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CAYLEY FORMS AND SELF-DUAL VARIETIES

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Dedicated to Slava Shokurov on the occasion of his 60th birthday

Abstract Generalized Chow forms were introduced by Cayley for the case of 3-space; their zero set on the Grassmannian $G(1,3)$ is either the set Z of lines touching a given space curve (the case of an 'honest' Cayley form), or the set of lines tangent to a surface. Cayley gave some equations for F to be a generalized Cayley form, which should hold modulo the ideal generated by F and by the quadratic equation Q for $G(1,3)$. Our main result is that F is a Cayley form if and only if $Z = G(1,3) \cap \{F = 0\}$ is equal to its dual variety. We also show that the variety of generalized Cayley forms is defined by quadratic equations, since there is a unique representative $F_0 + QF_1$ of F, with F_0, F_1 harmonic, such that the harmonic projection of the Cayley equation is identically 0. We also give new equations for honest Cayley forms, but show, with some calculations, that the variety of honest Cayley forms does not seem to be defined by quadratic and cubic equations.

Keywords: Cayley forms; Chow varieties; self-dual varieties; developable surfaces; harmonic representative polynomials

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1. Introduction

Cayley forms, according to V. Arnold's paradigm by which no mathematical discovery bears the name of the mathematician who made it first, are nowadays called Chow forms.

A Chow form is a polynomial F_X in the Plücker coordinates of a Grassmann manifold $G(m-n-1,m)$ such that its zero set

$$Z = G(m-n-1,m) \cap \{F = 0\}$$

is the locus of projective subspaces that intersect a given projective variety $X^n_d \subset \mathbb{P}^m$ (the classical notation X^n_d means that X has dimension n and degree d).

Cayley (see [**3**, **4**]) introduced this concept in the case where X is a curve in $\mathbb{P}^3$.

His work was later generalized by Bertini, Chow and van der Waerden (see [**1**, **2**, **8**, **9**, **15**] for partial accounts), and, nowadays, given a variety $X^n_d \subset \mathbb{P}^m$ as above, one defines its *Bertini form* $\Phi_X(H_0,\dots,H_n)$ as the minimal polynomial, multihomogeneous of degree d in each variable $H_i \in (\mathbb{P}^m)^\vee$ such that

$$\Phi_X(H_0,\dots,H_n) = 0 \quad \iff \quad X \cap H_0 \cap \cdots \cap H_n \neq \emptyset.$$

This polynomial is very important for applications to vision imaging, since it provides the 'photographic picture' of X for each projection to $\mathbb{P}^{n+1}$ (if the projection is given by independent linear forms $(H'_0, \dots, H'_{n+1})$, the hypersurface image of X is defined by the polynomial Ψ such that, if we take $H_i = \sum_j a_{ij} H'_j$, $\Psi(H_0 \wedge \dots \wedge H_n) = \Phi_X(H_0, \dots, H_n)$).

Moreover, X is completely determined by Φ_X, and there have been several characterizations of Bertini forms; for instance, there is the characterization by Chow and van der Waerden requiring that the following hold.

(1) There exists a polynomial F in the Plücker coordinates of the Grassmann manifold $G(m-n-1, m)$ such that $\Phi_X(H_0, \dots, H_n) = F(H_0 \wedge \dots \wedge H_n)$; any such polynomial F is called a Chow form.

(2) $\Phi_X(H_0, \dots, H_n)$ splits as a product of forms that are linear in H_n in an algebraic extension of $\mathbb{C}(H_0, H_1, \dots, H_{n-1})$.

Another characterization was given later in [**2**, Theorem 1.14].

In our opinion the most exciting characterization was given by Green and Morrison (see [**9**]), who extended the result of Cayley, showing that F is a Chow form if and only if certain equations of degree 2 or 3 hold identically on the hypersurface $Z = G(m-n-1, m) \cap \{F = 0\}$.

The first motivation for this paper was the attempt to see whether the Chow variety was indeed definable by equations of degree 2 and 3. The impulse for this came from the beautiful result of Cayley, which we now explain in more detail.

In this paper an honest Cayley form (respectively, a tangential Cayley form) is a polynomial F in the Plücker coordinates of $G(1, 3)$, whose zero set $Z \subset G(1, 3)$ is the set of the lines intersecting a given space curve C (respectively, the lines tangent to a given surface S).

$G(1, 3)$ is indeed Klein's quadric in $\mathbb{P} := \mathbb{P}^5$, defined by

$$Q(p) := p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0,$$

and this non-degenerate quadratic form identifies $\mathbb{P}$ with its dual space.

Cayley's equation is

$$\frac{1}{2}\{F, F\} := \frac{\partial F}{\partial p_{01}} \frac{\partial F}{\partial p_{23}} - \frac{\partial F}{\partial p_{02}} \frac{\partial F}{\partial p_{13}} + \frac{\partial F}{\partial p_{03}} \frac{\partial F}{\partial p_{12}} = 0,$$

and Cayley showed that the equation holds on the 3-fold $Z = G(1, 3) \cap \{F = 0\}$ if and only if F is a Cayley form, i.e. either the honest Cayley form of a curve, or the tangential Cayley form of a surface.

Our main result (see Theorem 3.3) is that this equation is equivalent, for a hypersurface $Z \subset G(1, 3)$, to the assertion that Z is self-dual, i.e. Z is equal to its dual variety $Z^\vee$.

Examples where a variety and its dual variety are not hypersurfaces have for a long time been considered, at least according to our knowledge, as sporadic (see [**11**]), and indeed if the variety X is smooth, then Ein (see [**6**, **7**]) has classified the finite number of cases where $\dim(X) = \dim(X^\vee)$.

From Ein's classification one can see that there are very few examples where X is smooth and X and $X^\vee$ are projectively equivalent.

Our result states, on the other hand, that, once we drop the requirement that X be smooth, there are countably many families of self-dual varieties, which are not hypersurfaces. Other examples of self-dual varieties have been constructed by Popov and Tevelev, using the geometry of semi-simple Lie algebras and symmetric spaces (see [**13, 14**]).

Our second result expands on a remark made as a footnote to [**9**], that a Cayley form (which is not unique) can be changed, by adding a multiple of Klein's quadric Q, obtaining another Cayley form for which the Cayley equation holds identically on $Q = G(1,3)$.

We show more precisely (see Theorem 4.2) that there exists a unique representative F_2 of the Cayley form such that $F_2 = F_0 + QF_1$ with F_0 and F_1 harmonic, and such that the Cayley equation for F_2 holds identically on the Klein quadric $Q = G(1,3)$ (i.e. the harmonic projection of the Cayley equation is 0).

This result has as a corollary that the variety of Cayley forms is a projective variety defined by quadratic equations.

In the same section we also dispose, via elementary examples of curves and surfaces of degree 2 or 3, of too optimistic guesses, that F_2 is just the unique harmonic representative, or that there exists some representative F such that the Cayley equation for F is identically 0.

In the final section, we describe (see Theorem 5.2) some equations that detect honest Cayley forms among Cayley forms. These equations appear to be rather simple; however, these are again equations that express that three polynomials vanish identically on the Cayley 3-fold Z. The same elementary examples show that one cannot alter the Cayley form so that these vanish identically on Q, thus showing that the variety of honest Cayley forms is not a projective variety defined by equations of degree 2 or 3.

