COQUASITRIANGULAR STRUCTURES FOR EXTENSIONS OF HOPF ALGEBRAS. APPLICATIONS

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Abstract. Let $A \subseteq E$ be an extension of Hopf algebras such that there exists a normal left A-module coalgebra map $\pi : E \to A$ that splits the inclusion. We shall describe the set of all coquasitriangular structures on the Hopf algebra E in terms of the datum (A, E, π) as follows: first, any such extension E is isomorphic to a unified product $A \ltimes H$, for some unitary subcoalgebra H of E (A. L. Agore and G. Militaru, Unified products and split extensions of Hopf algebras, to appear in AMS Contemp. Math.). Then, as a main theorem, we establish a bijective correspondence between the set of all coquasitriangular structures on an arbitrary unified product $A \ltimes H$ and a certain set of datum (p, τ, u, v) related to the components of the unified product. As the main application, we derive necessary and sufficient conditions for Majid's infinite-dimensional quantum double $D_\lambda(A, H) = A \bowtie_\tau H$ to be a coquasitriangular Hopf algebra. Several examples are worked out in detail.

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1. Introduction. An important class of Hopf algebras is that of quasitriangular Hopf algebras or strict quantum groups. They were introduced by Drinfeld in [**6**] as a remarkable tool for studying the quantum Yang–Baxter equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$. That is, if M is a representation of a quasitriangular Hopf algebra (H, R), then the canonical map $m \otimes n \mapsto \sum R^1 m \otimes R^2 n$ is a solution for the quantum Yang–Baxter equation. The dual concept, namely that of a coquasitriangular Hopf algebra (also called braided Hopf algebras in [**7**], [**8**] or [**11**]) was first introduced by Majid in [**12**] and independently by Larson and Towber in [**10**]. These are Hopf algebras A endowed with a linear map $p : A \otimes A \to k$ satisfying some compatibility conditions. There is, of course, a dual result concerning the quantum Yang–Baxter equation: if M is a co-representation of a coquasitriangular Hopf algebra (A, p), then the canonical map $R_p(m \otimes n) = p(m_{<1>}, n_{<1>})m_{<0>} \otimes n_{<0>}$ is a solution for the quantum Yang–Baxter equation. However, what makes the coquasitriangular Hopf algebras so important is the fact that the converse of the above statement is also true. Namely, by the celebrated FRT theorem for any solution R of the quantum Yang–Baxter equation, there exists a quasitriangular bialgebra $(A(R), p)$ such that $R = R_p$ ([**4**]).

Based on this background, (co)quasitriangular Hopf algebras generated an explosion of interest and were studied for their implications in quantum groups, the construction of invariants of knots and 3-manifolds, statistical mechanics and quantum mechanics, but they also became a subject of research in its own right. Complete descriptions of the coquasitriangular structures have already been obtained for several families of Hopf algebras, see, for instance, [**3**, **7**, **8**] or [**11**].

Among the many research topics related to coquasitriangular Hopf algebras, one is of particular interest: for a given Hopf algebra H, describe (if any) all coquasitriangular structures that can be defined on H. We can formulate the more general problem:

Let $A \subseteq E$ be an extension of Hopf algebras. What is the connection between the coquasitriangular structures of A and those of E?

Obviously, if (E, p) is a coquasitriangular Hopf algebra, then A is also a coquasitriangular Hopf algebra with the coquasitriangular structure given by the restriction of p to $A \otimes A$. The difficult part of the problem is the converse: if $\sigma : A \otimes A \to k$ is a coquasitriangular structure on A, could it be extended to a coquasitriangular structure on E? In this paper, we give a complete answer to this problem in the case when the extension $A \subseteq E$ splits in the sense of [2]: i.e. there exists $\pi : E \to A$ a normal left A-module coalgebra map such that $\pi(a) = a$, for all $a \in A$. It was proved in [2] that an extension $A \subseteq E$ splits in the above sense if and only if E is isomorphic to a unified product between A and a certain subcoalgebra H of E. The unified product was introduced in [1] as an answer to the restricted extending structures problem for Hopf algebras. Unified products characterize Hopf algebras that factorize through a Hopf subalgebra A and a subcoalgebra H such that $1 \in H$. As special cases of the unified product, we recover the double cross product or the crossed product of Hopf algebras (see Examples 1.1).

An outline of the paper is as follows. In Section 2, we recall the construction and some basic properties of unified products. In Section 3, the notions of generalized (p, f)—left/right skew pairing and generalized (u, v) braidings are introduced. The main result of the paper is Theorem 3.6 where a bijective correspondence between the set of all coquasitriangular structures σ on the unified product $A \ltimes H$ and the set of all quadruples (p, τ, u, v) satisfying some compatibilities is established. All coquasitriangular structures on the unified product are explicitly described in terms of this quadruple (p, τ, u, v). In particular, in Corollary 3.7, necessary and sufficient conditions for a double cross product $A \bowtie H$ associated to a matched pair $(A, H, \triangleright, \triangleleft)$ of Hopf algebras to be a coquasitriangular Hopf algebra are given.

Let $\lambda : H \otimes A \to k$ be a skew pairing between two Hopf algebras and consider $D_\lambda(A, H) := A \bowtie_\lambda H$ to be the *generalized quantum double* as constructed in ([13, Example 7.2.6]). As the main application of the results in Theorem 4.1, the set of all coquasitriangular structures on the generalized quantum double $D_\lambda(A, H)$ is completely described. In particular, it is proved that a generalized quantum double is a coquasitriangular Hopf algebra if and only if both Hopf algebras A and H are coquasitriangular. Several explicit examples are also provided.

2. Preliminaries. Throughout this paper, k denotes an arbitrary field. Unless specified otherwise, all algebras, coalgebras, tensor products and homomorphisms are over k. For a coalgebra C, we use Sweedler's Σ-notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc., with summation understood. For a k-linear map $f : H \otimes H \to A$, we denote $f(g, h) = f(g \otimes h)$.

Recall from [5] that if A and H are two Hopf algebras and $\lambda : A \otimes H \to k$ is a k-linear map which fulfills the compatibilities:

(BR1) $\lambda(xy, z) = \lambda(x, z_{(1)})\lambda(y, z_{(2)})$,
(BR2) $\lambda(1, z) = \varepsilon(z)$,

(BR3) $\lambda(x, lz) = \lambda(x_{(1)}, z)\lambda(x_{(2)}, l)$,

(BR4) $\lambda(y, 1) = \varepsilon(y)$,

for all $x, y \in A$, $l, z \in H$, then λ is called *skew pairing* on (A, H). Notice that a skew pairing λ is convolution invertible with $\lambda^{-1} = \lambda \circ (S \otimes Id)$. Also, by a straightforward computation, it can be seen that if λ is a skew pairing on (A, H), then $\lambda \circ (S \otimes Id) \circ \nu$ is also a skew pairing on (H, A) where ν is the flip map.

Moreover, recall from [**10**] that a Hopf algebra H is called *coquasitriangular* or *braided* if there exists a linear map $p : H \otimes H \to k$ such that relations $(BR1) - (BR4)$ are fulfilled and

(BR5) $p(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = y_{(1)}x_{(1)}p(x_{(2)}, y_{(2)})$

holds for all $x, y, z \in H$.

2.1. Unified products.

We recall from [**1**] the construction of the unified product. An *extending datum* of a bialgebra A is a system $\Omega(A) = (H, \triangleleft, \triangleright, f)$, where $H = (H, \Delta_H, \varepsilon_H, 1_H, \cdot)$ is a k-module such that $(H, \Delta_H, \varepsilon_H)$ is a coalgebra, $(H, 1_H, \cdot)$ is a unitary not necessarily associative k-algebra, the k-linear maps $\triangleleft : H \otimes A \to H$, $\triangleright : H \otimes A \to A$, $f : H \otimes H \to A$ are coalgebra maps such that the following normalization conditions hold:

$$h \triangleright 1_A = \varepsilon_H(h)1_A, \quad 1_H \triangleright a = a, \quad 1_H \triangleleft a = \varepsilon_A(a)1_H, \quad h \triangleleft 1_A = h, \qquad (1)$$

$$\Delta_H(1_H) = 1_H \otimes 1_H, \qquad f(h, 1_H) = f(1_H, h) = \varepsilon_H(h)1_A, \qquad (2)$$

for all $h \in H$, $a \in A$.

