Can. J. Math. Vol. **45** (6), 1993 pp. 1231–1244

A_∞^+ CONDITION

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ABSTRACT The good weights for the one-sided Hardy-Littlewood operators have been characterized by conditions A_p^+ (A_p^-) In this paper we introduce a new condition A_∞^+ which is analogous to A_∞ We show several characterizations of A_∞^+ For example, we prove that the class of A_∞^+ weights is the union of A_p^+ classes We also give a new characterization of A_p^+ weights Finally, as an application of A_∞^+ condition, we characterize the weights for one-sided fractional integrals and one-sided fractional maximal operators

1. **Introduction.** For f and g locally integrable functions and g positive on the real line, we define the one-sided Hardy-Littlewood maximal functions M_g^+f and M_g^-f at x by

$$M_g^+f(x) = \sup_{h>0} \frac{\int_x^{x+h} |f|g}{\int_x^{x+h} g}, \quad M_g^-f(x) = \sup_{h>0} \frac{\int_{x-h}^x |f|g}{\int_{x-h}^x g}.$$

Recently ([S], [M], [O1], [MOT]), weighted inequalities for these operators have been studied. In particular, the following characterization has been proved.

THEOREM ([S], [M], [O1], [MOT]). *Let g and w be positive, locally integrable functions on the real line. Let $1 < p < \infty$ and let p' be such that $p + p' = pp'$. Then the following are equivalent.*

(a) There exists a constant $K > 0$ such that for all $\lambda > 0$ and every $f \in L^p(w)$

$$\int_{\{x\, M_g^+f(x)>\lambda\}} w \le \frac{K}{\lambda^p} \int_{-\infty}^{\infty} |f|^p w.$$

(b) w satisfies $A_p^+(g)$ ($w \in A_p^+(g)$), i.e., there exists a constant $K > 0$ such that for all numbers $a < b < c$

$$\int_a^b w\Big(\int_b^c \Big(\frac{g}{w}\Big)^{p'-1} g\Big)^{p-1} \le K\Big(\int_a^c g\Big)^p.$$

(c) There exists a constant $K > 0$ such that for every $f \in L^p(w)$

$$\int_{-\infty}^{\infty} |M_g^+f|^p w \le K \int_{-\infty}^{\infty} |f|^p w.$$

This research has been partially supported by D G I C Y T grant (PB88-0324) and Junta de Andalucía

Received by the editors April 1, 1992

AMS subject classification 42B25

Key words and phrases one-sided Hardy-Littlewood maximal operators, weighted inequalities, A_∞^+ weights, one-sided fractional operators, one-sided fractional maximal operators, Riemann-Liouville fractional integral operator

If $p = 1$ then (a) is equivalent to saying that w satisfies $A_1^+(g)$, i.e., there exists a constant $K > 0$ such that $M_g^-(wg^{-1}) \leq Kwg^{-1}$.

The analogous theorem holds for $M_g^- f$ and the corresponding $A_p^-(g)$ classes. We say that w satisfies $A_p^-(g)$ $(1 < p < \infty)$ if there exists a constant $K > 0$ such that for all numbers $a < b < c$

$$\int_b^c w\left(\int_a^b \left(\frac{g}{w}\right)^{p'-1} g\right)^{p-1} \leq K\left(\int_a^c g\right)^p.$$

A weight w satisfies $A_1^-(g)$ if there exists a constant $K > 0$ such that $M_g^+(wg^{-1}) \leq Kwg^{-1}$.

The first aim of this note is to introduce some $A_\infty^+(g)$ $(A_\infty^-(g))$ condition similar to the $A_\infty(g)$ condition (for $g = 1$ see [CF], [GR] and their references). In particular, we obtain that $A_\infty^+(g)$ is the union of $A_p^+(g)$ classes and the equivalence with the weak reverse Hölder inequality (see (f) in Theorem 1) which was the key step in the proof of $A_p^+(g) \Rightarrow A_{p-\epsilon}^+(g)$ in [M] and [O1]. In addition, we show that $w \in A_\infty^+(g)$ if and only if $g \in A_\infty^-(w)$. Then, in Section 3 we apply $A_\infty^+(g)$ to obtain a new characterization of $A_p^+(g)$ weights. Finally, Section 4 is dedicated to the characterization of the good weights for one-sided fractional integrals and one-sided fractional maximal operators. The results for the fractional integral are consequences of those for the fractional maximal operator and a distribution function weighted inequality (Lemma 7) in which the $A_\infty^+(g)$ weights play an important role.

Before starting with the definition of $A_\infty^+(g)$ $(A_\infty^-(g))$ let us fix some notation. From now on, $h(E)$ stands for $\int_E h$ for a positive, locally integrable function h and a measurable set E. If E is an interval (a, b) then we will simply write $h(a, b)$. The letter K will mean a positive finite constant not necessarily the same at each occurrence, w and g will denote positive, locally integrable functions and if $1 < p < \infty$ then p' will be the number such that $p + p' = pp'$. Finally, for a locally integrable function f, we define

$$M_g f(x) = \sup_{s,h>0} \frac{\int_{x-s}^{x+h} |f| g}{\int_{x-s}^{x+h} g}.$$

2. $A_\infty^+(g)$ **condition.** In order to define $A_\infty^+(g)$, it is convenient to know that the restricted weak type (p,p) inequality for $M_g^+ f$ was characterized in the following way.