The above result suggests the question of whether the space of generalized Chow forms (honest and tangential Chow forms) is also defined by quadratic equations. It also suggests the investigation of the geometric deformations of honest Chow forms to tangential Chow forms. For the time being, before finding the solution to this and other questions, we decided to write up this paper.

2. Notation and preliminaries

Let V be a four-dimensional vector space over the field $\mathbb{C}$ (or over an algebraically closed field of characteristic 0), endowed with a volume element, i.e. the non-zero vector

$$\mathrm{Vol} \in \Lambda^4(V)^\vee.$$

The volume element defines a non-degenerate symmetric bilinear form

$$\langle\cdot,\cdot\rangle\colon \Lambda^2(V) \times \Lambda^2(V) \to \mathbb{C},$$
$$\langle\omega,\psi\rangle := \mathrm{Vol}(\omega \wedge \psi).$$

Remark 2.1. The same situation holds for $\Lambda^m(V)$ when $\dim(V) = 2m$, and $\langle\cdot,\cdot\rangle$ is symmetric if and only if m is even, skew symmetric if and only if m is odd.

In the case where $V = \mathbb{C}^4$, with canonical basis e_0, e_1, e_2, e_3, we have a canonical volume such that $\mathrm{Vol}(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = 1$, and we have, identifying $p \in \Lambda^2(V)$ to a skew symmetric 4×4-matrix (p_{ij}), that one half of the corresponding quadratic form is just the Pfaffian $Q(p)$ of the skew symmetric 4×4-matrix

$$Q(p) := \tfrac{1}{2}\langle p, p\rangle = Pf((p_{ij})) = p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}.$$

To the symmetric bilinear form $\langle \cdot, \cdot \rangle$ there corresponds the polarity isomorphism

$$\mathcal{P} \colon \Lambda^2(V) \to \Lambda^2(V)^\vee$$

whose inverse determines a quadratic form on $\Lambda^2(V)^\vee$, which is still denoted by Q (this is unambiguous in view of the polarity isomorphism).

When $V = \mathbb{C}^4$, with canonical basis e_0, e_1, e_2, e_3, $\Lambda^2(V)$ has canonical basis $\partial/\partial p_{ij} := e_i \wedge e_j$, and the quadratic form on $\Lambda^2(V)^\vee$ yields the Laplace operator

$$\Delta := \frac{\partial}{\partial p_{01}}\frac{\partial}{\partial p_{23}} - \frac{\partial}{\partial p_{02}}\frac{\partial}{\partial p_{13}} + \frac{\partial}{\partial p_{03}}\frac{\partial}{\partial p_{12}}.$$

Throughout, we consider polynomial functions $F(p_{ij})$ on $\Lambda^2(V)$, and using the polarity isomorphism we can define the gradient as the column vector ∇F transpose of the row vector

$${}^{\mathrm{T}}\nabla F := \left(\frac{\partial F}{\partial p_{23}}, -\frac{\partial F}{\partial p_{13}}, \frac{\partial F}{\partial p_{12}}, \frac{\partial F}{\partial p_{03}}, -\frac{\partial F}{\partial p_{02}}, \frac{\partial F}{\partial p_{01}}\right),$$

corresponding to the differential $\mathrm{d}F$, and define the Cayley bracket.

2.1. The Cayley bracket

Definition 2.2. Let $F(p_{ij})$, $G(p_{ij})$ be polynomial functions on $\Lambda^2(V)$; their *Cayley bracket* is then defined by the symmetric bilinear form

$$\{F, G\} := \langle \nabla F, \nabla G\rangle = \langle \mathrm{d}F, \mathrm{d}G\rangle.$$

The Cayley equation for F is then the differential equation

$$\frac{1}{2}\{F, F\} = Q(\nabla F) = \frac{\partial F}{\partial p_{01}}\frac{\partial F}{\partial p_{23}} - \frac{\partial F}{\partial p_{02}}\frac{\partial F}{\partial p_{13}} + \frac{\partial F}{\partial p_{03}}\frac{\partial F}{\partial p_{12}} = 0.$$

Turning now to geometry, to a homogeneous polynomial $F(p_{ij})$ on $\Lambda^2(V)$ there corresponds the hypersurface

$$F := \{(p_{ij}) \mid F(p_{ij}) = 0\} \subset \mathbb{P}(\Lambda^2(V)) = \mathrm{Proj}(\Lambda^2(V)^\vee) \cong \mathbb{P}^5,$$

which we denote by the same symbol F.

The hypersurface Q plays a particular role, since

$$\{(p_{ij}) \mid Q(p_{ij}) = 0\} \subset \mathbb{P}(\Lambda^2(V))$$

equals the Grassmann manifold

$$G(1,3) = \{p = (p_{ij}) \mid \exists v, v' \in V,\ p = v \wedge v'\}$$

parametrizing projective lines L in $\mathbb{P}(V) \cong \mathbb{P}^3$.

If p is then a point of the hypersurface F (i.e. $F(p) = 0$), then the tangent hyperplane to F at p is the hyperplane

$$TF_p := \left\{(\xi_{ij}) \,\middle|\, \sum_{ij} \frac{\partial F}{\partial p_{ij}} \xi_{ij} = 0\right\} = \{\xi \mid (\mathrm{d}F, \xi) = 0\},$$

where $(\cdot,\cdot)$ denotes the standard duality.

As usual, the non-degenerate scalar product $\langle\cdot,\cdot\rangle$ identifies TF_p to the zero set of the linear form $\mathrm{d}F$, hence to the orthogonal to the gradient $\nabla F = \mathcal{P}^{-1}(\mathrm{d}F)$.

In particular, if $p \in Q$, then TQ_p is the orthogonal hyperplane $p^\perp$ to p, since $\mathrm{d}Q = \mathcal{P}(p)$.

In particular, the next lemma follows immediately.

Lemma 2.3. *Let Z be the 3-fold in $\mathbb{P} := \mathbb{P}(\Lambda^2(V))$, which is the complete intersection of the Grassmann manifold $Q = G(1,3)$ with the hypersurface F. The Zariski tangent space to Z at $p \in Z$ is then*

$$TZ_p = p^\perp \cap (\nabla F(p))^\perp.$$

We now come to a key formula.

Lemma 2.4. *Let F be a homogeneous polynomial of degree m on $\Lambda^2(V)$. Then Euler's formula reads as*

$$\{F, Q\} = \langle \nabla F, \nabla Q\rangle = \sum_{ij} \frac{\partial F}{\partial p_{ij}} p_{ij} = mF.$$

Proof. We have that $\nabla Q(p) = p$, since $\mathrm{d}Q = \mathcal{P}(p)$; hence, $\{F, Q\} = \langle \nabla F, p\rangle = (\mathrm{d}F, p) = mF$. □

An important consequence of this is that, for p on the hypersurface F, one has that $\langle \nabla F, p\rangle = 0$.

2.2. Lines in the Grassmannian

In the following we denote $\mathbb{P}(V)$ by $\mathbb{P}^3$, and by $\mathbb{P}$ we denote the projective space $\mathbb{P}(\Lambda^2(V))$ containing the Grassmann manifold $Q = G(1,3)$ parametrizing lines $L \in \mathbb{P}^3$. We use the notation x, y for points in $\mathbb{P}^3$, and π, π' for planes in $\mathbb{P}^3$.