Let $\Omega(A) = (H, \triangleleft, \triangleright, f)$ be an extending datum of A. We denote by $A \ltimes_{\Omega(A)} H = A \ltimes H$ the k-module $A \otimes H$ together with the multiplication:

$$(a \ltimes h) \bullet (c \ltimes g) := a(h_{(1)} \triangleright c_{(1)})f(h_{(2)} \triangleleft c_{(2)}, g_{(1)}) \ltimes (h_{(3)} \triangleleft c_{(3)}) \cdot g_{(2)}, \qquad (3)$$

for all $a, c \in A$ and $h, g \in H$, where we denoted $a \otimes h \in A \otimes H$ by $a \ltimes h$. The object $A \ltimes H$ is called *the unified product of A and $\Omega(A)$* if $A \ltimes H$ is a bialgebra with the multiplication given by (3), the unit $1_A \ltimes 1_H$ and the coalgebra structure given by the tensor product of coalgebras, i.e.:

$$\Delta_{A \ltimes H}(a \ltimes h) = a_{(1)} \ltimes h_{(1)} \otimes a_{(2)} \ltimes h_{(2)}, \qquad (4)$$

$$\varepsilon_{A \ltimes H}(a \ltimes h) = \varepsilon_A(a)\varepsilon_H(h), \qquad (5)$$

for all $h \in H$, $a \in A$. We have proved in [**1**, Theorem 2.4] that $A \ltimes H$ is an unified product if and only if $\Delta_H : H \to H \otimes H$ and $\varepsilon_H : H \to k$ are k-algebra maps, $(H, \triangleleft)$ is a right A-module structure and the following compatibilities hold:

(BE1) $(g \cdot h) \cdot l = (g \triangleleft f(h_{(1)}, l_{(1)})) \cdot (h_{(2)} \cdot l_{(2)})$,

(BE2) $g \triangleright (ab) = (g_{(1)} \triangleright a_{(1)})[(g_{(2)} \triangleleft a_{(2)}) \triangleright b]$,

(BE3) $(g \cdot h) \triangleleft a = [g \triangleleft (h_{(1)} \triangleright a_{(1)})] \cdot (h_{(2)} \triangleleft a_{(2)})$,

(BE4) $[g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)})]f(g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)}), h_{(3)} \triangleleft a_{(3)}) = f(g_{(1)}, h_{(1)})[(g_{(2)} \cdot h_{(2)}) \triangleright a]$,

(BE5) $(g_{(1)} \triangleright f(h_{(1)}, l_{(1)}))f(g_{(2)} \triangleleft f(h_{(2)}, l_{(2)}), h_{(3)} \cdot l_{(3)}) = f(g_{(1)}, h_{(1)})f(g_{(2)} \cdot h_{(2)}, l)$,

(BE6) $g_{(1)} \triangleleft a_{(1)} \otimes g_{(2)} \triangleright a_{(2)} = g_{(2)} \triangleleft a_{(2)} \otimes g_{(1)} \triangleright a_{(1)}$,

(BE7) $g_{(1)} \cdot h_{(1)} \otimes f(g_{(2)}, h_{(2)}) = g_{(2)} \cdot h_{(2)} \otimes f(g_{(1)}, h_{(1)})$,

for all g, h, $l \in H$ and a, $b \in A$. In this case, $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is called a *bialgebra extending structure* of A. A bialgebra extending structure $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is called a *Hopf algebra extending structure* of A if $A \ltimes H$ has an antipode. If A is a Hopf algebra with an antipode S_A and H has an antipode S_H, then the unified product $A \ltimes H$ has an antipode given by

$$S(a \ltimes g) := (S_A[f(S_H(g_{(2)}), g_{(3)})] \ltimes S_H(g_{(1)})) \bullet (S_A(a) \ltimes 1_H),$$

for all $a \in A$ and $g \in H$ ([**1**, Proposition 2.8]).

In [**2**], it was proved that a Hopf algebra E is isomorphic to a unified product $A \ltimes H$ if and only if there exists a morphism of Hopf algebras $i : A \to E$, which has a retraction $\pi : E \to A$ that is a normal ([**2**, Definition 2.1]) left A-module coalgebra morphism.

EXAMPLE 2.1. (1) Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ an extending datum of A such that the cocycle f is trivial, that is $f(g, h) = \varepsilon_H(g)\varepsilon_H(h)1_A$, for all $g, h \in H$.

Then $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is a bialgebra extending structure of A if and only if H is a bialgebra and $(A, H, \triangleleft, \triangleright)$ is a matched pair of bialgebras in the sense of [**13**, Definition 7.2.1]. In this case, the associated unified product $A \ltimes H = A \bowtie H$ is the double cross product of bialgebras in Majid's terminology (also called bicrossed product of bialgebras in [**9**]). Perhaps, the most famous example of a double cross product is the generalized quantum double (see Section 4 below). If H is a finite-dimensional Hopf algebra, then the generalized quantum double coincides with the celebrated quantum double $D(H) = H^{*op} \bowtie H$ which is a double cross product by the mutual coadjoint actions:

$$h \triangleright \alpha = \alpha_{(2)}\langle h, S(\alpha_{(1)})\alpha_{(3)}\rangle, \quad h \triangleleft \alpha = h_{(2)}\langle \alpha, S(h_{(1)})h_{(3)}\rangle,$$

for all $h \in H$ and $\alpha \in H^*$.

(2) Let A be a bialgebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ an extending datum of A such that the action $\triangleleft$ is trivial, that is $h \triangleleft a = \varepsilon_A(a)h$, for all $h \in H$ and $a \in A$. In this case, the associated unified product $A \ltimes H = A\#_f^\triangleright H$ is called the crossed product of Hopf algebras. For more details on crossed products of Hopf algebras, we refer to [**3**].

3. Coquasitriangular structures on the unified products. In this section, we describe the coquasitriangular or braided structures on the unified product. In other words, we determine all braided structures that can be defined on the monoidal category of $A \ltimes H$ comodules. First, we introduce some new definitions as natural generalizations for the concepts of braiding and skew pairing.

DEFINITION 3.1. Let A be a Hopf algebra, $H = (H, \Delta_H, \varepsilon_H, 1_H, \cdot)$ a k-module such that $(H, \Delta_H, \varepsilon_H)$ is a coalgebra, $(H, 1_H, \cdot)$ is a unitary not necessarily associative k-algebra, $f : H \otimes H \to A$ a coalgebra map and $p : A \otimes A \to k$ a braiding on A. A linear map $u : A \otimes H \to k$ is called *generalized (p,f) right skew pairing* on (A, H) if the following compatibilities are fulfilled for any $a, b \in A, g, t \in H$:

(RS1) $u(ab, t) = u(a, t_{(1)})u(b, t_{(2)})$,
(RS2) $u(1, h) = \varepsilon(h)$,
(RS3) $u(a_{(1)}, g_{(2)} \cdot t_{(2)})p(a_{(2)}, f(g_{(1)}, t_{(1)})) = u(a_{(1)}, t)u(a_{(2)}, g)$,
(RS4) $u(a, 1) = \varepsilon(a)$.

DEFINITION 3.2. Let A be a Hopf algebra, $H = (H, \Delta_H, \varepsilon_H, 1_H, \cdot)$ a k-module such that $(H, \Delta_H, \varepsilon_H)$ is a coalgebra, $(H, 1_H, \cdot)$ is a unitary not necessarily associative k-algebra, $f : H \otimes H \to A$ a coalgebra map and $p : A \otimes A \to k$ a braiding on A. A linear map $v : H \otimes A \to k$ is called *generalized (p,f) left skew pairing on* (H, A) if the following compatibilities are fulfilled for any $b, c \in A$, $h, g \in H$:

(LS1) $p(f(h_{(1)}, g_{(1)}), c_{(1)})v(h_{(2)} \cdot g_{(2)}, c_{(2)}) = v(h, c_{(1)})v(g, c_{(2)})$,
(LS2) $v(h, 1) = \varepsilon(h)$,
(LS3) $v(h, bc) = v(h_{(1)}, c)v(h_{(2)}, b)$,
(LS4) $v(1, a) = \varepsilon(a)$.

