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CLASSICAL ISOL INCOMPARABILITY AND $\infty \cdot$ ON MANIFOLD RET'S

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Abstract

An infinite collection of indecomposable isols such that no isol is comparable to certain infinitary combinations of the others is constructed, extending a result of Dekker and Myhill. This collection is then used to investigate differences between the arithmetic of classical RET's and that of RET's on recursive manifolds, a difference relevant to the manifold equivalent of the Schröder–Bernstein Theorem.

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0. Introduction

Dekker and Myhill, in [1], developed the classical theory of recursive equivalence types (RET's) as a constructive analogue to the theory of cardinal numbers. In the context of RET's, they proved a parallel to the Schröder–Bernstein theorem by first establishing

0.1. PROPOSITION. *For any* RET*'s* $[C]$ *and* $[D]$, $[C] = [C] + [D]$ *if and only if* $[C] \geqslant \infty \cdot [D]$.

(This, of course, is a parallel to the property of cardinal numbers that $\lambda = \lambda + \kappa$ if and only if $\lambda \geqslant \aleph_0 \cdot \kappa$.)

The situation becomes different when the study of RET's is generalized to the setting of recursive manifolds. Manifolds own a theory of RET's very similar to

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the classical theory in many respects, but [2] showed that on a manifold there might not be an operation that behaves as $\infty\cdot$ does in Proposition 0.1. That result from [2], however, did not necessarily point to a deep difference between classical and manifold RET's. Rather, it was demonstrated by a trivial counterexample based on the simple fact that, in any reasonable manifold analogue of Proposition 0.1, two operations like $\infty\cdot$ should be involved. This paper strengthens the result in [2] by showing that even with two such operations, Proposition 0.1 does not carry over to manifolds. The construction of the counterexample here is not at all trivial; a significant difference in the workings of classical and manifold RET's seems to be indicated. Section 2 of this paper makes precise the ideas discussed above, and it details the counterexample's construction. This construction takes place over the simplest recursive manifold allowing a theory of RET's not identical to the classical theory: RET's on less simple manifolds will behave, if anything, even less like the classical ones. The results in Section 1 are necessary for Section 2, but Section 1 deals exclusively with classical recursion theory. In particular, Section 1 centers around the construction of a collection of isols with rather strong incomparability properties.

NOTATION. N, the set of natural numbers, is taken to include 0 as an element. For $T \subseteq N$, the RET of T is denoted $[T]$. The recursive pairing function $x + (x^2 + y^2 + 2xy + x + y)/2$ will be written $\langle x, y\rangle$. For S, $T \subseteq N$, $\langle S, T\rangle$ denotes $\{\langle m, n\rangle \mid m \in S \text{ and } n \in T\}$. Also, we use $\langle x, T\rangle$ and $\langle S, y\rangle$ as abbreviations for $\langle \{x\}, T\rangle$ and $\langle S, \{y\}\rangle$, respectively. Following [1], $\infty \cdot [T]$ is defined as $[\langle N, T\rangle]$ (or, equivalently, $[\langle T, N\rangle]$). With $\langle x, y, z\rangle$ defined as $\langle\langle x, y\rangle, z\rangle$, $\langle\cdot,\cdot,\cdot\rangle$ provides a bijection from N^3 to N. Other notation for classical recursion-theoretic notions will be as in [1]. We discuss recursive manifold notation in the next paragraphs.

MANIFOLDS. A recursive manifold, in general, consists of a base set and a collection of functions mapping N into that set. The manifold we will use in Section 2 has, at its base set, the Cartesian product N^2. For each $p \in N$, define the function α_p: $N \to N^2$ by $\alpha_p(n) = (p, n)$. The collection $\mathscr{A} = \{\alpha_p\}_{p \in N}$ is called the *atlas* of our manifold. A_p will denote the range of α_p; thus $A_p = \{p\} \times N$ and $N^2 = \bigcup_{p \in N} A_p$. The A_p's are called *patches* of the manifold.

A subset C of N^2 is *$\mathscr{A}$-recursive* (*$\mathscr{A}$-rec.*, for short), if and only if for every $p \in N$, $\alpha_p^{-1}(C)$ is recursive. Similarly, C is *$\mathscr{A}$-recursively enumerable* (*$\mathscr{A}$-r.e.*, for short) if and only if for every $p \in N$, $\alpha_p^{-1}(C)$ is r.e.. A function f: $N^2 \to N^2$ is *$\mathscr{A}$-$\mathscr{A}$-partial recursive* (*$\mathscr{A}$-$\mathscr{A}$-p.r.*, for short) if and only if both

(a) dom f, the domain of f, is $\mathscr{A}$-r.e., and

(b) given p, $q \in N$, the set $\alpha_p^{-1}(f^{-1}(A_q))$ is the domain of a p.r. function $f_{p,q}$ satisfying $\alpha_q \circ f_{p,q} = f \circ \alpha_p$.