THEOREM ([O1]). *Let $1 < p < \infty$. The following are equivalent.*

(a) There exists $K > 0$ such that for all $\lambda > 0$ and every measurable set E

$$\int_{\{x: M_g^+ \chi_E(x) > \lambda\}} w \leq \frac{K}{\lambda^p} \int_E w.$$

(b) There exists $K > 0$ such that for for all numbers $a < b < c$ and all sets $E \subset (b, c)$

$$\frac{g(E)}{g(a,c)} \leq K\left(\frac{w(E)}{w(a,b)}\right)^{\frac{1}{p}}.$$

Keeping in mind this result, Proposition 1 in [KT] and the definition of A_∞ (*cf.* [GR] for instance), we define $A_\infty^+(g)$ and $A_\infty^-(g)$.

DEFINITION 1. We say that w is in $A_\infty^+(g)$ if there exist positive numbers K and δ such that for all numbers $a < b < c$ and all measurable sets $E \subset (b, c)$

$$\frac{g(E)}{g(a,c)} \leq K\left(\frac{w(E)}{w(a,b)}\right)^{\delta}.$$

DEFINITION 2. We say that w is in $A_\infty^-(g)$ if there exist positive numbers K and δ such that for all numbers $a < b < c$ and all measurable sets $E \subset (a, b)$

$$\frac{g(E)}{g(a,c)} \leq K\left(\frac{w(E)}{w(b,c)}\right)^{\delta}.$$

THEOREM 1. *The following are equivalent.*

(a) $w \in A_\infty^+(g)$.

(b) There exists p such that $w \in A_p^+(g)$.

(c) For every α, $0 < \alpha < 1$, *there exists* $\beta > 0$ *such that, for all numbers* $a < b < c$ *and every* $E \subset (b, c)$ *with* $\frac{w(E)}{w(a,b)} < \beta$, *we have* $\frac{g(E)}{g(a,c)} < \alpha$.

(d) For every α, $0 < \alpha < 1$, *there exists* $\beta > 0$ *such that the following implication holds: given* $\lambda > 0$ *and an interval* (a, b) *such that* $\lambda \leq \frac{w(a,x)}{g(a,x)}$ *for all* $x \in (a, b)$, *then*

$$g\left(\left\{x \in (a,b) : \frac{w(x)}{g(x)} > \beta\lambda\right\}\right) > \alpha g(a,b).$$

(e) For every α, $0 < \alpha < 1$, *there exists* $\beta > 0$ *such that the following implication holds: given* $\lambda > 0$ *and an interval* (a, b) *such that* $\frac{w(a,b)}{g(a,b)} = \lambda \leq \frac{w(a,x)}{g(a,x)}$ *for all* $x \in (a, b)$, *then*

$$g\left(\left\{x \in (a,b) : \frac{w(x)}{g(x)} > \beta\lambda\right\}\right) > \alpha g(a,b).$$

(f) Weak reverse Hölder's inequality.

There exist positive numbers δ *and* K *such that for all numbers* $a < b$

$$\int_a^b \left(\frac{w}{g}\right)^{\delta} w \leq K \int_a^b w\left(M_g\left(\frac{w}{g}\chi_{(a,b)}\right)(b)\right)^{\delta}.$$

(g) There exist positive numbers δ *and* K *such that for all numbers* $a < b$

$$M_w\left(\left(\frac{w}{g}\right)^{\delta}\chi_{(a,b)}\right)(b) \leq K\left(M_g\left(\frac{w}{g}\chi_{(a,b)}\right)(b)\right)^{\delta}.$$

(h) There exists p such that $g \in A_p^-(w)$.

(i) $g \in A_\infty^-(w)$.

(j) There exist γ, $0 < \gamma \leq \frac{1}{2}$, *and* $K > 0$ *such that*

$$\frac{w(a,b)}{g(a,b)} \exp\left(\frac{1}{g(c,d)}\int_c^d g \log\frac{g}{w}\right) \leq K$$

for all numbers $a < b \le c < d$ *such that* $g(a,b) = g(c,d) = \gamma g(a,d)$

PROOF OF THEOREM 1 (b) $\Rightarrow$ (a) We may assume $p > 1$ (if $w \in A_1^+(g)$ then $w \in A_p^+(g)$ for every $p > 1$) Let $a < b < c$ and let E be any measurable set $E \subset (b,c)$ By Holder's inequality we have

$$\big(g(E)\big)^p \le w(E)\bigg(\int_E \Big(\frac{g}{w}\Big)^{p' \ 1} g\bigg)^{p \ 1}$$