REMARK 3.3. If H is a bialgebra and $f = \varepsilon \otimes \varepsilon$ is the trivial cocycle, then the notion of generalized (p,f) left/right skew pairing on (A, H) coincides with the notion of skew pairing on (A, H).

DEFINITION 3.4. Let A be a Hopf algebra, $H = (H, \Delta_H, \varepsilon_H, 1_H, \cdot)$ a k-module such that $(H, \Delta_H, \varepsilon_H)$ is a coalgebra, $(H, 1_H, \cdot)$ is a unitary not necessarily associative k-algebra, $f : H \otimes H \to A$ a coalgebra map and $p : A \otimes A \to k$ a braiding on A, $u : A \otimes H \to k$ a generalized (p, f) right skew pairing and $v : H \otimes A \to k$ a generalized (p, f) left skew pairing. A linear map $\tau : H \otimes H \to k$ is called a *generalized (u, v) skew braiding on* H if the following compatibilities are fulfilled for all $h, g, t \in H$:

(SBR1) $u(f(h_{(1)}, g_{(1)}), t_{(1)})\tau(h_{(2)} \cdot g_{(2)}, t_{(2)}) = \tau(h, t_{(1)})\tau(g, t_{(2)})$,
(SBR2) $\tau(1, g) = \varepsilon(g)$,
(SBR3) $\tau(h_{(1)}, g_{(2)} \cdot t_{(2)})v(h_{(2)}, f(g_{(1)}, t_{(1)})) = \tau(h_{(1)}, t)\tau(h_{(2)}, g)$,
(SBR4) $\tau(g, 1) = \varepsilon(g)$,
(SBR5) $\tau(h_{(1)}, g_{(1)})h_{(2)} \cdot g_{(2)} = g_{(1)} \cdot h_{(1)}\tau(h_{(2)}, g_{(2)})$.

REMARK 3.5. If H is a bialgebra and $f = \varepsilon \otimes \varepsilon$ is the trivial cocycle, then the notion of generalized (u, v) skew braiding on H coincides with the notion of coquasitriangular structure (or braiding) on H.

THEOREM 3.6. *Let A be a Hopf algebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ a Hopf algebra extending structure of A. There is a bijective correspondence between:*

(i) The set of all coquasitriangular structures σ on the unified product $A \ltimes H$.

(ii) The set of all quadruples (p, τ, u, v) where $p : A \otimes A \to k$, $\tau : H \otimes H \to k$, $u : A \otimes H \to k$, $v : H \otimes A \to k$ are linear maps such that (A, p) is a coquasitriangular Hopf algebra, u is a generalized (p, f) right skew pairing, v is a generalized (p, f) left skew pairing, (H, τ) is a generalized (u, v) skew braiding and the following compatibilities are fulfilled:

$$v(h_{(1)}, b_{(1)})(h_{(2)} \triangleright b_{(2)}) \otimes (h_{(3)} \triangleleft b_{(3)}) = b_{(1)} \otimes h_{(1)}v(h_{(2)}, b_{(2)}), \qquad (6)$$

$$(g_{(1)} \triangleright a_{(1)}) \otimes (g_{(2)} \triangleleft a_{(2)})u(a_{(3)}, g_{(3)}) = u(a_{(1)}, g_{(1)})a_{(2)} \otimes g_{(2)}, \qquad (7)$$

$$\tau(h_{(1)}, g_{(1)})f(h_{(2)}, g_{(2)}) = f(g_{(1)}, h_{(1)})\tau(h_{(2)}, g_{(2)}), \qquad (8)$$

$$u(a_{(1)}, g_{(2)} \triangleleft c_{(2)})p(a_{(2)}, g_{(1)} \triangleright c_{(1)}) = p(a_{(1)}, c)u(a_{(2)}, g), \qquad (9)$$

$$\tau(h_{(1)}, g_{(2)} \triangleleft c_{(2)})v(h_{(2)}, g_{(1)} \triangleright c_{(1)}) = v(h_{(1)}, c)\tau(h_{(2)}, g), \qquad (10)$$

$$p(h_{(1)} \triangleright b_{(1)}, c_{(1)})v(h_{(2)} \triangleleft b_{(2)}, c_{(2)}) = v(h, c_{(1)})p(b, c_{(2)}), \qquad (11)$$

$$u(h_{(1)} \triangleright b_{(1)}, t_{(1)})\tau(h_{(2)} \triangleleft b_{(2)}, t_{(2)}) = \tau(h, t_{(1)})u(b, t_{(2)}). \qquad (12)$$

Under the above bijection, the coquasitriangular structure $\sigma : (A \ltimes H) \otimes (A \ltimes H) \to k$
corresponding to (p, τ, u, v) *is given by*

$$\sigma(a \ltimes h, b \ltimes g) = u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})v(h_{(2)}, b_{(2)}), \qquad (13)$$

for all $a, b, c \in A$ *and* $h, g, t \in H$.

Proof. Suppose first that $(A \ltimes H, \sigma)$ is a coquasitriangular Hopf algebra. We define
the following linear maps:

$$\begin{aligned}
p &: A \otimes A \to k, & p(a, b) &= \sigma(a \otimes 1, b \otimes 1), \\
\tau &: H \otimes H \to k, & \tau(h, g) &= \sigma(1 \otimes h, 1 \otimes g), \\
u &: A \otimes H \to k, & u(a, h) &= \sigma(a \otimes 1, 1 \otimes h), \\
v &: H \otimes A \to k, & v(h, a) &= \sigma(1 \otimes h, a \otimes 1).
\end{aligned}$$

Before going into the proof, we collect here some compatibilities satisfied by the
maps defined above which will be useful in the sequel. The following are just easy
consequences of the fact that σ is a coquasitriangular structure on $A \ltimes H$, and hence,
it satisfies the normalizing relations (BR2) and (BR4):

$$p(1, b) = \varepsilon(b) = p(b, 1), \qquad (14)$$

$$\tau(1, h) = \varepsilon(h) = \tau(h, 1), \qquad (15)$$

$$u(1, h) = \varepsilon(h), \quad u(a, 1) = \varepsilon(a), \qquad (16)$$

$$v(1, a) = \varepsilon(a), \quad v(h, 1) = \varepsilon(h). \qquad (17)$$

Remark that from relation (15), it follows that τ fulfills (SBR2) and (SBR4), while
from relation (17), we can derive that v fulfills (LS2) and (LS4).

First, we prove that relation (13) indeed holds:

$$\begin{aligned}
\sigma(a\#h, \ b\#g) &= \sigma((a\#1)(1\#h), (b\#1)(1\#g)) = \\
&\overset{(BR1)}{=} \sigma(a\#1, (b_{(1)}\#1)(1\#g_{(1)}))\sigma((1\#h), (b_{(2)}\#1)(1\#g_{(2)})), \\
&\overset{(BR3)}{=} \sigma(a_{(1)}\#1, 1\#g_{(1)})\sigma(a_{(2)}\#1, b_{(1)}\#1)\sigma(1\#h_{(1)}, 1\#g_{(2)})\sigma(1\#h_{(2)}, b_{(2)}\#1) \\
&= u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})v(h_{(2)}, b_{(2)}).
\end{aligned}$$

Next, we prove that (A, p) is a coquasitriangular Hopf algebra, u is a generalized
(p, f) right skew pairing on (H, A), v is a generalized (p, f) left skew pairing on (A, H)
and τ is a generalized (u, v) skew braiding on H. Having in mind that $(A \ltimes H, \sigma)$
is a coquasitriangular Hopf algebra, it is straightforward to see that (A, p) is a
coquasitriangular Hopf algebra by considering $x = a\#1$, $y = b\#1$ and $z = c\#1$ in
(BR1)–(BR5).
Since σ satisfies (BR1), then for all $a, b, c \in A$ and $h, g, t \in H$, we have

$$\begin{aligned}
\sigma(a(g_{(1)} \triangleright b_{(1)})f(h_{(2)} &\triangleleft b_{(2)}, h_{(1)})\#(g_{(3)} \triangleleft b_{(3)}) \cdot h_{(2)}, c\#t) \\
&= \sigma(a \otimes g, c_{(1)} \otimes t_{(1)})\sigma(b \otimes h, c_{(2)} \otimes t_{(2)}). \qquad (18)
\end{aligned}$$

