic domination arguments to prove the theorems in their full generality. This can be seen by considering $\kappa(x, y) = xy$, and uncorrelated W and \bar{W} . In that case the marginal probability that any edge $uv \in E$ is open is the same for all measures for which $E(W \bar{W})$ is constant. However, the edge density in $P_{E(W \bar{W})}^{\text{bond}}$ is also $E(W \bar{W})$ and, hence, stochastic domination cannot be used to prove Theorem 1.1.

- We can compare some of our results with bounds given in [3]: any undirected 1-dependent edge percolation model on the two-dimensional square lattice, with marginal probability for an edge to be open at least 0.8639, is supercritical. Since this bound holds for all 1-dependent measures, it is to be expected that our bounds improve on this in our specific model. Since we can only compare undirected models, we can only compare in the symmetric case (which is perhaps not the most interesting case). Still, since P_p^{site} percolates above $p = 0.68$ [21] (this is a rigorous bound, the correct value of the critical probability is around 0.59), we see that our model percolates when $P(W = \bar{W}) = 1$, $\kappa(x, y) = xy$, and $E(W\bar{W}) = E(W^2) \geq 0.68$. Hence, we improve on the general bound when $E(W^2) \geq 0.68$ and $(E(W))^2 < 0.8639$.
- Our model is a generalization of many other percolation processes, such as the locally dependent random graph model [15], mixed percolation [8], generalized random graphs [7], and Poissonian random graphs [18], where the two latter ones were previously defined for only finite complete graphs G . The inhomogeneous random graphs of [6] are defined only on complete graphs, but they are more general than our model on the complete graph.

3. Proofs

Let S_1 be the projection of S in the first coordinate direction, and let S_2 be the projection of S in the second coordinate direction. Let E'_v be the set of all edges starting at v , and let E^*_v be the set of all edges ending at v . For any pair of (possibly empty) sets $A \subset E'_v$ and $B \subset E^*_v$, any $|A|$ -dimensional vector $\mathbf{x} = (x_1, x_2, \dots, x_{|A|}) \in (S_2)^{|A|}$, and any $|B|$ -dimensional vector $\mathbf{y} = (y_1, y_2, \dots, y_{|B|}) \in (S_1)^{|B|}$, we define the *zero function* $z_u(\mathbf{P}; A, B; \mathbf{x}, \mathbf{y})$ as the probability that either none of the edges in A are open or none of the edges in B are open if the weights assigned to the endpoints of the edges in A and B are the elements of the vectors \mathbf{x} and \mathbf{y} , respectively. This can be expressed as

$$z_u(\mathbf{P}; A, B; \mathbf{x}, \mathbf{y}) := E\left(1 - \left[1 - \prod_{i=1}^{|A|} (1 - \kappa(W, x_i))\right] \left[1 - \prod_{j=1}^{|B|} (1 - \kappa(y_j, \bar{W}))\right]\right),$$

if $|A||B| > 0$. For the remaining cases, we define

$$z_u(\mathbf{P}; A, \emptyset; \mathbf{x}, \emptyset) := E\left(\prod_{i=1}^{|A|} (1 - \kappa(W, x_i))\right) \quad \text{if } |A| > 0,$$

$$z_u(\mathbf{P}; \emptyset, B; \emptyset, \mathbf{y}) := E\left(\prod_{j=1}^{|B|} (1 - \kappa(y_j, \bar{W}))\right) \quad \text{if } |B| > 0,$$

$$z_u(\mathbf{P}; \emptyset, \emptyset; \emptyset, \emptyset) := 1.$$

If the graph is transitive, we do not need the reference to the vertex u in the zero function.

In the epidemiological setting, the zero function $z_u(\mathbf{P}; A, B; \mathbf{x}, \mathbf{y})$ is the (conditional) probability that if all endpoints of edges in B become infected and have ‘ W -weights’ $y_1, \dots, y_{|B|}$, either u will not get infected via an edge in B , or u will not transmit the disease to any of the endpoints of edges in A , if those endpoints have ‘ \bar{W} -weights’ $x_1, \dots, x_{|A|}$.