Since $E \subset (b,c)$ and w satisfies $A_p^+(g)$, we get from the last inequality

$$\big(g(E)\big)^p \le K\big(g(a,c)\big)^p \frac{w(E)}{w(a,b)},$$

what is $A_\infty^+(g)$ with $\delta = \frac{1}{p}$

(a) $\Rightarrow$ (c) is obvious

(c) $\Rightarrow$ (d) Let $a < b$ and let $\lambda > 0$ such that $\lambda \le \frac{w(a,x)}{g(a,x)}$ for all $x \in (a,b)$ Let $x_0 = b$ and for k, a negative integer, let x_k, $a < x_k < x_{k+1}$, be such that

$$\int_{x_k}^{x_{k+1}} w = \int_a^{x_k} w$$

Let $E' = \Big\{x \in (a,b) \quad \frac{w(x)}{g(x)} \le \beta\lambda\Big\}$, $E'_k = E' \cap [x_k, x_{k+1})$ and $I_k = [x_{k \ 1}, x_k)$ From our assumption and the definition of the sequence x_k we have

$$\lambda \le \frac{w(a, x_{k+1})}{g(a, x_{k+1})} = 4\frac{w(I_k)}{g(a, x_{k+1})},$$

and by the definition of E'_k

$$\frac{w(E'_k)}{w(I_k)} \le \frac{\beta\lambda g(E'_k)}{w(I_k)} \le 4\beta\frac{g(E'_k)}{w(I_k)}\frac{w(I_k)}{g(a, x_{k+1})} < 4\beta$$

Then, taking β small enough, we get from (c) that

$$\frac{g(E'_k)}{g(x_{k \ 1}, x_{k+1})} \le \gamma \quad \text{for some } \gamma \in \Big(0, \frac{1}{2}\Big)$$

Hence

$$\begin{aligned} g\bigg(\Big\{x \in (a,b) \quad \frac{w(x)}{g(x)} > \beta\lambda\Big\}\bigg) &= g(a,b) - \sum_{k- \ 1}^{\infty} g(E'_k) \\ &\ge g(a,b) - \gamma \sum_{k \ 1}^{\infty} g(x_{k \ 1}, x_{k+1}) \\ &\ge (1 - 2\gamma) g(a,b) \end{aligned}$$

(d) $\Rightarrow$ (e) is obvious

(e) $\Rightarrow$ (f) Let $a < b$ Put $\lambda_0 = M_g(\frac{w}{g}\chi_{(a\ b)})(b)$ and

$$O(\lambda) = \Big\{x \quad M_g\Big(\frac{w}{g}\chi_{(a\ b)}\Big)(x) > \lambda\Big\}$$

for $\lambda > \lambda_0$. Then $O(\lambda) = \cup_j (a_j, b_j)$ such that $(a_j, b_j) \subset (a, b)$, the intervals (a_j, b_j) are pairwise disjoint and

$$\lambda = \frac{w(a_j, b_j)}{g(a_j, b_j)} \leq \frac{w(a_j, x)}{g(a_j, x)} \text{ for all } x \in (a_j, b_j).$$

By (e), we have for some positive numbers β and α,

$$\begin{aligned} w\Big(\Big\{x \in (a,b) : \frac{w(x)}{g(x)} > \lambda\Big\}\Big) &\leq \sum_j w(a_j, b_j) = \lambda \sum_j g(a_j, b_j) \\ &\leq \frac{\lambda}{\alpha} \sum_j g\Big(\Big\{x \in (a_j, b_j) : \frac{w(x)}{g(x)} > \beta\lambda\Big\}\Big) \\ &\leq \frac{\lambda}{\alpha} g\Big(\Big\{x \in (a,b) : \frac{w(x)}{g(x)} > \beta\lambda\Big\}\Big). \end{aligned}$$

Multiplying by $\lambda^{\delta-1}$, integrating over (λ_0, ∞) and applying Tonnelli's theorem we get

$$\frac{1}{\delta} \int_{\{x \in (a,b)\ \frac{w}{g}(x) > \lambda_0\}} w\Big(\Big(\frac{w}{g}\Big)^\delta - \lambda_0^\delta\Big) \leq \frac{1}{(1+\delta)\alpha\beta^{1+\delta}} \int_a^b \Big(\frac{w}{g}\Big)^\delta w.$$

This inequality implies easily that

$$\frac{1}{\delta} \int_a^b \Big(\frac{w}{g}\Big)^\delta w - \frac{\lambda_0^\delta}{\delta} \int_a^b w \leq \frac{1}{(1+\delta)\alpha\beta^{1+\delta}} \int_a^b \Big(\frac{w}{g}\Big)^\delta w,$$

or

$$\Big(\frac{1}{\delta} - \frac{1}{(1+\delta)\alpha\beta^{1+\delta}}\Big) \int_a^b \Big(\frac{w}{g}\Big)^\delta w \leq \frac{1}{\delta} \int_a^b w\Big(M_g\Big(\frac{w}{g}\chi_{(a,b)}\Big)(b)\Big)^\delta,$$

which, for δ small enough, gives (f). Details can be found in [CF] or [M] and thus are omitted.

(f) $\Rightarrow$ (g). The statement (g) is a direct consequence of (f).