Moreover, since σ also fulfills (BR3), we have

$$\begin{aligned}
\sigma(a\#h, b(g_{(1)} &\triangleright c_{(1)})f(g_{(2)} \triangleleft c_{(2)}, t_{(1)})\#(g_{(3)} \triangleleft c_{(3)}) \cdot t_{(2)}) \\
&= \sigma(a_{(1)} \otimes h_{(1)}, c \otimes t)\sigma(a_{(2)} \otimes h_{(2)}, b \otimes g). \qquad (19)
\end{aligned}$$

Furthermore, by (BR5), we have

$$\sigma(a_{(1)} \otimes h_{(1)}, b_{(1)} \otimes g_{(1)})a_{(2)}(h_{(2)} \triangleright b_{(2)})f(h_{(3)} \triangleleft b_{(3)}, g_{(2)}) \otimes (h_{(4)} \triangleleft b_{(4)}) \cdot g_{(3)}$$
$$= b_{(1)}(g_{(1)} \triangleright a_{(1)})f(g_{(2)} \triangleleft a_{(2)}, h_{(1)}) \otimes (g_{(3)} \triangleleft a_{(3)}) \cdot h_{(2)}\sigma(a_{(4)} \otimes h_{(3)}, b_{(2)} \otimes g_{(4)}). \quad (20)$$

By considering $h = g = 1$ and $c = 1$ in (18), we get relation (RS1). If we let $b = c = 1$ and $h = 1$ in (19), it yields:

$$u(a_{(1)}, g_{(3)} \cdot t_{(3)})p(a_{(2)}, f(g_{(1)}, t_{(1)}))\tau(1, g_{(4)}t_{(4)})v(1, f(g_{(2)}, t_{(2)})) = u(a_{(1)}, t)u(a_{(2)}, g).$$

Now using relations (15) and (17), we get (RS3). Hence, we proved that u is a generalized (p, f) right skew pairing on (H, A). Considering $a = b = 1$ and $t = 1$ in (18) yields:

$$u(f(h_{(1)}, g_{(1)}), 1)p(f(h_{(2)}, g_{(2)}), c_{(1)})\tau(h_{(3)} \cdot g_{(3)}, 1)v(h_{(4)} \cdot g_{(4)}, c_{(2)}) = v(h, c_{(1)})v(g, c_{(2)}).$$

Using (15) and (16), we get that (LS1) holds for v. Moreover, from (19) applied to $g = t = 1$ and $a = 1$, we get that (LS3) also holds for v and we proved that v is indeed a generalized (p, f) left skew pairing on (A, H). Next, we apply (18) for $a = b = c = 1$:

$$u(f(h_{(1)}, g_{(1)}), t_{(1)})p(f(h_{(2)}, g_{(2)}), 1)\tau(h_{(3)} \cdot g_{(3)}, t_{(2)})v(h_{(4)} \cdot g_{(4)}, 1) = \tau(h, t_{(1)})\tau(g, t_{(2)}).$$

Using (14) and (17), we obtain (SBR1). Now equation (19) applied for $a = b = c = 1$ yields:

$$u(1, g_{(3)} \cdot t_{(3)})p(1, f(g_{(1)}, t_{(1)}))v(h_{(2)}, f(g_{(2)}, t_{(2)}))\tau(h_{(1)}, g_{(4)} \cdot t_{(4)}) = \tau(h_{(1)}, t)\tau(h_{(2)}, g).$$

From (14) and (16), we obtain that (SBR3) holds for τ. Considering $a = b = 1$ in (20), we get

$$\tau(h_{(1)}, g_{(1)})f(h_{(2)}, g_{(2)})\#h_{(3)} \cdot g_{(3)} = f(g_{(1)}, h_{(1)})\#g_{(2)} \cdot h_{(2)}\tau(h_{(3)}, g_{(3)}).$$

Having in mind that f is a coalgebra map, we obatin, by applying $\varepsilon \otimes Id$, that (SBR5) holds for τ, and therefore τ is a generalized (u, v) skew braiding.

We still need to prove that the compatibilities (6)–(12) hold. Compatibilities (6) and (7) are obtained from (20) by considering: $a = 1$ and $g = 1$, respectively, $b = 1$ and $h = 1$, while (8) can be derived from (20) by considering $a = b = 1$ and then applying $Id \otimes \varepsilon$. The next two compatibilities, (9) and (10), can be obtained by considering $h = t = 1$ and $b = 1$, respectively, $a = b = 1$ and $t = 1$ in (19). To this end, relations (11) and (12) can be derived from (18) by considering $g = t = 1$ and $a = 1$, respectively, $a = c = 1$ and $g = 1$.

Assume now that (A, p) is a coquasitriangular Hopf algebra, u is a generalized (p, f) right skew pairing, v is a generalized (p, f) left skew pairing, τ is a generalized (u, v) skew braiding and σ is given by (13) such that compatibilities (6)–(12) are fulfilled. Then, using relations (RS2), (SBR2), (LS2) and the fact that p is a coquasitriangular structure, we can prove that for all $a \in A, h \in H$, we have

$$\sigma(1\#1, a\#h) = u(1, h_{(1)})p(1, a_{(1)})\tau(1, h_{(2)})v(1, a_{(2)})$$
$$= \varepsilon(a)\varepsilon(h)$$
$$= \varepsilon(a\#h).$$

Moreover, using relations (RS4), (SBR4), (LS4) and again the fact that p is a coquasitriangular structure, we also have

$$\sigma(a\#h, 1\#1) = u(a_{(1)}, 1)p(a_{(2)}, 1)\tau(h_{(1)}, 1)v(h_{(2)}, 1)$$
$$= \varepsilon(a)\varepsilon(h)$$
$$= \varepsilon(a\#h),$$

for all $a \in A$, $h \in H$. Hence, σ also fulfills (BR4).

To prove that σ satisfies (BR1), we start by first computing the left-hand side. Thus, for all $a, b, c \in A$ and $h, g, t \in H$, we have