We write $z_v(\mathbf{P}^{(a)}) \geq z_v(\mathbf{P}^{(b)})$ if $z_v(\mathbf{P}^{(a)}; A, B; \mathbf{x}, \mathbf{y}) \geq z_v(\mathbf{P}^{(b)}; A, B; \mathbf{x}, \mathbf{y})$ for all $A \subset E'_v$, all $B \subset E_v^*$, all $\mathbf{x} \in (S_2)^{|A|}$, and all $\mathbf{y} \in (S_1)^{|B|}$. The following result is interesting in its own right, and will be the main tool to prove Theorem 1.1.

Theorem 3.1. *If the distributions of the weights with respect to the measures $\mathbf{P}^{(a)}$ and $\mathbf{P}^{(b)}$ are such that $z_v(\mathbf{P}^{(a)}) \leq z_v(\mathbf{P}^{(b)})$ for all $v \in V$ then, for any hoppable collection Ξ of paths,*

$$\mathbf{P}^{(b)}(\mathcal{C}^\Xi) \leq \mathbf{P}^{(a)}(\mathcal{C}^\Xi).$$

Remark 3.1. Theorem 3.1 does not hold if we would allow for all collections of paths Ξ . Here is a counterexample. Let G be the subgraph of the two-dimensional square lattice, consisting of the origin and its nearest neighbors (with nearest neighbor edges). Let $\kappa(x, y) = xy$, and suppose that the weights assigned to the neighbors of the origin are all equal to 1. We consider two measures $\mathbf{P}^{(a)}$ and $\mathbf{P}^{(b)}$ on the weights assigned to the origin: $\mathbf{P}^{(a)}(W = \bar{W}) = 1$, and W is uniformly $(0, 1)$ distributed under $\mathbf{P}^{(a)}$, while $\mathbf{P}^{(b)}(W = \bar{W} = 0) = \frac{17}{48}$, $\mathbf{P}^{(b)}(W = \bar{W} = \frac{2}{3}) = \frac{9}{16}$, and $\mathbf{P}^{(b)}(W = \bar{W} = 1) = \frac{1}{12}$.

Now let ξ be the path from $(0, -1)$ to $(0, 1)$, let ϕ be the path from $(-1, 0)$ to $(1, 0)$ in G , and let $\Xi = \{\xi, \phi\}$; this is not a hoppable collection. A quick computation gives

$$\mathbf{P}^{(a)}(\mathcal{C}^\Xi) = \frac{7}{15} < \frac{17}{36} = \mathbf{P}^{(b)}(\mathcal{C}^\Xi).$$

All weights of neighbors of the origin are 1, so we only have to consider $z_0(\mathbf{P}^{(a)}; A, B) := z_0(\mathbf{P}^{(a)}; A, B; \mathbf{1}, \mathbf{1})$ and $z_0(\mathbf{P}^{(b)}; A, B) := z_0(\mathbf{P}^{(b)}; A, B; \mathbf{1}, \mathbf{1})$, where $\mathbf{1}$ is the vector with all entries equal to 1. It follows from some tedious but straightforward computations that $z_0(\mathbf{P}^{(a)}) \leq z_0(\mathbf{P}^{(b)})$; we do not present the computation here.

In the proof of Theorem 3.1 we need the following lemma.

Lemma 3.1. *If Ξ is hoppable then Ξ_n is hoppable for all $n \in \mathbb{N}$.*

Proof. Since Ξ_n consists of finitely many edges only, it is enough to prove that Ξ_n is weakly hoppable, that is, for any v and any two paths $\xi_n, \phi_n \in \Xi_n$ going through v (say that v is the end vertex of the i th edge of ξ and the start vertex of the j th edge of ϕ), the conjunction $(\xi_n^s(i), \phi_n^t(j))$ is in Ξ_n as well. We distinguish various cases.