(g) $\Rightarrow$ (h). Let $a < b < c$. From the definition of M_w and (g) it follows that we have for all $x \in (b, c)$

$$\begin{aligned} \Big(\frac{1}{w(a,c)} \int_a^b \Big(\frac{w}{g}\Big)^\delta w\Big)^{\frac{1}{\delta}} &\leq \Big(M_w\Big(\Big(\frac{w}{g}\Big)^\delta \chi_{(a,x)}\Big)\Big)^{\frac{1}{\delta}}(x) \\ &\leq K M_g\Big(\frac{w}{g}\chi_{(a,x)}\Big)(x) \\ &\leq K M_g\Big(\frac{w}{g}\chi_{(a,c)}\Big)(x). \end{aligned}$$

Now the fact that M_g is of weak type $(1,1)$ with respect to $g(x)\,dx$ gives

$$g(b,c) \leq K\big(w(a,c)\big)^{\frac{1}{\delta}} \Big(\int_a^b \Big(\frac{w}{g}\Big)^\delta w\Big)^{-\frac{1}{\delta}} w(a,c).$$

Thus $g \in A_p^-(w)$ with $p = \frac{1+\delta}{\delta}$ and therefore (h) holds.

The chain of implications that we have just proved shows that (b) $\Rightarrow$ (h) via (a), (c), (d), (e), (f) and (g). In a symmetric way it is proved that (h) $\Rightarrow$ (b) and therefore (a), (b), (c), (d), (e), (f), (g) and (h) are equivalent, and in addition we get that each of these statements is equivalent to (i). To finish the proof of the Theorem, we will prove (b) $\Rightarrow$ (j) and (j) $\Rightarrow$ (e).

(b) $\Rightarrow$ (j). We will prove that (j) holds with $b = c$ and $\gamma = \frac{1}{2}$.

Since $w \in A_p^+(g)$ we have

$$\frac{w(a,b)}{g(a,b)}\left(\frac{1}{g(b,d)}\int_b^d g\left(\frac{g}{w}\right)^{p'-1}\right)^{p-1} \leq K$$

for all numbers $a < b < d$ such that $g(a,b) = g(b,d)$. On the other hand, by Jensen's inequality,

$$\exp\left(\frac{1}{g(b,d)}\int_b^d g\log\frac{g}{w}\right) = \left(\exp\left(\frac{1}{g(b,d)}\int_b^d g\log\left(\frac{g}{w}\right)^{p'-1}\right)\right)^{p-1}$$
$$\leq \left(\frac{1}{g(b,d)}\int_b^d g\left(\frac{g}{w}\right)^{p'-1}\right)^{p-1}.$$

Putting both inequalities together we get (j).

(j) $\Rightarrow$ (e). The proof of this implication follows the idea from [GR] (see pp. 405–406).

Let α, λ and (a,b) be as in statement (e). Let $x_0 = b$ and for every negative integer k let x_k be such that $a < x_k < x_{k+1}$ and $g(x_k, x_{k+1}) = \gamma g(a, x_{k+1})$. For fixed k, let y_k be such that $g(a, y_k) = g(x_k, x_{k+1})$. Now we choose for every negative integer k the number α_k such that

$$\int_{x_k}^{x_{k+1}} g\log\frac{g}{\alpha_k w} = 0.$$

Applying (j) to the quadruple (a, y_k, x_k, x_{k+1}) we get

$$\frac{\alpha_k w(a,y_k)}{g(a,y_k)} \leq K \quad \text{for every } k \leq -1.$$

Therefore, by the properties of the points of the interval (a,b) we get

$$\alpha_k\lambda \leq \frac{\alpha_k w(a,y_k)}{g(a,y_k)} \leq K \quad \text{for every } k \leq -1.$$

https://doi.org/10.4153/CJM-1993-069-9 Published online by Cambridge University Press

The last inequality and the way of choosing α_k give for every $\beta > 0$

$$
\begin{aligned}
g\Big(\Big\{x \in (x_k, x_{k+1}) : \frac{w(x)}{g(x)} \le \beta\lambda\Big\}\Big)&\\
&= g\Big(\Big\{x \in (x_k, x_{k+1}) : \log\Big(1 + \frac{1}{\alpha_k \beta\lambda}\Big) \le \log\Big(1 + \frac{g(x)}{\alpha_k w(x)}\Big)\Big\}\Big)\\
&\le \frac{1}{\log(1 + \frac{1}{\alpha_k\beta\lambda})} \int_{x_k}^{x_{k+1}} g \log\Big(1 + \frac{g}{\alpha_k w}\Big)\\
&= \frac{1}{\log(1 + \frac{1}{\alpha_k\beta\lambda})} \int_{x_k}^{x_{k+1}} g \log\Big(1 + \frac{\alpha_k w}{g}\Big)\\
&\le \frac{1}{\log(1 + \frac{1}{\alpha_k\beta\lambda})} \int_{x_k}^{x_{k+1}} \alpha_k w\\
&\le \frac{K}{\lambda \log(1 + \frac{1}{K\beta})} \int_{x_k}^{x_{k+1}} w.
\end{aligned}
$$

Summing in k and keeping in mind that $\frac{w(a,b)}{g(a,b)} = \lambda$ we get

$$
\begin{aligned}
g\Big(\Big\{x \in (a,b) : \frac{w(x)}{g(x)} \le \beta\lambda\Big\}\Big) &\le \frac{K}{\lambda \log(1 + \frac{1}{K\beta})} w(a,b)\\
&= \frac{K}{\log(1 + \frac{1}{K\beta})} g(a,b).
\end{aligned}
$$

Hence, given $\alpha \in (0,1)$, we can take β small enough to obtain

$$
g\Big(\Big\{x \in (a,b) : \frac{w(x)}{g(x)} \le \beta\lambda\Big\}\Big) < (1-\alpha) g(a,b),
$$

and therefore (e) holds.