$$
\begin{aligned}
LHS \;=\;& u(a_{(1)}(g_{(1)} \triangleright b_{(1)})f(g_{(3)} \triangleleft b_{(3)}, h_{(1)}), t_{(1)})v((g_{(6)} \triangleleft b_{(6)}) \cdot h_{(4)}, c_{(2)}) \\
& p(a_{(2)}(g_{(2)} \triangleright b_{(2)})f(g_{(4)} \triangleleft b_{(4)}, h_{(2)}), c_{(1)})\tau((g_{(5)} \triangleleft b_{(5)}) \cdot h_{(3)}, t_{(2)}) \\[4pt]
\overset{(RS1)}{=}\;& u(a_{(1)}, t_{(1)})u(g_{(1)} \triangleright b_{(1)}, t_{(2)})u(f(g_{(3)} \triangleleft b_{(3)}, h_{(1)}), t_{(3)})v((g_{(6)} \triangleleft b_{(6)}) \cdot h_{(4)}, c_{(4)}) \\
& p(a_{(2)}, c_{(1)})p(\underline{f(g_{(4)} \triangleleft b_{(4)}, h_{(2)})}, c_{(3)})\tau(\underline{(g_{(5)} \triangleleft b_{(5)})} \cdot h_{(3)}, t_{(4)})p(g_{(2)} \triangleright b_{(2)}, c_{(2)}) \\[4pt]
\overset{(BE7)}{=}\;& u(a_{(1)}, t_{(1)})u(g_{(1)} \triangleright b_{(1)}, t_{(2)})u(f(g_{(3)} \triangleleft b_{(3)}, h_{(1)}), t_{(3)})v(\underline{(g_{(6)} \triangleleft b_{(6)}) \cdot h_{(4)}, c_{(4)}}) \\
& p(a_{(2)}, c_{(1)})\underline{p(f(g_{(5)} \triangleleft b_{(5)}, h_{(3)}), c_{(3)})}\tau((g_{(4)} \triangleleft b_{(4)}) \cdot h_{(2)}, t_{(4)})p(g_{(2)} \triangleright b_{(2)}, c_{(2)}) \\[4pt]
\overset{(LS1)}{=}\;& u(a_{(1)}, t_{(1)})u(g_{(1)} \triangleright b_{(1)}, t_{(2)})u(f(g_{(3)} \triangleleft b_{(3)}, h_{(1)}), t_{(3)})p(a_{(2)}, c_{(1)}) \\
& p(g_{(2)} \triangleright b_{(2)}, c_{(2)})\tau(\underline{(g_{(4)} \triangleleft b_{(4)}) \cdot h_{(2)}, t_{(4)}})v(g_{(5)} \triangleleft b_{(5)}, c_{(3)})v(h_{(3)}, c_{(4)}) \\[8pt]
\overset{(SBR1)}{=}\;& u(a_{(1)}, t_{(1)})u(g_{(1)} \triangleright b_{(1)}, t_{(2)})p(a_{(2)}, c_{(1)})p(g_{(2)} \triangleright b_{(2)}, c_{(2)}) \\
& \tau(g_{(3)} \triangleleft b_{(3)}, t_{(3)})\tau(h_{(1)}, t_{(4)})v(g_{(4)} \triangleleft b_{(4)}, c_{(3)})v(h_{(2)}, c_{(4)}) \\[4pt]
\overset{(BE6)}{=}\;& u(a_{(1)}, t_{(1)})u(g_{(1)} \triangleright b_{(1)}, t_{(2)})p(a_{(2)}, c_{(1)})\underline{p(g_{(3)} \triangleright b_{(3)}, c_{(2)})} \\
& \tau(g_{(2)} \triangleleft b_{(2)}, t_{(3)})\tau(h_{(1)}, t_{(4)})\underline{v(g_{(4)} \triangleleft b_{(4)}, c_{(3)})}v(h_{(2)}, c_{(4)}) \\[4pt]
\overset{(11)}{=}\;& u(a_{(1)}, t_{(1)})\underline{u(g_{(1)} \triangleright b_{(1)}, t_{(2)})}p(a_{(2)}, c_{(1)})\underline{\tau(g_{(2)} \triangleleft b_{(2)}, t_{(3)})}\tau(h_{(1)}, t_{(4)}) \\
& v(g_{(3)}, c_{(2)})p(b_{(3)}, c_{(3)})v(h_{(2)}, c_{(4)}) \\[4pt]
\overset{(12)}{=}\;& u(a_{(1)}, t_{(1)})p(a_{(2)}, c_{(1)})\tau(g_{(1)}, t_{(2)})u(b_{(1)}, t_{(3)})\tau(h_{(1)}, t_{(4)}) \\
& v(g_{(2)}, c_{(2)})p(b_{(2)}, c_{(3)})v(h_{(2)}, c_{(4)}) \\[4pt]
=\;& RHS,
\end{aligned}
$$

where in the second equality, we also used the fact that p is a coquasitriangular structure. To prove (BR3) we start again by computing the left-hand side. Thus, for all $a, b, c \in A$ and $h, g, t \in H$, we have

$$
\begin{aligned}
LHS \;=\;& u(\underline{(a_{(1)}, (g_{(5)} \triangleleft c_{(5)}) \cdot t_{(3)})p(a_{(2)}, b_{(1)}(g_{(1)} \triangleright c_{(1)})f(g_{(3)} \triangleleft c_{(3)}, t_{(1)}))} \\
& \tau(h_{(1)}, (g_{(6)} \triangleleft c_{(6)}) \cdot t_{(4)})\underline{v(h_{(2)}, b_{(2)}(g_{(2)} \triangleright c_{(2)})f(g_{(4)} \triangleleft c_{(4)}, t_{(2)}))} \\[4pt]
\overset{(RS3)}{=}\;& u(a_{(1)}, \underline{(g_{(5)} \triangleleft c_{(5)}) \cdot t_{(3)}})p(a_{(2)}, f(g_{(3)} \triangleleft c_{(3)}, t_{(1)}))p(a_{(3)}, g_{(1)} \triangleright c_{(1)})p(a_{(4)}, b_{(1)}) \\
& \tau(h_{(1)}, \underline{(g_{(6)} \triangleleft c_{(6)}) \cdot t_{(4)}})v(h_{(2)}, \underline{f(g_{(4)} \triangleleft c_{(4)}, t_{(2)})})v(h_{(3)}, g_{(2)} \triangleright c_{(2)})v(h_{(4)}, b_{(2)}) \\[4pt]
\overset{(BE7)}{=}\;& u(a_{(1)}, (g_{(4)} \triangleleft c_{(4)}) \cdot t_{(2)})p(a_{(2)}, f(g_{(3)} \triangleleft c_{(3)}, t_{(1)}))p(a_{(3)}, g_{(1)} \triangleright c_{(1)})p(a_{(4)}, b_{(1)}) \\
& \tau(h_{(1)}, (g_{(6)} \triangleleft c_{(6)}) \cdot t_{(4)})v(h_{(2)}, \underline{f(g_{(5)} \triangleleft c_{(5)}, t_{(3)})})v(h_{(3)}, g_{(2)} \triangleright c_{(2)})v(h_{(4)}, b_{(2)})
\end{aligned}
$$

$$\overset{(SBR3)}{=}\; u(a_{(1)}, (g_{(4)} \triangleleft c_{(4)}) \cdot t_{(2)})p(a_{(2)}, f(g_{(3)} \triangleleft c_{(3)}, t_{(1)}))p(a_{(3)}, g_{(1)} \triangleright c_{(1)})p(a_{(4)}, b_{(1)})$$
$$\tau(h_{(1)}, t_{(3)})\tau(h_{(2)}, g_{(5)} \triangleleft c_{(5)})v(h_{(3)}, g_{(2)} \triangleright c_{(2)})v(h_{(4)}, b_{(2)})$$

$$\overset{(RS3)}{=}\; u(a_{(1)}, t_{(1)})u(a_{(2)}, g_{(3)} \triangleleft c_{(3)})p(a_{(3)}, g_{(1)} \triangleright c_{(1)})p(a_{(4)}, b_{(1)})\tau(h_{(1)}, t_{(2)})$$
$$\tau(h_{(2)}, g_{(4)} \triangleleft c_{(4)})v(h_{(3)}, g_{(2)} \triangleright c_{(2)})v(h_{(4)}, b_{(2)})$$

$$\overset{(BE6)}{=}\; u(a_{(1)}, t_{(1)})u(a_{(2)}, g_{(2)} \triangleleft c_{(2)})p(a_{(3)}, g_{(1)} \triangleright c_{(1)})p(a_{(4)}, b_{(1)})\tau(h_{(1)}, t_{(2)})$$
$$\tau(h_{(2)}, g_{(4)} \triangleleft c_{(4)})v(h_{(3)}, g_{(3)} \triangleright c_{(3)})v(h_{(4)}, b_{(2)})$$

$$\overset{(9)}{=}\; u(a_{(1)}, t_{(1)})p(a_{(2)}, c_{(1)})u(a_{(3)}, g_{(1)})p(a_{(4)}, b_{(1)})\tau(h_{(1)}, t_{(2)})$$
$$\tau(h_{(2)}, g_{(3)} \triangleleft c_{(3)})v(h_{(3)}, g_{(2)} \triangleright c_{(2)})v(h_{(4)}, b_{(2)})$$

$$\overset{(10)}{=}\; u(a_{(1)}, t_{(1)})p(a_{(2)}, c_{(1)})u(a_{(3)}, g_{(1)})p(a_{(4)}, b_{(1)})\tau(h_{(1)}, t_{(2)})$$
$$v(h_{(2)}, c_{(2)})\tau(h_{(3)}, g_{(2)})v(h_{(4)}, b_{(2)})$$

$$=\; RHS.$$

Note that in the second equality, we used the fact that p is a coquasitriangular structure. In order to show that σ also fulfills (BR5), we need the following compatibilities that can be easily derived from (6) and (7) by applying $\varepsilon \otimes Id$:

$$v(h_{(1)}, b_{(1)})(h_{(2)} \triangleleft b_{(2)}) = h_{(1)}v(h_{(2)}, b), \tag{21}$$
$$(g_{(1)} \triangleleft a_{(1)})u(a_{(2)}, g_{(2)}) = u(a, g_{(1)})g_{(2)}. \tag{22}$$