Also, $f\colon N^2 \to N^2$ is called *$\mathscr{A}$-$\mathscr{A}$-bounded* (respectively *$\mathscr{A}$-$\mathscr{A}$-compact*) if and only if for each $p \in N$, $f(A_p)$ (respectively, $f^{-1}(A_p)$) is contained in the union of finitely many A_q's. An *$\mathscr{A}$-$\mathscr{A}$-embedding* is a function which is 1-1, $\mathscr{A}$-$\mathscr{A}$-p.r., $\mathscr{A}$-$\mathscr{A}$-bounded, and $\mathscr{A}$-$\mathscr{A}$-compact.

For subsets C and D of N^2, C is *recursively equivalent* to D (denoted $C \simeq D$) if and only if there exists an $\mathscr{A}$-$\mathscr{A}$-embedding f with $C \subseteq \operatorname{dom} f$ and $f(C) = D$. (With such C, D, f, we also say that C is recursively equivalent to D *via* f.) The terminology is justified; $\simeq$ can be shown to be an equivalence relation. The equivalence class containing C is denoted $[C]$ and is referred to as C's *recursive equivalence type* (*RET*).

Subsets C and D of N^2 are called *separable* if and only if there exist disjoint $\mathscr{A}$-r.e. sets E and F with $C \subseteq E$ and $D \subseteq F$. Addition of RET's is defined by $[C] + [D] = [C^* \cup D^*]$, where C^* and D^* are separable sets with $C^* \in [C]$ and $D^* \in [D]$. It is easily shown that addition is well-defined. In particular, $[C] + [D] = [C \oplus D]$, where $C \oplus D$ is defined as $\bigcup_{p \in N} \alpha_p(\{2n \mid \alpha_p(n) \in C\} \cup \{2n + 1 \mid \alpha_p(n) \in D\})$.

Parallelling the classical notation, we write $[C] \leqslant [D]$ if and only if there exists E with $[C] + [E] = [D]$. However, despite the notation, it has not yet been established that this relation $\leqslant$ is actually a partial order. Specifically, the anti-symmetry of $\leqslant$ represents a yet-to-be-proved manifold analogue of the Schröder–Bernstein theorem that could have been proved easily if Proposition 0.1 carried over to manifolds.

A subset C of N^2 is called *$\mathscr{A}$-bounded* if and only if C is contained in the union of finitely many A_q's. If C is $\mathscr{A}$-bounded, then C is recursively equivalent to a set of the form $\alpha_p(D)$ for some $p \in N$ and $D \subseteq N$. Further, if (for p_1, $p_2 \in N$ and $D_1, D_2 \subseteq N$), $[C] = [\alpha_{p_1}(D_1)]$ and $[C] = [\alpha_{p_2}(D_2)]$, then D_1 and D_2 are (classically) recursively equivalent, i.e., $[D_1] = [D_2]$. Thus φ, given by $\varphi([\alpha_p(D)]) = [D]$, is a well-defined function from the collection of RET's of $\mathscr{A}$-bounded subsets of the manifold to the collection of (classical) RET's over N. In fact, φ is easily shown to be an additive isomorphism.

1. Incomparability properties of isols

As stated earlier, this section is concerned exclusively with classical recursion theory. The main construction in this section will use a typical "satisfy-requirements-in-stages" approach in order to obtain this proposition.

1.1. PROPOSITION. *There exists a sequence of sets of natural numbers $\{E_i\}_{i=0}^{\infty}$ such that*

(a) *for each i, $[E_i]$ is an infinite indecomposable isol;*

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(b) *for each i and each* $j \neq i$, $[E_i] \nleq \infty \cdot [E_j]$;
(c) *for each* i, $[E_i] \nleq [\bigcup_{j \neq i} \langle j, E_j \rangle]$.

From Proposition 1.1, the following result, central to the main construction of Section 2, will follow routinely.