REMARKS. (1) Because of the symmetry between (a) and (i) or between (b) and (h) all the other statements can be written changing the roles of a and w by the corresponding ones of b and g. More equivalent conditions can be obtained keeping in mind that $w \in A_p^+(g)$ if and only if $(\frac{g}{w})^{p'-1} g \in A_{p'}^-(g)$.

(2) The implication (b) $\Rightarrow$ (j) can be obtained by letting p tend to ∞ in the inequality of the $A_p^+(g)$ condition (see [GR]). In this way we can consider $A_\infty^+(w)$ as the limit of $A_p^+(g)$.

(3) As we said in the introduction, the weak reverse Hölder inequality is the key step to prove that if $w \in A_p^+(g)$, $1 < p < \infty$, then there exists s, $1 < s < p$, such that $w \in A_s^+(g)$. See [M] for a proof in the case $g = 1$.

(4) The proof of (b) $\Rightarrow$ (j) shows that the number γ in (j) can be taken equal to $1/2$. This remark will be used in the proof of Theorem 2. Statement (j) with other values of γ is useful in a forthcoming paper about one-sided BMO spaces.

3. **A characterization of $A_p^+(g)$.** If w satisfies $A_p^+(g)$ then by Theorem 1 we know that $w \in A_\infty^+(g)$. Of course, we also have the corresponding result for $A_p^-(g)$ classes. On the other hand, $w \in A_p^+(g)$ if and only if $(\frac{g}{w})^{p'-1}g \in A_{p'}^-(g)$. Therefore, if w satisfies $A_p^+(g)$ then $w \in A_\infty^+(g)$ and $(\frac{g}{w})^{p'-1}g \in A_\infty^-(g)$. The question is if the converse is true as in Muckenhoupt's classes. The purpose of the next theorem is to give an affirmative answer to this question. It includes another characterization of $A_p^+(g)$.

THEOREM 2. *Let $1 < p < \infty$. The following are equivalent.*
(a) $w \in A_\infty^+(g)$ and $(\frac{g}{w})^{p'-1}g \in A_\infty^-(g)$.
(b) There exists a positive constant K such that for all numbers $a < b$

$$M_g\Big(\frac{w}{g}\chi_{(a,b)}\Big)(b) \le K\Big(M_\sigma\Big(\frac{g}{\sigma}\chi_{(a,b)}\Big)(b)\Big)^{p-1}$$

where $\sigma = (\frac{g}{w})^{p'-1}g$.
(c) $w \in A_p^+(g)$.

PROOF OF THEOREM 2. We only have to prove (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b). Let $a < b$. Let $x_0 = a$ and for a nonnegative integer k let $x_k < x_{k+1} < b$ such that

$$g(x_k, x_{k+1}) = \frac{1}{3}g(x_k, b).$$

It is clear that $(a, b) = \cup_{k=0}^\infty (x_k, x_{k+1}]$. For fixed k let y be the point such that

$$g(x_k, x_{k+1}) = g(y, b).$$

Therefore $g(x_k, x_{k+1}) = g(x_{k+1}, y) = g(y, b)$. Since $w \in A_\infty^+(g)$, statement (j) of Theorem 1 holds with $\gamma = 1/2$ (see the remark after the proof of Theorem 1). Thus

$$\frac{w(x_k, x_{k+1})}{g(x_k, x_{k+1})}\exp\Big(\frac{1}{g(x_{k+1}, y)}\int_{x_{k+1}}^y g\log\frac{g}{w}\Big) \le K.$$

Similarly, the version of (j) equivalent to $\sigma \in A_\infty(g)$ applied to (x_{k+1}, y, b) gives

$$\frac{\sigma(y, b)}{g(y, b)}\exp\Big(\frac{1}{g(x_{k+1}, y)}\int_{x_{k+1}}^y g\log\frac{g}{\sigma}\Big) \le K.$$

Raising this inequality to $p-1$ and multiplying the last two inequalities we get

$$\begin{aligned} w(x_k, x_{k+1}) &\le Kg(x_k, x_{k+1})\Big(\frac{g(y, b)}{\sigma(y, b)}\Big)^{p-1} \\ &\le Kg(x_k, x_{k+1})\Big(M_\sigma\Big(\frac{g}{\sigma}\chi_{(a,b)}\Big)(b)\Big)^{p-1}. \end{aligned}$$

Summing in k yields

$$\frac{w(a, b)}{g(a, b)} \le K\Big(M_\sigma\Big(\frac{g}{\sigma}\chi_{(a,b)}\Big)(b)\Big)^{p-1}.$$

Now, the statement (b) follows, as a was an arbitrary number less than b.