Given $x \in \mathbb{P}^3$, $\mathbb{P}^2_x \subset Q$ is defined as the projective plane in $\mathbb{P}$,

$$\mathbb{P}^2_x := \{L \mid x \in L\} \cong \mathbb{P}^2,$$

and given a plane $\pi \subset \mathbb{P}^3$, $\mathbb{P}^2_\pi := \{LL \subset \pi\} \cong \mathbb{P}^2$.

Given x, π, one has $\mathbb{P}^2_\pi \cap \mathbb{P}^2_x = \emptyset$ unless $x \in \pi$, and in this case one obtains a Schubert line in $\mathbb{P}$:

$$\Gamma(x, \pi) := \mathbb{P}^2_\pi \cap \mathbb{P}^2_x = \{L \mid x \in L \subset \pi\} \ (\cong \mathbb{P}^1 \text{ for } x \in \pi).$$

Observe that any line $\Gamma \subset Q$ is of this form, and one can find x, π as follows. Let L, L' be two points of Γ, so the corresponding lines $L, L' \subset \mathbb{P}^3$ are not skew (else, $\langle L, L' \rangle \neq 0$); hence, x is the intersection point of the corresponding two lines, and $\pi \subset \mathbb{P}^3$ is the plane spanned by L, L'.

We recover the planes $\mathbb{P}^2_x$ and $\mathbb{P}^2_\pi$ starting from Γ in the following way. Intersect Q with the orthogonal $\Gamma^\perp$, and observe that $\Gamma \subset \Gamma^\perp$ is then the vertex of the quadric $Q' := Q \cap \Gamma^\perp \subset \Gamma^\perp \cong \mathbb{P}^3$.

Hence, Q' splits as the union of two planes meeting along Γ, which, therefore, are of the form $\mathbb{P}^2_x$ for $x \in \mathbb{P}^3$ as above, respectively, $\mathbb{P}^2_\pi$ for the above plane $\pi \subset \mathbb{P}^3$.

2.3. Harmonic polynomials

Consider the coordinate ring of $\mathbb{P}$, namely, the symmetric algebra of $\Lambda^2(V)^\vee$:

$$\mathcal{A} = \bigoplus_{m\geqslant 0} \mathcal{A}_m := \bigoplus_{m\geqslant 0} S^m(\Lambda^2(V)^\vee).$$

Inside $\mathcal{A}_m$ there is the linear subspace of harmonic polynomials

$$\mathcal{H}_m := \{F \in \mathcal{A}_m \mid \Delta(F) = 0\},$$

where Δ is, as above, the Laplace operator

$$\Delta := \frac{\partial}{\partial p_{01}}\frac{\partial}{\partial p_{23}} - \frac{\partial}{\partial p_{02}}\frac{\partial}{\partial p_{13}} + \frac{\partial}{\partial p_{03}}\frac{\partial}{\partial p_{12}}.$$

We recall some basic formulae, which are easy to establish, for homogeneous polynomials A, B (indeed, (3) was proved in Lemma 2.4):

(1) $\Delta(AB) = \Delta(A)B + A\Delta(B) + \langle \nabla A, \nabla B\rangle$,

(2) $\Delta(Q) = 3$,

(3) $\langle \nabla A, \nabla Q\rangle = \deg(A) \cdot A$;

hence, finally,

(1*) $\Delta(GQ) = (\deg(G) + 3) \cdot G + Q\Delta(G)$,

which is the main tool to prove the following.

Lemma 2.5. *There is an isomorphism $\mathcal{H}_m \cong H^0(\mathcal{O}_Q(m)) := W_m$, and, moreover, one has the direct sum decomposition*

$$\mathcal{A}_m = \bigoplus_{i\geqslant 0} Q^i \mathcal{H}_{m-2i}.$$

Proof. One shows the assertion by induction on m, using that $\mathcal{A}_m \cong W_m \oplus Q\mathcal{A}_{m-2}$.

Assume that G is harmonic and let $\deg(G) = m - 2i$; then, by induction on i, we easily get that

$$\Delta(GQ^i) = i(m + 2 - i) \cdot G \cdot Q^{i-1}.$$

This formula, and the induction assumption, shows that the subspaces $Q^i\mathcal{H}_{m-2i}$ build a direct sum inside $\mathcal{A}_m$, since no harmonic polynomial can belong to the subspace $Q\mathcal{A}_{m-2}$.

Hence, there is an injective linear map $\mathcal{H}_m \to W_m$, and to conclude that it is an isomorphism it suffices (either to show that both spaces have the same dimension, or) to use that both spaces are representations of $\mathrm{GL}(V)$, and that W_m is irreducible (being the space of sections of a linearized line bundle on an homogeneous variety). □

3. Cayley forms and self-dual 3-folds

Definition 3.1. We say that $F \in H^0(\mathcal{O}_{\mathbb{P}}(m))$ is a Cayley form if the 3-fold $Z := Q \cap F = G(1,3) \cap F$ is such that each of its irreducible components W is either

(i) an honest Cayley 3-fold, consisting of the lines L that intersect an irreducible curve $C \subset \mathbb{P}^3$ ($W = \bigcup_{x \in C} \mathbb{P}^2_x$) or

(ii) a tangential Cayley 3-fold, consisting of the closure of the set of lines L that are tangent to an irreducible non-degenerate surface $S \subset \mathbb{P}^3$ (i.e. S is not a plane) at a smooth point $x \in S$ ($W = \overline{\bigcup_{x \in S \setminus \mathrm{Sing}(S)} \Gamma(x, TS_x)}$).

Remark 3.2. In the case where F is an honest Cayley form, $m = \deg(F) = \deg(C)$ (we identify here the polynomial F with the hypersurface $\{F = 0\}$ it defines).

If F is a tangential Cayley form associated to a surface $S \subset \mathbb{P}^3$, then $m = \deg(F)$ is the intersection number of $Z := Q \cap F = G(1,3) \cap F$ with a line Γ contained in Q, which is then of the form $\Gamma(x, \pi)$.

If one denotes by C' the intersection of S with a general plane π, one sees, therefore, that m is the class of the plane curve C'. Thus, we have that

$$m = n(n-1) - \sum_{y \in \mathrm{Sing}(C')} c(y),$$

where $n = \deg(S)$, and $c(y)$ is the Plücker defect of the singular point $y \in C'$.

The following is our first result.

Theorem 3.3. *Let $F \in H^0(\mathcal{O}_{\mathbb{P}}(m))$, and assume that $Z := Q \cap F$ is reduced. The following conditions are then equivalent:*

(1) *F is a Cayley form,*

(2) *F satisfies the weak Cayley equation $\{F, F\} \equiv 0 \ (\mathrm{mod}(Q, F))$,*

(3) *the 3-fold $Z := Q \cap F = G(1,3) \cap F$ is self-dual, i.e. $Z = Z^\vee$.*

The structure of the proof runs as follows: first we show that we can restrict to the case where Z is irreducible, and we prove that (1) $\Rightarrow$ (2); then we show (2) $\iff$ (3), and finally (3) $\Rightarrow$ (1).

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Proof of Theorem 3.3, part I.