1.2. PROPOSITION. *There exists a sequence of sets of natural numbers* $\{B_i\}_{i=0}^{\infty}$ *such that*

(a) *for each* i, $[B_i]$ *is an infinite indecomposable isol with* $B_i \subseteq \langle i, N \rangle$;
(b) *for each* i, *if* $m \in N$ *and* J *is a finite subset of* $N \setminus \{i\}$, *then* $[B_i] \nleq \infty \cdot [\bigcup_{j \in J} B_j] + m \cdot [\bigcup_{j=i+1}^{\infty} B_j]$.

We begin by recalling a definition and proving an easy lemma.

1.3. DEFINITION. An infinite set B is *cohesive* if and only if for every r.e. set T, either $B \cap T$ or $B \setminus T$ is finite.

1.4. LEMMA. *Let* C^* *be cohesive, and let* $C = \{2n \mid n \in C^*\}$. *Then* C *is also cohesive.*

PROOF. The proof is trivial. Let an r.e. set T be given. If we define T^* as $\{n \mid 2n \in T\}$, then T^* is r.e. Since C^* is cohesive, either $C^* \cap T^*$ or $C^* \setminus T^*$ is finite. But $C \cap T = \{2n \mid n \in C^* \cap T^*\}$, and $C \setminus T = \{2n \mid n \in C^* \setminus T^*\}$. Thus, either $C \cap T$ or $C \setminus T$ is finite.

By Theorem 43 of [1], cohesive sets exist. Fix a cohesive set C^*, and let C be defined as in Lemma 1.4. The letter C will be reserved for this particular set throughout the rest of Section 1. Notice that if $B \subseteq C$, then B is either finite or cohesive. In particular we have

1.5. REMARK. If $B \subseteq C$, then $[B]$ will be an indecomposable isol.

One other piece of notation needs to be introduced. Throughout Section 1, $\{f_n\}_{n=0}^{\infty}$ will denote a fixed (non-effective) listing of all 1-1 p.r. functions f such that $C \cap \operatorname{dom} f$ is infinite.

Before proceeding to the formal details of the construction of the sequence $\{E_i\}$ of Proposition 1.1, we first make a few general remarks about the plan of attack. The sets E_i will be subsets of C and will be defined in stages inductively (by a non-effective construction, of course). At stage r, we will define two sequences of sets, $\{E_{i,r}\}_{i=0}^{\infty}$ and $\{F_{i,r}\}_{i=0}^{\infty}$. The idea is that $E_{i,r}$ (respectively $F_{i,r}$) will be the set of numbers which, up through stage r, have been earmarked to belong to E_i (respectively, to be *forbidden* membership in E_i.) Naturally, the action taken at stage r will help to ensure that the sets E_i have the desired properties.

At stage 0, each $E_{i,0}$ will be empty. Thereafter at stage $s+1$, $E_{i,s+1}$ will equal $E_{i,s}$ for all but at most one choice of i. Further, if $E_{i,s+1} \neq E_{i,s}$, then $E_{i,s+1}$ will equal $E_{i,s} \cup \{x\}$ for some $x \in N$. Thus for any fixed stage r, we have obtained inductively that only finitely many $E_{i,r}$ will be nonempty and that all $E_{i,r}$ will be finite.

The preceding paragraph will also apply, *mutatis mutandis*, to the sets $F_{i,r}$.

Here are the details.

1.6. Inductive construction of the sequences $\{E_{i,r}\}$ and $\{F_{i,r}\}$.

Stage $r = 0$.

For all i, let $E_{i,0}$ and $F_{i,0}$ be empty.

Stage $r = s + 1$.

There will be several cases, but in all cases, start by tentatively setting $E_{k,r} = E_{k,s}$ and $F_{k,r} = F_{k,s}$ for each k.

Case I. $s = \langle i, j, 0\rangle$ for some i, j.

In this case, pick any $x \in C \setminus (E_{i,s} \cup F_{i,s})$. (Since C is infinite and $E_{i,s} \cup F_{i,s}$ is finite, such an x will exist.) Change $E_{i,r}$ to $E_{i,s} \cup \{x\}$.

Case II. $s = \langle i, n, 1\rangle$ for some i, n.

In this case, pick any $x \in C \setminus F_{i,s}$ such that $f_n(x) \notin \bigcup_{j \neq i} \langle j, E_{j,s}\rangle$. (Such an x will exist since $C \cap \operatorname{dom} f_n$ is infinite, $F_{i,s}$ is finite, $\bigcup_{j \neq i} \langle j, E_{j,s}\rangle$ is finite, and f_n is 1-1.) Thus $f_n(x) = \langle y, z\rangle$ for some y, z with either $y = i$ or $z \notin E_{y,s}$. Change $E_{i,r}$ to $E_{i,s} \cup \{x\}$, and, if $y \neq i$, change $F_{y,r}$ to $F_{y,s} \cup \{z\}$.