(b) $\Rightarrow$ (c). Let $a < b < c$. For every $x \in (b, c)$ we have

$$\frac{w(a,b)}{g(a,c)} \leq M_g\Big(\frac{w}{g}\chi_{(a,x)}\Big)(x).$$

This inequality and (b) give for all $x \in (b,c)$

$$\frac{w(a,b)}{g(a,c)} \leq K\Big(M_\sigma\Big(\frac{g}{\sigma}\chi_{(a,x)}\Big)(x)\Big)^{p-1} \leq K\Big(M_\sigma\Big(\frac{g}{\sigma}\chi_{(a,c)}\Big)(x)\Big)^{p-1}.$$

Then since M_σ is of weak type $(1,1)$ with respect to $\sigma(x)\,dx$, we obtain

$$\begin{aligned}\sigma(b,c) &\leq \sigma\left(\left\{x : M_\sigma\Big(\frac{g}{\sigma}\chi_{(a,c)}\Big)(x) > K\Big(\frac{w(a,b)}{g(a,c)}\Big)^{p'-1}\right\}\right)\\ &\leq K\Big(\frac{g(a,c)}{w(a,b)}\Big)^{p'-1} g(a,c)\end{aligned}$$

which means that (c) holds.

4. **Fractional integrals.** This section is devoted to the study of weighted inequalities for one-sided fractional integrals and one-sided fractional maximal operators. More precisely, we consider, for $0 < \alpha < 1$ and g as above, the following operators:

$$I^+_{\alpha,g}f(x) = \int_x^\infty \frac{f(y)g(y)}{\big(g(x,y)\big)^{1-\alpha}}\,dy \quad \text{and} \quad M^+_{\alpha,g}(f)(x) = \sup_{h>0}\frac{\int_x^{x+h}|f(y)|g(y)\,dy}{\big(g(x,x+h)\big)^{1-\alpha}}$$

The good weights for these operators ($g = 1$) were studied in [AS] and [MT] and the pairs of weights for $M^+_{\alpha,g}$ to be of strong type $(p,q), p \leq q$, can be found in [O2]. We shall characterize the weights w for which the operators $I^+_{\alpha,g}$ and $M^+_{\alpha,g}$ take the space $L^p(w^pg)$ either into $L^q(w^qg)$ or into the weak $L^q(w^qg)$, where $\alpha = p^{-1} - q^{-1}$. Our proofs are new even in the case $g = 1$.

Observe that $M^+_{\alpha,g}f \leq I^+_{\alpha,g}(|f|)$. We do not have an opposite pointwise inequality, but we can obtain the following integral inequalities where the $A^+_\infty(g)$ weights play a crucial role.

THEOREM 3. *If w satisfies $A^+_\infty(g)$, $0 < q < \infty$ and $0 < \alpha < 1$, then there exists K such that for every non negative function f*

$$\int_{-\infty}^\infty |I^+_{\alpha,g}f(x)|^q w \leq K\int_{-\infty}^\infty |M^+_{\alpha,g}f(x)|^q w$$

and

$$\sup_{\lambda>0}\lambda^q w\big(\{x : I^+_{\alpha,g}f(x) > \lambda\}\big) \leq K\sup_{\lambda>0}\lambda^q w\big(\{x : M^+_{\alpha,g}f(x) > \lambda\}\big).$$

This theorem reduces the study of weights for $I^+_{\alpha,g}$ to the corresponding ones for $M^+_{\alpha,g}$. For that reason, we will first study the weights for the fractional maximal operators.

THEOREM 4. *Let* $0 < \alpha < 1$, $1 \le p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. *Let w be a positive measurable function. The following are equivalent.*

(a) There exists K such that for every $\lambda > 0$ and all measurable functions f

$$\Big(w^q g\big(\{x : M^+_{\alpha,g}f(x) > \lambda\}\big)\Big)^{\frac{1}{q}} \le \frac{K}{\lambda}\Big(\int_{-\infty}^{\infty} |f|^p w^p g\Big)^{\frac{1}{p}}.$$

(b) The function $w^q g$ satisfies $A^+_r(g)$ where $r = 1 + \frac{q}{p'}$.

If $1 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$ then (a) and (b) are equivalent to

(c) There exists K such that for all measurable functions

$$\Big(\int_{-\infty}^{\infty} |M^+_{\alpha,g}f|^q w^q g\Big)^{\frac{1}{q}} \le K\Big(\int_{-\infty}^{\infty} |f|^p w^p g\Big)^{\frac{1}{p}}.$$

PROOF OF THEOREM 4. (b) follows from (a) in the usual way, *i.e.*, fixed $a < b < c$, we test the inequality by functions $f = w^{-p'}\chi_E$ where $E \subset (b, c)$ if $p > 1$ and by functions $f = \chi_E$ if $p = 1$. For the converse we will need the following lemma.