Assume that the hypersurface Z is reducible; we can then write $Z = Z_1 \cup Z_2$. Hence, since $\mathrm{Pic}(Q) \cong \mathbb{Z}$, changing F modulo Q, we may assume that $F = F_1F_2$, with F_1, F_2 relatively prime.

Then,

$$\begin{aligned}\{F,F\} &= \langle \mathrm{d}F, \mathrm{d}F\rangle \\ &= \langle F_1\,\mathrm{d}F_2 + F_2\,\mathrm{d}F_1, F_1\,\mathrm{d}F_2 + F_2\,\mathrm{d}F_1\rangle \\ &= F_1^2\{F_2,F_2\} + 2F_1F_2\{F_1,F_2\} + F_2^2\{F_1,F_1\}.\end{aligned}$$

Hence, F_1 and F_2 satisfy (2) if and only if F does. Therefore, we may restrict ourselves to showing the theorem in the case where Z is irreducible.

(1) $\Rightarrow$ (2).

Case (i) where F is a honest Cayley form of an irreducible curve C.

Let $L \in Z$; there is then $x \in C$ such that $L \in \mathbb{P}^2_x \subset Z$. Hence, F vanishes on $\mathbb{P}^2_x$. Now take coordinates on $\mathbb{P}^3$ such that $x = e_0$; hence, $\mathbb{P}^2_x = \{p \mid p_{12} = p_{13} = p_{23} = 0\}$, whence $\nabla F(L)$ has components that satisfy

$$\frac{\partial F}{\partial p_{0i}}(L) = 0,\ i = 1,2,3 \quad \Rightarrow \quad \{F,F\}(L) = 0.$$

Thus, $\{F,F\}$ vanishes on Z; equivalently, the weak Cayley equation (2) holds.

Case (ii) where F is a tangential Cayley form.

Let $L \in Z$ be general; there is then $x \in S$ that is a smooth point and is such that L is tangent to S at x. Now take coordinates on $\mathbb{P}^3$ such that $x = e_0$, $L = e_0 \wedge e_1$, and the tangent space TS_x is the plane $\{x \mid x_3 = 0\}$.

There exists a local parametrization of S with

$$x = (1, u, v, \phi(u,v)),$$

where ϕ has order at least 2 at the origin $u = v = 0$.

A local parametrization for the variety of tangent lines is then given by the wedge product of the two (row) vectors

$$\begin{gathered}(1, u, v, \phi(u,v)),\\ (0, 1, \lambda, \phi_u(u,v) + \lambda\phi_v(u,v));\end{gathered}$$

hence, the lines are parametrized by (u, v, λ), L corresponds to the origin in this system of coordinates, and we have that

$$\begin{gathered}p_{01} = 1, \qquad p_{02} = \lambda, \qquad p_{03} = \phi_u(u,v) + \lambda\phi_v(u,v), \qquad p_{12} = u\lambda - v,\\ p_{13} = u(\phi_u(u,v) + \lambda\phi_v(u,v)) - \phi(u,v).\end{gathered}$$

Note that, since $p_{01} = 1$, $p_{23} = p_{02}p_{13} - p_{03}p_{12}$ on Q, and looking at the Taylor development of the function

$$F(p(u,v,\lambda)) = \frac{\partial F}{\partial p_{02}}(L)\lambda + \frac{\partial F}{\partial p_{03}}(L)\phi_u(u,v) - \frac{\partial F}{\partial p_{12}}(L)v + \text{terms of order} \geqslant 2,$$

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which is identically 0, we obtain that, at the point L, $\partial F/\partial p_{02}$ vanishes, and $\partial F/\partial p_{03}$ vanishes too unless

$$\phi_{uu}(0,0) := \frac{\partial^2 \phi}{\partial u^2}(0,0) = 0.$$

Moreover, $(\partial F/\partial p_{01})(L)$ vanishes by Euler's formula.

The conclusion is that $\{F, F\}(L) = 0$ unless the tangent line L is a zero of the second fundamental form of S (a so-called asymptotic direction). But since the surface is non-degenerate, for general L we have that L is not a zero of the second fundamental form of S.

Hence, $\{F, F\}$ vanishes on Z; equivalently, the weak Cayley equation (2) holds. □

The above calculation in local coordinates shows that, if L is a smooth point of Z, the tangent space TZ_L is the subspace $\{p \mid p_{13} = p_{23} = 0\}$, which contains the $\mathbb{P}^2_x$ of lines passing through x.

It also shows the following.

Proposition 3.4. *If the line L is not an asymptotic direction at $x \in S$, then the second derivative of F does not identically vanish on $\mathbb{P}^2_x$.*

Proof. $\mathbb{P}^2_x$ is the subspace $\{p \mid p_{12} = p_{13} = p_{23} = 0\}$, and we are claiming that the second fundamental form of Z does not vanish on it.

Intersecting Z with this subspace we obtain the subvariety defined by

$$\begin{aligned} &v = \lambda u, \quad u(\phi_u(u,v) + \lambda\phi_v(u,v)) = \phi(u,v) \\ &\qquad\qquad \iff \quad v = \lambda u, \quad u\phi_u(u,\lambda u) + \lambda u\phi_v(u,\lambda u) - \phi(u,\lambda u) = 0. \end{aligned}$$

All we have to show is that at the origin the function

$$u\phi_u(u,\lambda u) + \lambda u\phi_v(u,\lambda u) - \phi(u,\lambda u)$$

has a quadratic term that is not identically 0.

But this quadratic term equals the quadratic term of

$$u\phi_u(u,\lambda u) - \phi(u,\lambda u).$$

Letting $\phi(u,v) = au^2 + buv + cv^2 \pmod{(u,v)^3}$, we obtain that

$$u(2au) - au^2 = au^2 \equiv 0;$$

hence, $0 = 2a = \phi_{uu}(0,0)$, contradicting our assumption. □

Proof of Theorem 3.3, part II.

(2) $\iff$ (3).

Condition (2) just says that, for $L \in Z$, $Q(\nabla F(L)) = 0$; this means that $\nabla F(L)$ is a point in Q.

However, since

$$\langle \nabla F(L), \nabla F(L)\rangle = 0, \qquad \langle L, L\rangle = 0, \qquad \langle \nabla F(L), L\rangle = 0,$$

where the last equality is nothing other than the Euler formula (see Lemma 2.4), we see that (2) is equivalent to saying that the line $\Gamma_L := L * \nabla F(L)$ joining L and $\nabla F(L)$ is fully contained in the Grassmannian Q.

Observe now that, identifying $\mathbb{P}$ with its dual space via the polarity $\mathcal{P}$, the line $\Gamma_L := L * \nabla F(L)$ is dual to the pencil of tangent hyperplanes to Z at L, since $TZ_L = L^{\perp} \cap \nabla F(L)^{\perp}$.

We have, therefore, shown the following.

Claim: (2) holds if and only if we have the inclusion of the dual variety of Z in Q:

$$Z^{\vee} \subset Q.$$

We conclude the proof of this step via part (2) of the following lemma.

Lemma 3.5. *Assume that $Z \subset Q$. Then,*

(1) $Z \subset Z^{\vee}$,

(2) $Z^{\vee} \subset Q \quad \iff \quad Z = Z^{\vee}$.