Case III. $s = \langle i, j, n+2\rangle$ for some i, j, n.

Subcase (i). $i = j$.

If $i = j$, just proceed to the next stage.

Subcase (ii). $i \neq j$, and for all $x \in C \setminus F_{i,s}$, $f_n(x)$ belongs to $\langle E_{j,s}, N\rangle$. If this subcase holds, also just go to the next stage.

Subcase (iii). $i \neq j$, and there exists $x \in C \setminus F_{i,s}$ with $f_n(x) \notin \langle E_{j,s}, N\rangle$. If this condition holds, fix one such x. Thus $f_n(x) = \langle y, z\rangle$ for some y, z with $y \notin E_{j,s}$. Change $E_{i,r}$ to $E_{i,s} \cup \{x\}$ and change $F_{j,r}$ to $F_{j,s} \cup \{y\}$.

This completes the description of the construction of the sequences $\{E_{i,r}\}$ and $\{F_{i,r}\}$. We are now in a position to establish Proposition 1.1.

1.7. Proof of Proposition 1.1. Let $\{E_{i,r}\}$ and $\{F_{i,r}\}$ be as in Construction 1.6. For each i, define E_i to be $\bigcup_{r=0}^{\infty} E_{i,r}$. We shall verify that the sets E_i satisfy the properties listed in Proposition 1.1.

(a) Let $i \in N$ be given. For each j, stage $\langle i, j, 0\rangle$ contributes a new element towards E_i according to the specifications of Construction 1.6, Case I. Thus E_i is infinite. Furthermore, $E_i \subseteq C$ by the construction. Therefore, by Remark 1.5, $[E_i]$ is an indecomposable isol.

(b) Let $i, j \in N$ (with $j \neq i$) be given. We shall assume that $[E_i] \leqslant \infty \cdot [E_j]$ and derive a contradiction. If $[E_i] \leqslant \infty \cdot [E_j]$, then there exists D^* with $[E_i] + [D^*] = \infty \cdot [E_j]$. Let $D = \{2m + 1 | m \in D^*\}$. Then $[D^*] = [D]$, and E_i and D are separable (since all elements of C, and thus of E_i, are even.) So $\infty \cdot [E_j] = [E_i] + [D] = [E_i \cup D]$, and there exists a 1-1 p.r. function f with $\operatorname{dom} f \supseteq E_i \cup D$ and $f(E_i \cup D) = \langle E_j, N\rangle$. Since $\operatorname{dom} f$ contains E_i, an infinite subset of C, it follows that f equals f_n for some value of n. Let $s = \langle i, j, n+2\rangle$ and consider what happened at stage $r = s + 1$ according to Construction 1.6, Case III.

Obviously, subcase (i) did not hold since $i \neq j$.

If subcase (ii) held, we know that $x \in C \setminus F_{i,s}$ implies that $f(x) = f_n(x) \in \langle E_{j,s}, N\rangle$. But $F_{i,s} \cap E_i$ is empty, so $f(E_i) \subseteq \langle E_{j,s}, N\rangle$. Define B as $D \cap f^{-1}(\langle E_{j,s}, N\rangle)$. Then E_i and B are separable, since $B \subseteq D$, and D is separable from E_i. Furthermore, $E_i \cup B = f^{-1}(\langle E_{j,s}, N\rangle)$, which is r.e. because $E_{j,s}$ is finite. So E_i itself is r.e., contradicting the fact that $[E_i]$ is an infinite isol.

If subcase (iii) held, let x, y, and z be as in the description of this subcase in Construction 1.6. At stage r we ensured that $x \in E_i$. Also, at this stage, y had not yet been included as a contribution towards E_j; by placing y on the "forbidden list" $F_{j,r}$, we kept y out of E_j. But then $f(x) = f_n(x) = \langle y, z\rangle \notin \langle E_j, N\rangle$. This contradicts the statement that $f(E_i \cup D) = \langle E_j, N\rangle$.

So all possibilities lead to contradictions, and the assumption $[E_i] \leqslant \infty \cdot [E_j]$ is false.