LEMMA 5. *Let α, p and q be as in Theorem 4. If the function $w^q g$ satisfies $A^+_r(g)$ where $r = 1 + \frac{q}{p'}$ then*

$$(M^+_{\alpha,g}f)^q \le K\|f\|^{q-p}_{L^p(w^p g)} M^+_{w^q g}(f^p w^{p-q})$$

for all measurable functions.

Assume that (b) holds. By Lemma 5,

$$w^q g\big(\{x : M^+_{\alpha,g}f(x) > \lambda\}\big) \le w^q g\big(\{x : M^+_{w^q g}(f^p w^{p-q})(x) > K\lambda^q \|f\|^{p-q}_{L^p(w^p g)}\}\big),$$

and because $M^+_{w^q g}$ is of weak type $(1, 1)$ with respect to the measure $w^q g$ we obtain

$$w^q g\big(\{x : M^+_{\alpha,g}f(x) > \lambda\}\big) \le \frac{K}{\lambda^q \|f\|^{p-q}_{L^p(w^p g)}} \|f\|^p_{L^p(w^p g)} = \frac{K}{\lambda^q}\|f\|^q_{L^p(w^p g)}$$

which is (a).

Now assume $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. It is clear that (c) $\Rightarrow$ (a). The implication (b) $\Rightarrow$ (c) is a consequence of (b) $\Rightarrow$ (a) and the fact that $w^q g \in A^+_r(g)$ implies $w^q g \in A^+_s(g)$ for some s, $1 < s < r$ (see Remarks after Theorem 1). Details can be found in [MW].

PROOF OF LEMMA 5. For x and h fixed, we choose a decreasing sequence $\{x_k\}$ such that

$$x_0 = x + h \text{ and } w^q g(x_{k+1}, x_k) = w^q g(x, x_{k+1}).$$

Observe that $w^q g(x_{k+2}, x_{k+1})$ is comparable to $w^q g(x, x_k)$. More precisely

$$w^q g(x, x_k) = 4w^q g(x_{k+2}, x_{k+1}).$$

Assume $p > 1$. Then, from the Hölder inequality and $w^q g \in A_r^+(g)$,

$$\int_x^{x+h} |f|g = \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} |f|g \leq \sum_{k=0}^{\infty} \Big(\int_{x_{k+1}}^{x_k} |f|^p w^p g \Big)^{\frac{1}{p}} \Big(\int_{x_{k+1}}^{x_k} w^{-p'} g \Big)^{\frac{1}{p'}}$$

$$\leq K \sum_{k=0}^{\infty} \frac{(\int_{x_{k+1}}^{x_k} |f|^p w^p g)^{\frac{1}{p}}}{(w^q g(x_{k+2}, x_{k+1}))^{\frac{1}{q}}} \big(g(x_{k+2}, x_k) \big)^{1-\alpha}$$

$$\leq K \sum_{k=0}^{\infty} \Big(\frac{\int_x^{x_k} |f|^p w^p g}{w^q g(x, x_k)} \Big)^{\frac{1}{q}} \Big(\int_{x_{k+1}}^{x_k} |f|^p w^p g \Big)^{\frac{1}{p} - \frac{1}{q}} \big(g(x_{k+2}, x_k) \big)^{1-\alpha}.$$

Now, from the definition of $M_{w^q g}^+$ and the Hölder inequality applied to the sum, we obtain

$$\int_x^{x+h} |f|g \leq K (M_{w^q g}^+ |f|^p w^{p-q})^{\frac{1}{q}}(x) \big(g(x, x+h) \big)^{1-\alpha} \|f\|_{L^p(w^p g)}^{\alpha p},$$

which proves the case $p > 1$ of the lemma taking into account the relation between α, p and q.

Now assume p=1. Then $w^q g \in A_1^+(g)$ and therefore

$$\int_x^{x+h} |f|g = \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} |f| g w w^{-1} \leq K \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} |f| w g \Big(\frac{g(x_{k+2}, x_k)}{w^q g(x_{k+2}, x_{k+1})} \Big)^{\frac{1}{q}}$$

$$\leq K \sum_{k=0}^{\infty} \Big(\frac{\int_x^{x_k} |f| g w}{w^q g(x, x_k)} \Big)^{\frac{1}{q}} \Big(\int_{x_{k+1}}^{x_k} |f| g w \Big)^{1 - \frac{1}{q}} \big(g(x_{k+2}, x_k) \big)^{1-\alpha}.$$

As before, this inequality proves the case $p = 1$ of the lemma.

Once we have studied weights for fractional maximal operators, we can state the results for fractional integrals.

THEOREM 6. *Let $0 < \alpha < 1$, $1 \leq p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Let w be a positive measurable function. The following are equivalent.*

(a) There exists K such that for every $\lambda > 0$ and all measurable functions f

$$\Big(w^q g \big(\{ x : I_{\alpha,g}^+ f(x) > \lambda \} \big) \Big)^{\frac{1}{q}} \leq \frac{K}{\lambda} \Big(\int_{-\infty}^{\infty} |f|^p w^p g \Big)^{\frac{1}{p}}.$$

(b) The function $w^q g$ satisfies $A_r^+(g)$ where $r = 1 + \frac{q}{p'}$.