Proof of Lemma 3.5. (1) Assume that $L \in Z$ is a smooth point; then, $TZ_L \subset TQ_L = L^{\perp}$. Hence, $L \in Z^{\vee}$.

(2) $Z^{\vee} \subset Q$ implies, by (1), that $Z^{\vee} \subset (Z^{\vee})^{\vee} = Z$, where the last equality is the biduality theorem. Again by (1), $Z \subset Z^{\vee}$; hence, $Z^{\vee} \subset Q$ implies that $Z = Z^{\vee}$, while the converse is obvious. □

□

The following proposition explains the geometrical background for the last step of proof of Theorem 3.3. It involves the concept of the Segre dual curve, which we need to recall (see [**12**]; however, for the reader's benefit, we give an elementary proof).

Definition 3.6. Let C be a non-degenerate curve in $\mathbb{P}^n$, which means that, if $\gamma(t)$ is a parametrization of C, for general t the n vectors $\gamma(t), \gamma'(t), \dots, \gamma^{(n-1)}(t)$ are linearly independent.

The Segre dual curve $C^* \subset (\mathbb{P}^n)^{\vee}$ is then the curve of osculating $(n-1)$-dimensional spaces, so C^* is parametrized by

$$\gamma^*(t) := \gamma(t) \wedge \gamma'(t) \wedge \cdots \wedge \gamma^{(n-1)}(t).$$

More generally, the k-th associated curve $C[k]$ is the curve of osculating (k)-dimensional spaces, a curve in the Grassmann manifold $G(k, n)$, parametrized by

$$\gamma[k](t) := \gamma(t) \wedge \gamma'(t) \wedge \cdots \wedge \gamma^{(k)}(t).$$

Lemma 3.7. *If C is a non-degenerate curve in $\mathbb{P}^n$, then*

(a) $(C^*)^* = C$;

(b) *for each value of the parameter t, $\gamma^*[n-1-k](t)$ is the annullator subspace of $\gamma[k](t)$;*

(c) *$C^{\vee}$ is the tangential developable hypersurface of C^*.*

Proof. Observe that (a) is the special case of the more general statement (b), obtained by taking $k = 0$.

In order to prove (b), we use the method of moving frames. Namely, we let $A(t)$ be the matrix with columns the $n+1$ vectors

$$\gamma(t), \gamma'(t), \dots, \gamma^{(n-1)}(t), \gamma^{(n)}(t).$$

$A(t)$ determines a flag in $\mathbb{C}^{n+1}$, and we may also take a unitary matrix $U(t)$ determining the same flag.

Then the 'dual flag', given by the annullators of these subspaces in the dual space $\mathbb{C}^{n+1}$, corresponds to the matrices $B(t)$, $V(t)$ where one takes the respective dual bases in the opposite order.

One considers, as usual, the Cartan matrix $C(t)$, the skew symmetric matrix defined by

$$U^{\cdot}(t) := \frac{\mathrm{d}U(t)}{\mathrm{d}t} = C(t)U(t).$$

We have that ${}^{\mathrm{T}}V(t)U(t) \equiv J$, where J is the anti-identity matrix; whence, taking the derivative of both sides,

$$\begin{aligned} {}^{\mathrm{T}}V(t)U^{\cdot}(t) + {}^{\mathrm{T}}V^{\cdot}(t)U(t) = 0 \quad &\Rightarrow \quad {}^{\mathrm{T}}V(t)C(t) + {}^{\mathrm{T}}V^{\cdot}(t) = 0 \\ &\Rightarrow \quad V^{\cdot}(t) = C(t)V(t). \end{aligned}$$

This formula shows that the dual flag is the osculating flag of the curve $\gamma^*(t)$.

One can also avoid the use of the complex numbers, and work with the moving frame $A(t)$, defining the companion matrix $M(t)$ such that $A^{\cdot}(t) = M(t)A(t)$, and the proof follows similarly.

To prove the last statement, observe that

$$\begin{aligned} C^{\vee} &= \{H \mid \exists x \in C,\ TC_x \subset H\} \\ &= \{H \mid H \in \operatorname{Ann}\gamma[1](x)\} \\ &= \{H \mid H \in \gamma^*[n-2](x)\} \\ &= \{H \mid \exists x \in C,\ H \in \text{Linear span}(\gamma^*(x), \dots, \gamma^{*(n-2)}(x))\}. \end{aligned}$$

□

Proposition 3.8. *Consider the (involutory) polarity isomorphism identifying $\mathbb{P}$ with its dual space, which geometrically corresponds to the mapping associating to a line $L \subset \mathbb{P}^3$ the pencil of planes containing it (a line in $(\mathbb{P}^3)^{\vee}$).*

It sends the tangential Cayley 3-fold of a surface S to the tangential Cayley 3-fold of the dual variety $S^{\vee}$ when the latter is a surface S, else to the honest Cayley 3-fold of the dual variety $S^{\vee}$ when the latter is a curve.

It sends the honest Cayley 3-fold of a curve C to the tangential Cayley 3-fold of the dual variety $C^{\vee}$, which is the tangential developable surface of the Segre dual curve C^*.

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Proof. We use the standard notation, by which the projectively dual subspace of a projective subspace $L \subset \mathbb{P}^n$, i.e. the projective subspace corresponding to the annullator, is denoted by L^*.

Now, if L is a tangent line to the surface S at a point x, then $x \in L \subset TS_x$; hence, defining $H := TS_x$, we have $H^* \in L^* \subset x^*$. Thus, L^* is tangent to $S^\vee$, which settles the proof in the case where $S^\vee$ is a surface (in view of biduality).

Again by biduality, it suffices to consider the honest Cayley 3-fold of a curve $C \subset \mathbb{P}^3$. It consists of the lines L intersecting the curve C in a point x; the dual subspace L^* then satisfies $H^* \in L^* \subset x^*$, whenever the plane H contains L. We choose H to also contain TC_x, so $H^* \in C^\vee$, and L^* is tangent to $C^\vee$ at H^*.

Conversely, if L^* is tangent to $C^\vee$ at H^*, then there exists x such that $H^* \in L^* \subset x^*$, and $x \in L$. □

Proof of Theorem 3.3, part III.

(3) ⇒ (1).

For each smooth point $L \in Z$, the line $\Gamma_L := (L * \nabla F(L))$ corresponds to the pencil of tangent hyperplanes to Z in L; hence, it is contained in $Z^\vee = Z$.

Being a line in the Grassmannian, it determines a point $x \in \mathbb{P}^3$ and a plane $\pi \subset \mathbb{P}^3$ such that $\Gamma_L = (L * \nabla F(L)) = \Gamma(x, \pi)$.

Hence, we get a rational map of Z onto a correspondence

$$\Sigma \subset \mathbb{P}^3 \times (\mathbb{P}^3)^\vee := \overline{\{(x,\pi) \mid \exists L \in Z \setminus \mathrm{Sing}(Z),\ \text{s.t. } \Gamma_L = \Gamma(x,\pi)\}}.$$

Lemma 3.9. *Σ has dimension 2 and is a duality correspondence with respect to the two projections.*

Proof. For each point $L \in Z$, we have the line $\Gamma_L := (L * \nabla F(L)) = \Gamma(x, \pi)$, which is contained in Z. Assume that there is another line Γ' passing through L, contained in Z and different from Γ_L. Then Γ' is contained in $TZ_L = \Gamma_L^\perp$. Hence, the plane Π spanned by Γ_L and by Γ' is contained in TZ_L, and we then have $\Pi \subset Q$, since $\Gamma' \subset TZ_L = \Gamma_L^\perp$.