Computing the left-hand side of (BR5), we obtain

$$LHS \;=\; u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})v(h_{(2)}, b_{(2)})a_{(3)}(h_{(3)} \triangleright b_{(3)})$$
$$f(h_{(4)} \triangleleft b_{(4)}, g_{(3)})\#(h_{(5)} \triangleleft b_{(5)}) \cdot g_{(4)}$$

$$\overset{(BE7)}{=}\; u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})v(h_{(2)}, b_{(2)})a_{(3)}(h_{(3)} \triangleright b_{(3)})$$
$$f(h_{(5)} \triangleleft b_{(5)}, g_{(4)})\#(h_{(4)} \triangleleft b_{(4)}) \cdot g_{(3)}$$

$$\overset{(6)}{=}\; u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})a_{(3)}b_{(2)}f(h_{(4)} \triangleleft b_{(4)}, g_{(4)})\#$$
$$h_{(2)} \cdot g_{(3)}v(h_{(3)}, b_{(3)})$$

$$=\; u(a_{(1)}, g_{(1)})\tau(h_{(1)}, g_{(2)})b_{(1)}a_{(2)}p(a_{(3)}, b_{(2)})f(h_{(4)} \triangleleft b_{(4)}, g_{(4)})\#$$
$$h_{(2)} \cdot g_{(3)}v(h_{(3)}, b_{(3)})$$

$$\overset{(SBR5)}{=}\; u(a_{(1)}, g_{(1)})b_{(1)}a_{(2)}p(a_{(3)}, b_{(2)})f(h_{(4)} \triangleleft b_{(4)}, g_{(4)})\#(g_{(2)} \cdot h_{(1)})$$
$$\tau(h_{(2)}, g_{(3)})v(h_{(3)}, b_{(3)})$$

$$\overset{(7)}{=}\; b_{(1)}(g_{(1)} \triangleright a_{(1)})p(a_{(4)}, b_{(2)})f(h_{(4)} \triangleleft b_{(4)}, g_{(5)})\#(g_{(2)} \triangleleft a_{(2)}) \cdot h_{(1)}$$
$$u(a_{(3)}, g_{(3)})\tau(h_{(2)}, g_{(4)})v(h_{(3)}, b_{(3)})$$

$$\overset{(22)}{=}\; b_{(1)}(g_{(1)} \triangleright a_{(1)})p(a_{(3)}, b_{(2)})f(h_{(4)} \triangleleft b_{(4)}, g_{(5)})\#(g_{(3)} \cdot h_{(1)})u(a_{(2)}, g_{(2)})$$
$$\tau(h_{(2)}, g_{(4)})v(h_{(3)}, b_{(3)})$$

$$\overset{(21)}{=}\ b_{(1)}(g_{(1)} \triangleright a_{(1)})p(a_{(3)}, b_{(2)})\underline{f(h_{(3)}, g_{(5)})}\#(g_{(3)} \cdot h_{(1)})u(a_{(2)}, g_{(2)})$$
$$\tau(h_{(2)}, g_{(4)})v(h_{(4)}, b_{(3)})$$

$$\overset{(8)}{=}\ b_{(1)}(g_{(1)} \triangleright a_{(1)})p(a_{(3)}, b_{(2)})v(h_{(4)}, b_{(3)})\tau(h_{(3)}, g_{(5)})\underline{f(g_{(4)}, h_{(2)})}\#$$
$$(g_{(3)} \cdot h_{(1)})u(a_{(2)}, g_{(2)})$$

$$\overset{(BE7)}{=}\ b_{(1)}(g_{(1)} \triangleright a_{(1)})p(a_{(3)}, b_{(2)})v(h_{(4)}, b_{(3)})\tau(h_{(3)}, g_{(5)})f(\underline{g_{(3)}}, h_{(1)})\#$$
$$(g_{(4)} \cdot h_{(2)})\underline{u(a_{(2)}, g_{(2)})}$$

$$\overset{(22)}{=}\ b_{(1)}(g_{(1)} \triangleright a_{(1)})p(a_{(4)}, b_{(2)})v(h_{(4)}, b_{(3)})\tau(h_{(3)}, g_{(5)})f(g_{(2)} \triangleleft a_{(2)}, h_{(1)})\#$$
$$(\underline{g_{(4)}} \cdot h_{(2)})\underline{u(a_{(3)}, g_{(3)})}$$

$$\overset{(22)}{=}\ b_{(1)}(g_{(1)} \triangleright a_{(1)})p(a_{(5)}, b_{(2)})v(h_{(4)}, b_{(3)})\tau(h_{(3)}, g_{(5)})f(g_{(2)} \triangleleft a_{(2)}, h_{(1)})\#$$
$$((g_{(3)} \triangleleft a_{(3)}) \cdot h_{(2)})u(a_{(4)}, g_{(4)})$$

$$=\ RHS.$$

In the forth equality, we used the fact that p is a coquasitriangular structure. Thus, (BR5) holds for σ and this ends the proof. $\qquad\square$

The following result that characterizes the coquasitriangular structures on a double cross product can be obtained from Theorem 3.6 by considering $f = \varepsilon \otimes \varepsilon$ to be the trivial cocycle.

COROLLARY 3.7. *Let $A \bowtie H$ be a double cross product of Hopf algebras. There is a bijective correspondence between:*

(i) The set of all coquasitriangular structures σ on the double cross product $A \bowtie H$.

(ii) The set of all quadruples (p, τ, u, v), where $p : A \otimes A \to k$, $\tau : H \otimes H \to k$, $u : A \otimes H \to k$ and $v : H \otimes A \to k$ are linear maps such that (A, p) and (H, τ) are coquasitriangular Hopf algebras, u and v are skew pairings on (A, H), respectively, on (H, A) and the following compatibilities are fulfilled:

$$v(h_{(1)}, b_{(1)})(h_{(2)} \triangleright b_{(2)}) \otimes (h_{(3)} \triangleleft b_{(3)}) = b_{(1)} \otimes h_{(1)}v(h_{(2)}, b_{(2)}),$$
$$(g_{(1)} \triangleright a_{(1)}) \otimes (g_{(2)} \triangleleft a_{(2)})u(a_{(3)}, g_{(3)}) = u(a_{(1)}, g_{(1)})a_{(2)} \otimes g_{(2)},$$
$$u(a_{(1)}, g_{(2)} \triangleleft c_{(2)})p(a_{(2)}, g_{(1)} \triangleright c_{(1)}) = p(a_{(1)}, c)u(a_{(2)}, g),$$
$$\tau(h_{(1)}, g_{(2)} \triangleleft c_{(2)})v(h_{(2)}, g_{(1)} \triangleright c_{(1)}) = v(h_{(1)}, c)\tau(h_{(2)}, g),$$
$$p(h_{(1)} \triangleright b_{(1)}, c_{(1)})v(h_{(2)} \triangleleft b_{(2)}, c_{(2)}) = v(h, c_{(1)})p(b, c_{(2)}),$$
$$u(h_{(1)} \triangleright b_{(1)}, t_{(1)})\tau(h_{(2)} \triangleleft b_{(2)}, t_{(2)}) = \tau(h, t_{(1)})u(b, t_{(2)}).$$

Under the above bijection, the coquasitriangular structure $\sigma : (A \bowtie H) \otimes (A \bowtie H) \to k$ corresponding to (p, τ, u, v) is given by

$$\sigma(a \bowtie h, b \bowtie g) = u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})v(h_{(2)}, b_{(2)}), \qquad (23)$$

for all $a, b, c \in A$ and $h, g, t \in H$.