1. If the ϕ_n are the truncations of the infinite path $\phi \in \Xi$ then $(\xi_n^s(i), \phi_n^t(j))$ is the truncation of $(\xi_n^s(i), \phi^t(j))$ at the first time this paths leaves E_n . All of its edges are in E_n . By the weak hoppability of Ξ , this conjunction is in Ξ_n as well.
2. If the ξ_n are the truncations of the infinite path $\xi \in \Xi$ and $\phi_n \in \Xi$, then $(\xi_n^s(i), \phi_n^t(j))$ contains only edges in E_n and is in Ξ by the weak hoppability of Ξ and in Ξ_n by the definition of Ξ_n .
3. If ξ_n and ϕ_n are both finite paths in Ξ , then $(\xi_n^s(i), \phi_n^t(j))$ contains only edges in E_n and is in Ξ by the weak hoppability of Ξ , and in Ξ_n by definition.

Proof of Theorem 3.1. (i) First we assume that there is a $u \in V$ such that $z_u(\mathbf{P}^{(a)}) \leq z_u(\mathbf{P}^{(b)})$, and that the distribution of (W_v, \bar{W}_v) is the same under $\mathbf{P}^{(a)}$ and $\mathbf{P}^{(b)}$ (and, therefore, $z_v(\mathbf{P}^{(a)}) = z_v(\mathbf{P}^{(b)})$) for all $v \in V \setminus \{u\}$. Under these assumptions, we will now prove that, for all $n \geq 1$, $\mathbf{P}^{(a)}(\mathcal{C}^{\Xi_n}) \geq \mathbf{P}^{(b)}(\mathcal{C}^{\Xi_n})$.

Let $E_u^{(n)}$ be the set of edges in $E^{(n)}$ with u as the start or end vertex. Let $\sigma_u^{(n)}$ denote a typical realization of the weights outside u , together with the states (open or closed) of the edges in

$E^{(n)} \setminus E_u^{(n)}$. Informally, $\sigma_u^{(n)}$ contains all information which does not depend on (W_u, \bar{W}_u) , and we will proceed by conditioning on $\sigma_u^{(n)}$. By assumption, the distribution (W_v, \bar{W}_v) is the same with respect to $P^{(a)}$ and $P^{(b)}$ for all $v \in V \setminus \{u\}$. This implies that

$$P^{(a)}(\mathcal{C}^{\Xi_n}) - P^{(b)}(\mathcal{C}^{\Xi_n}) = \int (P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) - P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)})) dP^{(a)}(\sigma_u^{(n)}).$$

We now split the $\sigma_u^{(n)}$ into three cases.

1. $\sigma_u^{(n)}$ is such that $P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = 0$, in which case

$$P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) - P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = 0.$$

This case occurs if $\sigma_u^{(n)}$ is such that no matter the weights assigned to u , the probability of an open path in Ξ_n is 0.

2. $\sigma_u^{(n)}$ is such that $P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = 1$, in which case

$$P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) - P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = 0.$$

This case occurs if $\sigma_u^{(n)}$ is such that no matter the weights assigned to u , the probability of an open path in Ξ_n is 1.

3. The only case remaining is that $\sigma_u^{(n)}$ is such that neither

$$P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = 1$$

nor

$$P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) = 0.$$

In this last case there are three possibilities.

- (i) If u is the starting vertex of some path in Ξ_n then—since Ξ_n is weakly hopable—the tail starting at u of a self-avoiding path through u is also in Ξ_n . Therefore, with the assumptions on $\sigma_u^{(n)}$, Ξ_n contains an open path if and only if it contains a self-avoiding open path starting at u . Let $A \subset E_u^{(n)}$ be the set of starting edges of open paths in Ξ_n starting at u , if all edges in $E_u^{(n)}$ would have been open. We note that, because we need to only consider self-avoiding paths in Ξ_n , at least one of the paths in Ξ_n is open if and only if at least one edge in A is open. This probability is given by 1 minus the zero function. Since, $z_u(P^{(a)}; A, \emptyset; \mathbf{x}, \emptyset) \leq z_u(P^{(b)}; A, \emptyset; \mathbf{x}, \emptyset)$ for all A and \mathbf{x} , it follows that

$$P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) - P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) \geq 0.$$