Since $\Gamma(x, \pi) = \Gamma_L \subset \Pi \subset Q$, it follows that either

[1] $\Pi = \mathbb{P}^2_x$ or

[2] $\Pi = \mathbb{P}^2_\pi$.

We separate our analysis according to different cases:

(i) for general $L \in Z$, there are only a finite number of lines passing through L and contained in Z;

(ii) for general $L \in Z$, there are an infinite number of lines contained in Z and passing through L.

Condition (ii) implies, by the above consideration, that one of the following holds:

[1] for general $L \in Z$, $L \in \mathbb{P}^2_x \subset Z$;

[2] for general $L \in Z$, $L \in \mathbb{P}^2_\pi \subset Z$.

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Therefore, if (ii) holds true, then necessarily Z is a honest Cayley 3-fold, or a dual honest Cayley 3-fold.

Consider now the tangential correspondence W for $Z' := Z \setminus \mathrm{Sing}(Z)$:

$$W := \{(L_1, L_2) \in Z' \times Z' \mid TZ_{L_1} \subset L_2^{\perp}\} = \{(L_1, L_2) \in Z \times Z \mid L_2 \in \Gamma_{L_1}\}.$$

Since $\dim(W) = 4$, and Z has dimension 3, the general fibre $Y := W_{L_2}$ of the second projection is irreducible of dimension 1. And, for each $L_1 \in Y$, $L_2 \in \Gamma_{L_1}$. Since (i) holds and Y is irreducible, it follows that all the lines Γ_{L_1} are equal, and the fibre Y equals Γ_{L_1}. In particular, the tangent space to Z is constant along Γ_{L_1}. We also obtain that the map onto Σ is constant over Γ_{L_1}; hence, Σ is a surface.

Moreover, since (ii) does not hold, the two projections of Σ yield two surfaces, $S \subset \mathbb{P}^3$, $S' \subset (\mathbb{P}^3)^{\vee}$.

There remains to show that S and S' are dual to each other. Now, for each general point $x \in S$, x is the image of a line $\Gamma(x, \pi) \subset Z$. If we show that the lines $L \in \Gamma$ are tangent to S, then this proves that $\pi = \bigcup_{L \in \Gamma} L$ is tangent to S in x; hence, S' is dual to S.

This assertion is proven in the forthcoming lemma. □

Lemma 3.10. *Let $f\colon Z \setminus \mathrm{Sing}(Z) \to S$ be the above morphism, such that $f(L) = x$, where x is the intersection point of the lines $L, \nabla \subset \mathbb{P}^3$, $\nabla := \nabla F(L)$.*

Then, $\mathbb{P}^2_x \subset TZ_L$, and if Df is of maximal rank at L, then $Df(\mathbb{P}^2_x) = L$.

Proof. Letting Γ, as usual, be the line joining L with ∇, we know that $TZ_L = \Gamma^{\perp}$, that $\Gamma \subset \Gamma^{\perp}$, $\Gamma \subset Z \subset Q$.

Then, $TZ_L \cap Q = \mathbb{P}^2_x \cup \mathbb{P}^2_{\pi}$, where π is the plane spanned by the lines $L, \nabla \subset \mathbb{P}^3$.

Now view L and ∇ as 4×4 skew-symmetric matrices, so x is the solution of the system

$$Lx = 0, \qquad \nabla x = 0.$$

Consider a tangent vector to L with direction $L' \subset \mathbb{P}^2_x$; then, if we work as usual with the ring $\mathbb{C}[\epsilon]/(\epsilon^2)$, we obtain the equations

$$(L + \epsilon L')(x + \epsilon x') = 0, \qquad (\nabla + \epsilon \nabla')(x + \epsilon x') = 0$$

for the first-order variation of f along the tangent direction L'.

Hence, we obtain

$$L'x + Lx' = 0, \quad \nabla' x + \nabla x' = 0 \quad \Rightarrow \quad Lx' = 0,$$

since $L'x = 0$.

The conclusion is that $x' = Df(L')$ lies in the line L. On the other hand, Df has maximal rank (equal to 2), and Γ lies in the kernel; hence, Df satisfies $Df(\mathbb{P}^2_x) = L$. □

□

Remark 3.11. The Cayley 3-folds Z, considered above, are all singular. In fact Ein (see [**7**]) classified the smooth projective varieties X such that $\dim(X) = \dim(X^\vee)$ (he actually forgot to explicitly mention the assumption of smoothness, but this is clearly used; see [**7**, Corollary 1.4]).

Remark 3.12. Igor Dolgachev pointed out another characterization of Cayley forms in terms of singular loci of line complexes (see [**10**, p. 308], [**5**, p. 534]).

Since it is related to the previous discussion, we give a brief account in our terminology. A line complex is a subvariety $Z \subset Q = G(1,3)$.

We denote by $\Lambda = \mathbb{P}(U)$ the projectivization of the tautological subbundle on the Grassmannian $G(1,3)$. Hence,

$$\Lambda = \{(x, L) \mid x \in L\} \subset \mathbb{P}^3 \times G(1,3).$$

Denote by

$$\Lambda_Z := \{(x, L) \mid x \in L \in Z\} \subset \mathbb{P}^3 \times Z$$

the restriction of the bundle to Z, and denote by f the projection on $\mathbb{P}^3$.

While Λ is the fibre bundle $\mathbb{P}(T_{\mathbb{P}^3})$, with fibre over $x \in \mathbb{P}^3$ equal to $\mathbb{P}^2_x$, the same does not occur for Λ_Z.

The *singular locus* of the line complex is defined to be the critical set $\mathcal{C}$ of $f\colon \Lambda_Z \to \mathbb{P}^3$, while the *focal locus* is by definition $\mathcal{F} := f(\mathcal{C})$, the set of critical values of f.

Therefore, the singular locus equals the closure of the set of pairs (x, L), L being a smooth point of Z, where the fibre of f is not smooth of the right codimension; i.e. such that $\mathbb{P}^2_x \cap Z$ is not a transversal intersection at L.

In the case where $\dim(Z) = 3$, this means that

$$\mathbb{P}^2_x \subset TZ_L = L^\perp \cap \nabla^\perp \quad \iff \quad L, \nabla \in \mathbb{P}^2_x \quad \Rightarrow \quad \nabla \in Q.$$

In particular, $\mathcal{C} \subset \Lambda_{Z \cap \{\{F,F\}=0\}}$. Conversely, proceeding as in the first two lines of the proof of Lemma 3.10, one sees that if $\nabla \in Q$, then $TZ_L \cap Q = \mathbb{P}^2_x \cup \mathbb{P}^2_\pi$; thus, $\mathcal{C}$ projects birationally onto $Z \cap \{\{F,F\} = 0\}$.

The interpretation pointed out by Dolgachev is, therefore, that Z is a Cayley 3-fold if and only if it equals the projection of its singular locus.

4. Quadratic equations for the variety of Cayley forms

A Cayley 3-fold is the divisor Z on the Grassmann manifold $Q = G(1,3)$ of a section $\zeta \in H^0(Q, \mathcal{O}_Q(m))$.