4. Applications: Coquasitriangular structures on generalized quantum doubles. Let A and H be two Hopf algebras and $\lambda : H \otimes A \to k$ be a skew pairing. Then (A, H) is

a matched pair of Hopf algebras with the following two actions:

$$h \triangleleft a = h_{(2)}\lambda^{-1}(h_{(1)}, a_{(1)})\lambda(h_{(3)}, a_{(2)}),$$
$$h \triangleright a = a_{(2)}\lambda^{-1}(h_{(1)}, a_{(1)})\lambda(h_{(2)}, a_{(3)}).$$

The corresponding double cross product is called the *generalized quantum double* and it will be denoted by $A \bowtie_\lambda H$ ([**13**, Example 7.2.6]). As a special case of Corollary 3.7, we get

THEOREM 4.1. *Let A and H be two Hopf algebras and $\lambda : H \otimes A \to k$ be a skew pairing. There is a bijective correspondence between:*

(i) The set of all coquasitriangular structures σ on the generalized quantum double $A \bowtie_\lambda H$.

(ii) The set of all quadruples (p, τ, u, v), where $p : A \otimes A \to k$, $\tau : H \otimes H \to k$, $u : A \otimes H \to k$ and $v : H \otimes A \to k$ are linear maps such that (A, p) and (H, τ) are coquasitriangular Hopf algebras, u and v are skew pairings on (A, H), respectively, on (H, A) and the following compatibilities are fulfilled:

$$v(h_{(1)}, b_{(1)})b_{(3)} \otimes \lambda^{-1}(h_{(2)}, b_{(2)})\lambda(h_{(4)}, b_{(4)})h_{(3)} = b_{(1)} \otimes h_{(1)}v(h_{(2)}, b_{(2)}), \quad (24)$$

$$a_{(2)}\lambda^{-1}(g_{(1)}, a_{(1)})\lambda(g_{(3)}, a_{(3)}) \otimes g_{(2)}u(a_{(4)}, g_{(4)}) = u(a_{(1)}, g_{(1)})a_{(2)} \otimes g_{(2)}, \quad (25)$$

$$u(a_{(1)}, g_{(2)})\lambda(g_{(3)}, c_{(3)})p(a_{(2)}, c_{(2)})\lambda^{-1}(g_{(1)}, c_{(1)}) = p(a_{(1)}, c)u(a_{(2)}, g), \quad (26)$$

$$\tau(h_{(1)}, g_{(2)})\lambda(g_{(3)}, c_{(3)})v(h_{(2)}, c_{(2)})\lambda^{-1}(g_{(1)}, c_{(1)}) = v(h_{(1)}, c)\tau(h_{(2)}, g), \quad (27)$$

$$p(b_{(2)}, c_{(1)})\lambda^{-1}(h_{(1)}, b_{(1)})v(h_{(2)}, c_{(2)})\lambda(h_{(3)}, b_{(3)}) = p(b, c_{(2)})\lambda(h, c_{(1)}), \quad (28)$$

$$u(b_{(2)}, t_{(1)})\lambda^{-1}(h_{(1)}, b_{(1)})\tau(h_{(2)}, t_{(2)})\lambda(h_{(3)}, b_{(3)}) = \tau(h, t_{(1)})u(b, t_{(2)}). \quad (29)$$

Under this correspondence, the coquasitriangular structure $\sigma : (A \bowtie_\lambda H) \otimes (A \bowtie_\lambda H) \to k$ corresponding to (p, τ, u, v) is given by

$$\sigma(a \otimes h, b \otimes g) = u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})v(h_{(2)}, b_{(2)}), \quad (30)$$

for all $a, b, c \in A$ and $h, g, t \in H$.

THEOREM 4.2. *Let (A, p) and (H, τ) be two coquasitriangular Hopf algebras and $\lambda : H \otimes A \to k$ be a skew pairing. Then the generalized quantum double $A \bowtie_\lambda H$ is a coquasitriangular Hopf algebra with the coquasitriangular structure given by*

$$\sigma(a \bowtie h, b \bowtie g) = \lambda(S(g_{(1)}), a_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})\lambda(h_{(2)}, b_{(2)}). \quad (31)$$

Proof. We make use of Theorem 4.1: take $v := \lambda$ and $u := \lambda^{-1} \circ \nu$, where ν is the flip map. We need to prove that relations (24)–(29) are fulfilled. We have

$$LHS(24) = b_{(3)} \otimes \underline{\lambda(h_{(1)}, b_{(1)})\lambda(S(h_{(2)}), b_{(2)})}\lambda(h_{(4)}, b_{(4)})h_{(3)}$$
$$= b_{(1)} \otimes \underline{h_{(1)}\lambda(h_{(2)}, b_{(2)})} = RHS(24),$$

$$LHS(25) = a_{(2)}\lambda(S(g_{(1)}), a_{(1)})\underline{\lambda(g_{(3)}, a_{(3)})\lambda(S(g_{(4)}), a_{(4)})} \otimes g_{(2)}$$
$$= a_{(2)}\lambda(S(g_{(1)}), a_{(1)}) \otimes g_{(2)} = RHS(25),$$

$$LHS(26) = \lambda(S(g_{(2)}), a_{(1)})\lambda(g_{(3)}, c_{(3)})p(a_{(2)}, c_{(2)})\lambda(S(g_{(1)}), c_{(1)})$$
$$= \lambda(S(g_{(1)}), c_{(1)}a_{(1)})p(a_{(2)}, c_{(2)})\lambda(g_{(2)}, c_{(3)})$$
$$= \lambda(S(g_{(1)}), a_{(2)}c_{(2)})p(a_{(1)}, c_{(1)})\lambda(g_{(2)}, c_{(3)})$$
$$= \lambda(S(g_{(2)}), c_{(2)})\lambda(S(g_{(1)}), a_{(2)})p(a_{(1)}, c_{(1)})\lambda(g_{(3)}, c_{(3)})$$
$$= \lambda(S(g_{(2)})g_{(3)}, c_{(2)})\lambda(S(g_{(1)}), a_{(2)})p(a_{(1)}, c_{(1)})$$
$$= \lambda(S(g), a_{(2)})p(a_{(1)}, c) = RHS(26),$$

$$LHS(27) = \tau(h_{(1)}, g_{(2)})\lambda(g_{(3)}, c_{(3)})\lambda(h_{(2)}, c_{(2)})\lambda(S(g_{(1)}), c_{(1)})$$
$$= \tau(h_{(1)}, g_{(2)})\lambda(h_{(2)}g_{(3)}, c_{(2)})\lambda(S(g_{(1)}), c_{(1)})$$
$$= \tau(h_{(2)}, g_{(3)})\lambda(g_{(2)}h_{(1)}, c_{(2)})\lambda(S(g_{(1)}), c_{(1)})$$
$$= \tau(h_{(2)}, g_{(3)})\lambda(g_{(2)}, c_{(2)})\lambda(h_{(1)}, c_{(3)})\lambda(S(g_{(1)}), c_{(1)})$$
$$= \tau(h_{(2)}, g)\lambda(h_{(1)}, c) = RHS(28),$$

$$LHS(28) = p(b_{(2)}, c_{(1)})\lambda(S(h_{(1)}), b_{(1)})\lambda(h_{(2)}, c_{(2)})\lambda(h_{(3)}, b_{(3)})$$
$$= p(b_{(2)}, c_{(1)})\lambda(S(h_{(1)}), b_{(1)})\lambda(h_{(2)}, b_{(3)}c_{(2)})$$
$$= p(b_{(3)}, c_{(2)})\lambda(S(h_{(1)}), b_{(1)})\lambda(h_{(2)}, c_{(1)}b_{(2)})$$
$$= p(b_{(3)}, c_{(2)})\lambda(S(h_{(1)}), b_{(1)})\lambda(h_{(2)}, b_{(2)})\lambda(h_{(3)}, c_{(1)})$$
$$= p(b, c_{(2)})\lambda(h, c_{(1)}) = RHS(28),$$

$$LHS(29) = \lambda(S(t_{(1)}), b_{(2)})\lambda(S(h_{(1)}), b_{(1)})\tau(h_{(2)}, t_{(2)})\lambda(h_{(3)}, b_{(3)})$$
$$= \lambda(S(t_{(1)}h_{(1)}), b_{(1)})\tau(h_{(2)}, t_{(2)})\lambda(h_{(3)}, b_{(2)})$$
$$= \lambda(S(t_{(2)})S(h_{(2)}), b_{(1)})\tau(h_{(1)}, t_{(1)})\lambda(h_{(3)}, b_{(2)})$$
$$= \lambda(S(t_{(2)}), b_{(1)})\lambda(S(h_{(2)}), b_{(2)})\tau(h_{(1)}, t_{(1)})\lambda(h_{(3)}, b_{(3)})$$
$$= \tau(h, t_{(1)})\lambda(S(t_{(2)}), b) = RHS(29).$$

$\square$

As a consequence, we derive the necessary and sufficient conditions for the generalized quantum double to be a coquasitriangular Hopf algebra.