- (ii) If u is the end vertex of some path in Ξ_n then—since Ξ_n is weakly hopable—the truncation at vertex u of a self-avoiding path through u is also in Ξ_n . Therefore, with the assumptions on $\sigma_u^{(n)}$, Ξ_n contains a self-avoiding open path if and only if it contains an open path ending at u . Let $B \subset E_u^{(n)}$ be the set of end edges of open paths in Ξ_n ending at u , if all edges in $E_u^{(n)}$ would have been open. Since $z_u(P^{(a)}; \emptyset, B; \emptyset, \mathbf{y}) \leq z_u(P^{(b)}; \emptyset, B; \emptyset, \mathbf{y})$ for all B and \mathbf{y} , it follows in the same way as for the previous case that

$$P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) - P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) \geq 0.$$

(iii) If u is neither a starting vertex nor an end vertex of any path in Ξ_n , then let A be the set of starting edges of tails of self-avoiding paths in Ξ_n , cut off at u , for which all edges in the tail are open, if all edges in $E_u^{(n)}$ would have been open. Similarly, let B be the set of end edges of truncations of self-avoiding paths in Ξ_n , cut off at u , for which all edges in the truncation are open, if all edges in $E_u^{(n)}$ would have been open. Since Ξ_n is weakly hopplable, it follows that there is no open path in Ξ_n if either all edges in A are closed or all edges in B are closed. By $z_u(P^{(a)}; A, B; \mathbf{x}, \mathbf{y}) \leq z_u(P^{(b)}; A, B; \mathbf{x}, \mathbf{y})$ for all A, B, \mathbf{x} , and \mathbf{y} , it follows that

$$P^{(a)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) - P^{(b)}(\mathcal{C}^{\Xi_n} \mid \sigma_u^{(n)}) \geq 0.$$

This concludes the first step of the proof.

(ii) In this second step, we relax the condition that the zero functions differ in one place only. Since the event \mathcal{C}^{Ξ_n} depends on the weights of at most $2n$ vertices in V , it is straightforward to construct a sequence of probability measures, $(P^{(i)}; 1 \leq i \leq 2n)$, such that $P^{(1)}(\mathcal{C}^{\Xi_n}) = P^{(a)}(\mathcal{C}^{\Xi_n})$ and $P^{(2n)}(\mathcal{C}^{\Xi_n}) = P^{(b)}(\mathcal{C}^{\Xi_n})$, and such that, for $1 \leq i \leq 2n - 1$, the distribution of (W_v, \bar{W}_v) with respect to $P^{(i)}$ and $P^{(i+1)}$ differ in at most one vertex, say v_i , for which $z_{v_i}(P^{(i)}) \leq z_{v_i}(P^{(i+1)})$. Repeatedly applying part (i) completes this part of the proof.

(iii) To complete the proof, we simply note that from the definition of hopplable we have, for $i = a, b$,

$$P^{(i)}(\mathcal{C}^{\Xi}) = \lim_{n \rightarrow \infty} P^{(i)}(\mathcal{C}^{\Xi_n}),$$

and the result follows.

For the proof of Theorem 1.1, we need the following elementary fact.

Lemma 3.2. *Let $f_i(x)$, $i = 1, \dots, n$, be convex nonincreasing, nonnegative functions on some domain $D \subset \mathbb{R}$. Then $\prod_{i=1}^n f_i(x)$ is also a convex nonincreasing, nonnegative function on D .*

Proof. For $n = 2$, observe that, for $0 < c < 1$ and $y > x$,

$$\begin{aligned} cf_1(x)f_2(x) + (1-c)f_1(y)f_2(y) &= [cf_1(x) + (1-c)f_1(y)][cf_2(x) + (1-c)f_2(y)] \\ &\quad + c(1-c)[f_1(y) - f_1(x)][f_2(y) - f_2(x)] \\ &\geq f_1(cx + (1-c)y)f_2(cx + (1-c)y), \end{aligned}$$

where in the inequality we have used the fact that both f_1 and f_2 are nonincreasing and convex. The proof of the lemma can be completed with induction on n —we leave the details to the reader.