A Cayley form F is a homogeneous polynomial of degree m, $F \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m))$ such that the restriction of F to the quadric Q is precisely ζ. Hence, we may change a given Cayley form F by adding a multiple of Q to it, trying to see whether one could obtain a Cayley form satisfying the strong Cayley equation $\{F, F\} \equiv 0$. We show that this cannot be achieved, but at least (as stated in [**9**]) one can obtain that $\{F, F\} \equiv 0 \pmod{Q}$.

Indeed, we show a more precise result, which has as a consequence that Cayley 3-folds are parametrized by a projective variety that is the intersection of quadrics.

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Proposition 4.1. *Assume that F is homogeneous of degree m and satisfies the weak Cayley equation*

$$\{F, F\} \equiv 0 \ (\mathrm{mod}(F, Q)).$$

There then exists another Cayley form F_2, defining the same Cayley 3-fold Z as F, such that

$$\{F_2, F_2\} \equiv 0 \ (\mathrm{mod}\, Q).$$

Moreover, F_2 is unique $(\mathrm{mod}(Q^2))$.

Proof. We seek for $F_2 = F + QG$ and calculate (using the formula $\{F, Q\} = mF$)

$$\begin{aligned}\{F_2, F_2\} &= \{F + QG, F + QG\} \\ &= \{F, F\} + 2Q\{F, G\} + 2mGF + 2(m-2)QG^2 + Q^2\{G, G\} + 2G^2Q.\end{aligned}$$

Hence, if

$$\{F, F\} = AQ + BF,$$

it suffices to take $G = -\frac{1}{2m}B$, and the solution G is unique modulo Q; hence, F_2 is unique modulo Q^2. □

As an important consequence, we then reach the following.

Theorem 4.2. *The variety $\mathcal{C}_m$ of Cayley 3-folds $Z \in \mathbb{P}(H^0(\mathcal{O}_Q(m)))$ is isomorphic to the subvariety $\mathcal{C}'_m \subset \mathbb{P}(\mathcal{H}_m \oplus Q\mathcal{H}_{m-2})$ defined by quadratic equations:*

$$\mathcal{C}'_m := \{Z = Q \cap F \mid F \in (\mathcal{H}_m \oplus Q\mathcal{H}_{m-2}),\ h_{2m-2}(\{F, F\}) = 0\},$$

(here $h_m \colon \mathcal{A}_m \to \mathcal{H}_m$ is the harmonic projector).

Remark 4.3. Let $F = F_0 + QF_1 \in \mathcal{H}_m \oplus Q\mathcal{H}_{m-2}$; the equation $h_{2m-2}(\{F, F\}) = 0$ can then be rewritten as

$$h_{2m-2}(\{F_0, F_0\} + 2mF_0F_1) = 0.$$

4.1. The easiest examples

Let F be a Cayley form, so there are polynomials A, B such that $\{F, F\} = AQ + BF$.

Then take $F_2 = F + QG$ as above, where $G = -\frac{1}{2m}B + CQ$ is as above.

In the special case where $\deg(F) \leqslant 3$, we then have the unicity of F_2, since $G = -\frac{1}{2m}B$ by degree considerations ($\deg(C) < 0$).

Moreover, A, B are both unique.

We then have

$$\{F_2, F_2\} = AQ - \frac{1}{m}Q\{F, B\} + \frac{m-1}{2m^2}QB^2 + \frac{1}{2^2m^2}Q^2\{B, B\}.$$

Let us start by considering the case $\deg(F) = 2$.

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Corollary 4.4. *In the case of a smooth quadric surface $S_2 \subset \mathbb{P}^3$ there is no tangential Cayley form F satisfying the strong Cayley equation*

$$\{F, F\} \equiv 0.$$

The unique Cayley form F_2 such that $\{F_2, F_2\} \equiv 0 \pmod{Q}$ is harmonic.

Even worse occurs for the honest Cayley forms of two skew lines, or of the twisted cubic curve: there is no tangential Cayley form F satisfying the strong Cayley equation

$$\{F, F\} \equiv 0;$$

moreover, the unique Cayley form F_2 such that $\{F_2, F_2\} \equiv 0 \pmod{Q}$ is not harmonic.

In the case of a smooth plane conic curve, instead, the harmonic representative satisfies the strong Cayley equation.

Proof. Take the tangential Cayley form of the quadric surface

$$S = \{x \mid x_0x_1 - x_2x_3 = 0\}.$$

A direct calculation shows that a Cayley form is given by

$$F := (p_{01} + p_{23})^2 + 4p_{03}p_{12},$$

and that

$$\{F, F\} = 8F.$$

We obtain (since then $A = 0$, $G = -2$) that

$$\{F_2, F_2\} = 8Q.$$

Hence, $F_2 = F - 2Q$, and $\Delta(F_2) = \Delta(F - 2Q) = 6 - 6 = 0$.

Actually, as pointed out by Dolgachev, if we are starting from a quadric surface that is diagonal with equation $\sum_i a_i x_i^2 = 0$, the corresponding form is $F = \sum_{ij} a_i a_j p_{ij}^2$, which is directly seen to be harmonic.

In the case of the honest Cayley form of a conic, a Cayley form is easily calculated as

$$F := p_{02}^2 + 4p_{01}p_{12},$$

which is easily seen to be harmonic and to satisfy the strong Cayley equation.

If we instead take two skew lines, then a Cayley form is

$$F := p_{01}p_{23},$$

satisfying $\Delta F = 1$, $\{F, F\} = 2F$. Hence, its harmonic representative is $F - \frac{1}{3}Q$, while $F_2 = F - \frac{1}{2}Q$, which satisfies $\{F_2, F_2\} = \frac{1}{2}Q$.

In the case of the twisted cubic curve, a Cayley form F is obtained as the determinant of the following symmetric matrix:

$$\begin{pmatrix} p_{01} & p_{02} & p_{03} \\ p_{02} & p_{12} + p_{03} & p_{13} \\ p_{03} & p_{13} & p_{23} \end{pmatrix}.$$

An easy calculation shows that

$$\Delta(F) = p_{12} + p_{03}.$$

Hence, if $F = F_0 + QF_1$ is the harmonic decomposition of F, then $4F_1 = \Delta(F) = p_{12} + p_{03}$.

We skip the rest of the explicit calculations, using a limiting argument: the twisted cubic admits as a limit a chain of three lines, with Cayley form

$$F := p_{01}p_{02}p_{23},$$

we get that

$$\{F, F\} = 2Fp_{02} \quad \Rightarrow \quad F_2 = F - \tfrac{1}{3}p_{02}Q,$$

and, hence,

$$\{F_2, F_2\} = -\tfrac{2}{3}Q\{F, p_{02}\} + \tfrac{4}{9}Qp_{02}^2 = 0 + \tfrac{4}{9}Qp_{02}^2.$$

Finally, observing that (since F has degree 3) F_2 is here unique,

$$\Delta(F_2) = \Delta(F - \tfrac{1}{3}p_{02}Q) = p_{12} + p_{03} - \tfrac{4}{3}p_{02}.$$

□

5. Equations for honest Cayley forms

In the previous sections we have shown that the space of Cayley forms is a projective variety defined by quadratic equations.