If $1 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$ then (a) and (b) are equivalent to

(c) There exists K such that for all measurable functions

$$\Big(\int_{-\infty}^{\infty} |I_{\alpha,g}^+ f|^q w^q g \Big)^{\frac{1}{q}} \leq K \Big(\int_{-\infty}^{\infty} |f|^p w^p g \Big)^{\frac{1}{p}}.$$

This theorem follows from Theorems 3 and 4 (*cf.* also [MW]). Therefore we will only prove Theorem 3 . In fact (see [MW]), it is well known that the inequalities in Theorem 3 are consequences of distribution function inequalities. More precisely, Theorem 3 is a corollary of the following lemma.

https://doi.org/10.4153/CJM-1993-069-9 Published online by Cambridge University Press

LEMMA 7 *Let* $0 < \alpha < 1$ *and let* $w \in A^+_\infty(g)$ *Then there exist positive constants* K *and* δ *such that for every non negative measurable function* f *all* $\lambda > 0$ *and each* γ $0 < \gamma < 1$

$$w\big(\{x \; I^+_{\alpha g}f(x) > 2\lambda, M^+_{\alpha g}f(x) \le \gamma\lambda\}\big) \le K\gamma^{\frac{\delta}{1}} w\big(\{x \; I^+_{\alpha g}f(x) > \lambda\}\big)$$

PROOF OF LEMMA 7 We may assume without loss of generality that f is bounded with compact support Let $\{I_i\}$ be the connected components of $\{x \; I^+_{\alpha g}f(x) > \lambda\}$ Then it is enough to prove

$$w\big(\{x \in I_i \; I^+_{\alpha g}f(x) > 2\lambda, M^+_{\alpha g}f(x) \le \gamma\lambda\}\big) \le K\gamma^{\frac{\delta}{1 \alpha}} w(I_i)$$

Fix $I_i = (a, b)$ Let $\{x_k\}$ be the sequence defined by

$$x_0 = a \text{ and } g(x_k, x_{k+1}) = g(x_{k+1}, b)$$

Observe that

$$g(x_k, b) = 4g(x_{k+1}, x_{k+2})$$

We will prove that if

$$E_k = \{x \in (x_k, x_{k+1}) \; I^+_{\alpha g}f(x) > 2\lambda, M^+_{\alpha g}f(x) \le \gamma\lambda\}$$

then

$$g(E_k) \le K\gamma^{\frac{1}{1 \alpha}} g(x_{k+1}, x_{k+2}) \tag{4 1}$$

Keeping in mind this inequality and the fact that $w \in A^+_\infty(g)$ if and only if $g \in A_\infty(w)$ (see Theorem 1), we apply $A_\infty(w)$ to the weight g, the points x_k, x_{k+1}, x_{k+2}, and the set E_k Then we obtain for some $\delta > 0$

$$w(E_k) \le K\gamma^{\frac{\delta}{1 \alpha}} w(x_k, x_{k+2})$$

Summing over k we obtain the desired inequality

Now we will prove (4 1) Fix k and let $f_1 = f$ on (x_k, b) and 0 elsewhere, let $f_2 = f - f_1$ Assume that there is a $t \in (x_k, x_{k+1})$ such that $M^+_{\alpha g}f(t) \le \gamma\lambda$, otherwise (4 1) is obvious Let t_k be the infimum of such t's Let $x \in (x_k, x_{k+1})$ Then

$$I^+_{\alpha g}f_2(x) = \int_b^\infty \frac{f(y)g(y)}{\big(g(x,y)\big)^{1 \alpha}}\,dy \le \int_b^\infty \frac{f(y)g(y)}{\big(g(b,y)\big)^{1 \alpha}}\,dy = I^+_{\alpha g}f(b) \le \lambda$$

Therefore

$$E_k \subset \{x \in (t_k, x_{k+1}) \; I^+_{\alpha g}f_1(x) > \lambda\}$$

Since the operator $I_{\alpha,g}^+$ is of weak type $(1, \frac{1}{1-\alpha})$ with respect to g (this can be done as in the classical case [St]), we have

$$g(E_k) \le g\Big(\{x : I_{\alpha,g}^+(f_1\chi_{(t_k,\infty)})(x) > \lambda\}\Big) \le K\Big(\frac{1}{\lambda}\int_{t_k}^b fg\Big)^{\frac{1}{1-\alpha}}$$
$$\le K\Big(\frac{1}{\lambda}\big(g(t_k,b)\big)^{1-\alpha} M_{\alpha,g}^+ f(t_k)\Big)^{\frac{1}{1-\alpha}} \le Kg(t_k,b)\gamma^{\frac{1}{1-\alpha}}$$
$$\le Kg(x_k,b)\gamma^{\frac{1}{1-\alpha}} = 4Kg(x_{k+1},x_{k+2})\gamma^{\frac{1}{1-\alpha}}.$$

REMARK. Using the methods of Bagby and Kurtz (see [BK] and [K]) instead of Lemma 7, the second author has obtained another proof of Theorem 3, restricted to $q \ge 1$, with a better constant. The proof will appear elsewhere.

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