Proof of Theorem 1.1. Let $(w^*, \bar{w}^*) \in S$ be such that

- $E(W) = w^*, E(\bar{W}) = \bar{w}^*$, and $E(W\bar{W}) \leq w^*\bar{w}^*$, or
- $E(W) \leq w^*, E(\bar{W}) \leq \bar{w}^*$, and $E(W\bar{W}) = w^*\bar{w}^*$.

Let \hat{P}^{bond} be a measure that satisfies $\hat{P}^{\text{bond}}(W = w^*, \bar{W} = \bar{w}^*) = 1$.

We proceed by proving that, for all A, B, \mathbf{x} , and \mathbf{y} ,

$$z_v(P; A, B; \mathbf{x}, \mathbf{y}) \geq z_v(\hat{P}^{\text{bond}}; A, B; \mathbf{x}, \mathbf{y}).$$

From Theorem 3.1 and the observation that the induced measure of \hat{P}^{bond} on Ω is just P_p^{bond} , where $p = \kappa(w^*\bar{w}^*)$, Theorem 1.1 then follows.

If $|B| > 0$ then $z_v(\mathbf{P}; \emptyset, B; \emptyset, \mathbf{y}) = E(\prod_{i=1}^{|B|} [1 - \kappa(y_i \bar{W})])$. The functions $1 - \kappa(y_i \bar{W})$ are by assumption nonnegative, convex, and decreasing. Therefore, by Lemma 3.2 and Jensen's inequality,

$$E\left(\prod_{i=1}^{|B|} [1 - \kappa(y_i \bar{W})]\right) \geq \prod_{i=1}^{|B|} [1 - \kappa(y_i E(\bar{W}))] \geq \prod_{i=1}^{|B|} [1 - \kappa(y_i \bar{w}^*)],$$

which is equal to $z_v(\hat{\mathbf{P}}^{\text{bond}}; \emptyset, B; \emptyset, \mathbf{y})$. For $|A| > 0$, we can use similar arguments to show that

$$z_v(\mathbf{P}; A, \emptyset; \mathbf{x}, \emptyset) \geq z_v(\hat{\mathbf{P}}^{\text{bond}}; A, \emptyset; \mathbf{x}, \emptyset).$$

If $|A||B| \geq 1$ then we need to show that

$$\begin{aligned} & E\left(\left(1 - \prod_{i=1}^{|A|} [1 - \kappa(Wx_i)]\right)\left(1 - \prod_{j=1}^{|B|} [1 - \kappa(\bar{W}y_j)]\right)\right) \\ & \leq \left(1 - \prod_{i=1}^{|A|} [1 - \kappa(w^*x_i)]\right)\left(1 - \prod_{j=1}^{|B|} [1 - \kappa(\bar{w}^*y_j)]\right). \end{aligned}$$

By the assumption that $\kappa(z)$ is increasing concave and taking values in $[0, 1]$, and by Lemma 3.2, it follows that $1 - \prod_{i=1}^{|A|} [1 - \kappa(x_i u)]$ is concave and increasing in u . By this concavity and $\kappa(0) \geq 0$, it is possible to choose $a := a(\mathbf{x}) \geq 0$ and $b := b(\mathbf{x}) \geq 0$ such that

$$a + bu \geq 1 - \prod_{i=1}^{|A|} [1 - \kappa(x_i u)]$$

for all $u \in S_1$ and $a + bw^* = 1 - \prod_{i=1}^{|A|} [1 - \kappa(x_i w^*)]$. Similarly, $1 - \prod_{j=1}^{|B|} [1 - \kappa(y_j u)]$ is concave and increasing in u , and it is possible to choose $\bar{a} := \bar{a}(\mathbf{y}) \geq 0$ and $\bar{b} := \bar{b}(\mathbf{y}) \geq 0$ such that

$$\bar{a} + \bar{b}u \geq 1 - \prod_{j=1}^{|B|} [1 - \kappa(y_j u)]$$

for all $u \in S_1$ and $\bar{a} + \bar{b}\bar{w}^* = 1 - \prod_{j=1}^{|B|} [1 - \kappa(y_j \bar{w}^*)]$.