Our geometrical explanation shows also that in this variety there are three sets:

(1) the closed set of honest Cayley forms (the Cayley forms of some curve C in $\mathbb{P}^3$);

(2) the closed set of dual honest Cayley forms (the Cayley forms of the developable surface S dual to some curve C' in $(\mathbb{P}^3)^\vee$);

(3) the open set of tangential and dual tangential Cayley forms (here S, $S^\vee$ are both surfaces).

We are, therefore, looking for equations that define the smaller closed sets, in particular the first one.

A simple way to obtain such equations is to observe that, while for honest Cayley forms the Cayley 3-fold Z contains the $\mathbb{P}^2_x$ determined by L, for a tangential Cayley 3-fold this space is contained in TZ_L (indeed, $TZ_L \cap Q = \mathbb{P}^2_x \cup \mathbb{P}^2_\pi$), but, according to Proposition 3.4, the second derivative of F does not vanish on $\mathbb{P}^2_x$ for general $L \in Z$.

Therefore, we want that, for $L \in Z = \{L \mid Q(L) = F(L) = 0\}$, the quadratic form $D^2F(L)(p, p)$ associated to the Hessian matrix of F vanishes identically on

$$\mathbb{P}^2_x = \{p \mid p \wedge x(L) = 0\}.$$

To have explicit equations use the following elementary lemma.

Lemma 5.1. *Let $L, L' \in Q$ be two coplanar lines in $\mathbb{P}^3$ such that the plane π spanned by them does not contain the point e_0. Then, letting x be the intersection point of the two lines, the plane $\mathbb{P}^2_x$ has as basis L, L' and $L'' = e_o \wedge x$.*

Writing $L'' = \sum_{i=1}^{3} y_i e_0 \wedge e_i = e_0 \wedge y$, we obtain that the Plücker coordinates y_i of L'' are bilinear functions of L, L'.

Proof. $e_o \wedge x$ is not contained in π, hence does not belong to the line $\Gamma = L * L'$, and the first assertion is proven.

Write $L'' = \sum_{i=1}^{3} y_i e_0 \wedge e_i = e_0 \wedge y$; then, $L'' = e_o \wedge x$ if and only if it contains x, or, equivalently, if and only if L'' is coplanar with L and with L', i.e. we have that

$$\begin{aligned} y_1 L_{23} - y_2 L_{13} + y_3 L_{12} = 0, \\ y_1 L'_{23} - y_2 L'_{13} + y_3 L'_{12} = 0. \end{aligned}$$

The second assertion then follows from Cramer's rule,

$$y_1 = L_{13} L'_{12} - L'_{13} L_{12}, \qquad y_2 = L_{23} L'_{12} - L'_{23} L_{12}, \\ y_3 = L_{13} L'_{12} - L'_{13} L_{12}.$$

□

We can now apply the lemma for the lines $L \in Z$, $L' := \nabla F(L)$, obtain a third line L'', which together with L, L' yields a basis of $\mathbb{P}^2_x$, under the assumption that F satisfies the weak Cayley equation, i.e. is a Cayley form.

Then, since the line $\Gamma = L * L'$ is contained in Z, we automatically obtain that

$$D^2 F(L)(L, L) = D^2 F(L)(L, L') = D^2 F(L)(L', L') = 0.$$

Hence, the next theorem follows immediately.

Theorem 5.2. *Let F be a Cayley form. Then, F is a honest Cayley form if, moreover, for each $L \in Z$, the following equation holds:*

$$D^2 F(L)(L'', L) = D^2 F(L)(L'', L') = D^2 F(L)(L'', L'') = 0.$$

I.e. if and only if the above three polynomials, whose coefficients have degree 2 or 3 in the coefficients of F, belong to the ideal (Q, F) of Z.

Proof. The entries of the matrix $D^2 F(L)$ are linear in the coefficients of F, as well as the coordinates of L', while the coordinates of L are homogeneous of degree 0 in the coefficients of F. Since the Plücker coordinates y_i of L'' are bilinear functions of L, L', they are linear in the coefficients of F.

Hence, the three equations are homogeneous in the coefficients of F, of respective degrees 2, 3, 3. □

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The next natural question is whether we can obtain from the above theorem equations which hold mod(Q): we show that the answer is negative already in the example of a chain of three lines.

In this case, as we observed, a Cayley form is

$$F := p_{01}p_{02}p_{23},$$

and F_2 is here unique, equal to

$$F_2 = F - \tfrac{1}{3}p_{02}Q.$$

We set $L := p$; hence, $L' = \nabla F - \frac{1}{3}p_{02}\nabla Q - \frac{1}{3}Q\nabla p_{02}$, and the equations determining L'' are

$$\langle L'', L\rangle = 0, \qquad 0 = \langle L'', \nabla F - \tfrac{1}{3}Q\nabla p_{02}\rangle = y_1p_{02}p_{23} + y_2p_{01}p_{23} - \tfrac{1}{3}Qy_2.$$

These yield (modulo Q) that

$$y_1p_{02} + y_2p_{01} = 0, \qquad y_3p_{12} + (p_{01}p_{23} + p_{02}p_{13}) = 0;$$

hence, as solution (modulo Q),

$$y_1 = p_{01}p_{12}, \qquad y_2 = -p_{02}p_{12}, \qquad y_3 = -(p_{01}p_{23} + p_{02}p_{13}).$$

Observe now that, denoting by $Q(q, q')$ the bilinear form associated to Q, namely, $Q(q, q') := \langle q, q'\rangle$, we have $Q(L'', L'') \equiv 0$ and also $Q(L, L'') \equiv Q(L', L'') \equiv 0 \pmod Q$. Furthermore, $Q(L, L') \equiv 0$ on Q (and also $Q(L', L') \equiv 0$, since we use F_2 for defining L'), while $Q(L, L) \equiv 0$ holds tautologically on Q.

Since we are considering a point $L = p \in Q$, when we look at the equation $D^2F_2(L)(L'', L) = D^2F_2(L)(L'', L') = D^2F_2(L)(L'', L'') = 0$, we may replace it by the simpler equation $D^2F(L)(L'', L) = D^2F(L)(L'', L') = D^2F(L)(L'', L'') = 0$. Because

$$D^2(p_{02}Q)(q, q') = 2p_{02}Q(q, q') + q_{02}Q(p, q') + q'_{02}Q(p, q).$$

Now, whereas

$$\begin{aligned}\tfrac{1}{2}D^2F(L)(L'', L) &= p_{01}[y_2p_{23}] + p_{02}[y_1p_{23}] + p_{23}[y_2p_{01} + y_1p_{02}] \equiv 0,\\ \tfrac{1}{2}D^2F(L)(L'', L'') &= p_{23}[y_1y_2] = -p_{12}^2p_{23}p_{01}p_{02} = -p_{12}^2F,\end{aligned}$$

which is not identically 0 modulo Q. We have therefore shown the following.

Proposition 5.3. *Consider the equation in Theorem 5.2 for a honest Cayley form:*

$$D^2F(L)(L'', L) = D^2F(L)(L'', L') = D^2F(L)(L'', L'') = 0.$$

If we take a chain C of three lines in $\mathbb{P}^3$, then the representative F_2 is unique, and for any choice of a Cayley form for C these equations belong to the ideal (Q, F) of Z, but not to the ideal of Q.

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