COROLLARY 4.3. *Let A and H be two Hopf algebras and $\tau : H \otimes A \to k$ be a skew pairing. Then the generalized quantum double $A \bowtie_\tau H$ is a coquasitriangular Hopf algebra if and only if both Hopf algebras A and H are coquasitriangular.*

Also, as a special case of Theorem 4.2, we recover Majid's result [**13**, Proposition 7.3.1].

COROLLARY 4.4. *Let (A, p) be a coquasitriangular Hopf algebra. Then the generalized quantum double $A \bowtie_p A$ has a coquasitriangular structure given by*

$$\sigma(a \otimes b, c \otimes d) = p(S(d_{(1)}), a_{(1)})p(a_{(2)}, c_{(1)})p(b_{(1)}, d_{(2)})p(b_{(2)}, c_{(2)}).$$

Proof. Consider $A = H$ and $\sigma = \tau := p$ in Theorem 4.2. $\square$

EXAMPLE 4.5. (1) Consider the group algebra $k\mathbb{Z}$ with the obvious Hopf algebra structure and let g be a generator of $\mathbb{Z}$ in multiplicative notation. We have a coquasitriangular structure $p : k\mathbb{Z} \otimes k\mathbb{Z} \to k$ given by: $p(g^t, g^l) = q^{tl}$.

Now consider the polynomial algebra $k[X]$ with the coalgebra structure and the antipode given by

$$\Delta(X^n) = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k}, \quad \varepsilon(X^n) = 0, \quad S(X^n) = (-1)^n X^n, \quad \text{for all} \quad n > 0.$$

Any element, $\alpha \in k$, induces a coquasitriangular structure τ on $k[X]$ as follows:

$$\tau(X^i, X^j) = \begin{cases} 0, & \text{if } i \neq j, \\ i! \alpha^i, & \text{if } i = j. \end{cases}$$

Moreover, there is a skew pairing λ between the two Hopf algebras $k[X]$ and $k\mathbb{Z}$ given by

$$\lambda(X^m, g^t) = t^m,$$

with the convention that $t^0 = 1$ even if $t = 0$. Thus, by applying Theorem 4.2, we obtain a coquasitriangular structure σ on the generalized quantum double $k\mathbb{Z} \bowtie_\lambda k[X]$:

$$\sigma(g^t \otimes X^n, g^l \otimes X^m) = \sum_{\substack{r+k=m \\ k \in \overline{0,m}, r \in \overline{0,n}}} (-1)^k \binom{m}{k} \binom{n}{r} t^k q^{tl} l^{m-r} r! \alpha^r.$$

(2) Let k be a field with $\text{char} k \neq 2$ and H_4 be Sweedler's Hopf algebra. That is, H_4 is generated as an algebra by elements g and x subject to relations:

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx.$$

The coalgebra structure and the antipode are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$
$$S(g) = g, \quad S(x) = gx.$$

For any $\alpha \in k$, the map $p_\alpha : H_4 \otimes H_4 \to k$ is a coquasitriangular structure on H_4, where p_α is defined as

p_α	1	g	x	gx
1	1	1	0	0
g	1	-1	0	0
x	0	0	α	α
gx	0	0	α	α

Let $\alpha, \beta, \gamma \in k$ and consider $p_\alpha, p_\beta, p_\gamma$ the corresponding coquasitriangular structures on H_4. Since any coquasitriangular structure is in particular a skew pairing, we can construct the generalized quantum double $H_4 \bowtie_{p_\gamma} H_4$. In view of Theorem 4.2, there is a coquasitriangular structure on $H_4 \bowtie_{p_\gamma} H_4$ given by:

$$\sigma(a \otimes h, b \otimes g) = p_\gamma(S(g_{(1)}), a_{(1)}) p_\alpha(a_{(2)}, b_{(1)}) p_\beta(h_{(1)}, g_{(2)}) p_\gamma(h_{(2)}, b_{(2)}).$$

(3) Let k be a field with $\mathrm{char}(k) \neq 2$ and consider the k-algebra $\widetilde{U(n)}$ defined by generators $\{c, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and relations:

$$c^2 = 1, \quad x_i^2 = y_i^2 = 0, \quad cx_i + x_i c = 0, \quad cy_i + y_i c = 0,$$
$$x_i x_j + x_j x_i = 0, \quad y_i y_j + y_j y_i = 0, \quad x_i y_j = y_j x_i, \quad 1 \leq i \leq n.$$

$\widetilde{U(n)}$ has a Hopf algebra structure given by

$$\Delta(c) = c \otimes c, \quad \Delta(x_i) = 1 \otimes x_i + x_i \otimes c, \quad \Delta(y_i) = c \otimes y_i + y_i \otimes 1,$$
$$\varepsilon(c) = 1, \quad \varepsilon(x_i) = \varepsilon(y_i) = 0, \quad S(c) = c, \quad S(x_i) = cx_i, \quad S(y_i) = y_i c, \quad 1 \leq i \leq n.$$

$\widetilde{U(n)}$ is a quotient of the Hopf algebra $U(n)$ introduced by Takeuchi in [14]. Now consider $\widetilde{B_-}$ and $\widetilde{B_+}$ to be the Hopf subalgebras of $\widetilde{U(n)}$ generated by $\{c, y_1, \ldots, y_n\}$, respectively, $\{c, x_1, \ldots, x_n\}$. These are the so-called *Borel subalgebras*. $\widetilde{B_-}$ and $\widetilde{B_+}$ are coquasitriangular Hopf algebras with

$$\tau : \widetilde{B_-} \otimes \widetilde{B_-} \to k, \quad \tau(c, c) = -1, \quad \tau(c, y_i) = \tau(y_j, c) = 0,$$
$$\tau(y_i, y_j) = \alpha_{ij}, \quad 1 \leq i, j \leq n,$$

$$p : \widetilde{B_+} \otimes \widetilde{B_+} \to k, \quad p(c, c) = -1, \quad p(c, x_i) = p(x_j, c) = 0,$$
$$p(x_i, x_j) = \beta_{ij}, \quad 1 \leq i, j \leq n.$$

Moreover, there is a skew pairing $\lambda : \widetilde{B_-} \otimes \widetilde{B_+} \to k$ given by

$$\lambda(c, c) = -1, \quad \lambda(c, x_j) = \lambda(y_i, c) = 0, \quad \lambda(y_i, x_j) = \delta_{ij}, \quad 1 \leq i, j \leq n,$$

where $\delta_{i,j}$ is the Kronecker delta. Therefore, using Theorem 4.2, the generalized quantum double $\widetilde{B_+} \bowtie_\tau \widetilde{B_-}$ is a coquasitriangular Hopf algebra with the coquasitriangular structure $\sigma : (\widetilde{B_+} \bowtie_\tau \widetilde{B_-}) \otimes (\widetilde{B_+} \bowtie_\tau \widetilde{B_-}) \to k$ given by

σ	$c \bowtie c$	$c \bowtie y_i$	$x_j \bowtie c$	$x_k \bowtie y_l$
$c \bowtie c$	1	0	0	0
$c \bowtie y_s$	0	α_{si}	δ_{sj}	0
$x_m \bowtie c$	0	$-\delta_{mj}$	β_{mj}	0
$x_n \bowtie y_r$	0	0	0	$\alpha_{rl}\beta_{nk} - \delta_{rk}\delta_{ln}$

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