It follows that

$$\begin{aligned} & E\left(\left(1 - \prod_{i=1}^{|A|} [1 - \kappa(Wx_i)]\right)\left(1 - \prod_{j=1}^{|B|} [1 - \kappa(\bar{W}y_j)]\right)\right) \\ & \leq E((a + bW)(\bar{a} + \bar{b}\bar{W})) \\ & = a\bar{a} + a\bar{b}E(\bar{W}) + \bar{a}bE(W) + b\bar{b}E(W\bar{W}) \\ & \leq a\bar{a} + a\bar{b}\bar{w}^* + \bar{a}bw^* + b\bar{b}w^*\bar{w}^* \\ & = (a + bw^*)(\bar{a} + \bar{b}\bar{w}^*) \\ & = \left(1 - \prod_{i=1}^{\infty} [1 - \kappa(w^*x_i)]\right)\left(1 - \prod_{j=1}^{|B|} [1 - \kappa(\bar{w}^*y_j)]\right), \end{aligned}$$

where the second inequality follows from the definitions of w^* and \bar{w}^* . This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. In order to use Theorem 3.1, we have to show that, for all $P, A, B \subset E_v^{(n)}$, all $\mathbf{x} \in (S_1)^{|A|}$, and all $\mathbf{y} \in (S_2)^{|B|}$, we have

$$z_v(P; A, B; \mathbf{x}, \mathbf{y}) \leq z_v(P_p^{\text{site}}; A, B; \mathbf{x}, \mathbf{y}),$$

where $p = E(W\bar{W})$.

For $\kappa(x, y) = xy$, this means that we have to prove that, for $|A| \geq 1$ and $|B| \geq 1$, it holds that

$$\begin{aligned} & E\left(1 - \left[1 - \prod_{i=1}^{|A|}(1 - Wx_i)\right] \left[1 - \prod_{j=1}^{|B|}(1 - y_j\bar{W})\right]\right) \\ & \leq 1 - E(W\bar{W}) \left[1 - \prod_{i=1}^{|A|}(1 - x_i)\right] \left[1 - \prod_{j=1}^{|B|}(1 - y_j)\right], \\ & E\left(\prod_{i=1}^{|A|}(1 - Wx_i)\right) \leq 1 - E(W\bar{W}) \left[1 - \prod_{i=1}^{|A|}(1 - x_i)\right], \\ & E\left(\prod_{j=1}^{|B|}(1 - y_j\bar{W})\right) \leq 1 - E(W\bar{W}) \left[1 - \prod_{j=1}^{|B|}(1 - y_j)\right], \end{aligned}$$

where respectively the cases $|A||B| > 0, B = \emptyset$, and $A = \emptyset$ are considered.

Observe that, for $u \in [0, 1]$ and $\mathbf{x} \in [0, 1]^{|A|}$, $f(u) = 1 - \prod_{i=1}^{|A|}(1 - ux_i)$ is concave and increasing by Lemma 3.2, so

$$1 - \prod_{i=1}^{|A|}(1 - ux_i) \geq uf(1) + (1 - u)f(0) = u\left(1 - \prod_{i=1}^{|A|}(1 - x_i)\right).$$

Similarly, for $\bar{u} \in [0, 1]$ and $\mathbf{y} \in [0, 1]^{|B|}$, we have

$$1 - \prod_{j=1}^{|B|}(1 - \bar{u}y_j) \geq \bar{u}\left[1 - \prod_{j=1}^{|B|}(1 - y_j)\right].$$

Replacing u by W and \bar{u} by \bar{W} in the above two equations completes the proof of Theorem 1.2.

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