(c) Let $i \in N$ be given. We shall use a proof by contradiction similar to that used in handling subcase (iii) above. Assume that $[E_i] \leqslant [\bigcup_{j \neq i} \langle j, E_j\rangle]$. Then there exists a 1-1 p.r. function f with $\operatorname{dom} f \supseteq E_i$ and $f(E_i) \subseteq \bigcup_{j \neq i} \langle j, E_j\rangle$. Since $C \cap \operatorname{dom} f$ contains the infinitely many members of E_i, f must equal f_n for some n. Consider stage $r = s + 1$, where $s = \langle i, n, 1\rangle$. At this stage, Construction 1.6, Case II applied; let x, y, and z be as specified in the description of that case. Thus either $y = i$ or $z \notin E_{y,s}$.

If $y = i$, then $f(x) = f_n(x) = \langle i, z\rangle \notin \bigcup_{j \neq i} \langle j, E_j\rangle$; since at stage r we ensured that $x \in E_i$, this contradicts the statement that $f(E_i) \subseteq \bigcup_{j \neq i} \langle j, E_j\rangle$.

On the other hand, if $y \neq i$, then $z \notin E_{y,s}$. Furthermore, at this stage z was added to the forbidden list so that $z \notin E_y$. But then $f(x) = \langle y, z\rangle \notin \bigcup_{j \neq i} \langle j, E_j\rangle$, and again we have a contradiction to $f(E_i) \subseteq \bigcup_{j \neq i} \langle j, E_j\rangle$.

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Thus, the assumption $[E_i] \leqslant [\bigcup_{j\neq i}\langle j, E_j\rangle]$ ultimately leads to contradiction and is therefore false; the proof of Proposition 1.1 is now complete.

Proposition 1.2 is now a straightforward consequence of Proposition 1.1.

1.8. Proof of Proposition 1.2. Let $\{E_i\}$ be as given by Proposition 1.1 and, for each i, define B_i to be $\langle i, E_i\rangle$.

(a) Clearly, each $B_i \subseteq \langle i, N\rangle$, and $[B_i] = [E_i]$, so that each $[B_i]$ is an infinite indecomposable isol.

(b) Let $i, m \in N$ and let $J = \{j_1, \dots, j_p\} \subseteq N \setminus \{i\}$ be given. We shall reach a contradiction from the assumption that $[B_i] \leqslant \infty\cdot[\bigcup_{j\in J} B_j] + m\cdot[\bigcup_{j=i+1}^{\infty} B_j]$. If this assumption holds, we may rewrite the right-hand side and obtain

$$[B_i] \leqslant \infty\cdot\left[B_{j_1}\right] + \cdots + \infty\cdot\left[B_{j_p}\right] + \left[\bigcup_{j=i+1}^{\infty} B_j\right] + \cdots + \left[\bigcup_{j=i+1}^{\infty} B_j\right].$$

Then Theorem 15(m) of [1] implies the existence of sets $D_1, \dots, D_{p+m}$ such that

$$[B_i] = [D_1] + \cdots + \left[D_{p+m}\right], \tag{1.9}$$

$$[D_k] \leqslant \infty\cdot\left[B_{j_k}\right] \quad \text{for } 1 \leqslant k \leqslant p, \text{ and} \tag{1.10}$$

$$[D_k] \leqslant \left[\bigcup_{j=i+1}^{\infty} B_j\right] \quad \text{for } p+1 \leqslant k \leqslant p+m. \tag{1.11}$$

Since $[B_i]$ is an infinite indecomposable isol, (1.9) implies that exactly one D_k is infinite. For this k, then, $[D_k] = [B_i] - [S]$, where S is some finite set.

If $1 \leqslant k \leqslant p$, then we have, by (1.10), that $[B_i] - [S] = [D_k] \leqslant \infty\cdot[B_{j_k}]$; thus $[B_i] \leqslant \infty\cdot[B_{j_k}] + [S] = \infty\cdot[B_{j_k}]$. But then $[E_i] \leqslant \infty\cdot[E_{j_k}]$ with $j_k \neq i$, contradicting Proposition 1.1(b).

On the other hand, if $p+1 \leqslant k \leqslant p+m$, then (1.11) says that $[B_i] - [S] = [D_k] \leqslant [\bigcup_{j=i+1}^{\infty} B_j]$. By standard RET arithmetic, one of two alternatives follows: either $[B_i] \leqslant [\bigcup_{j=i+1}^{\infty} B_j]$, or $[B_i] = [\bigcup_{j=i+1}^{\infty} B_j] +$ a finite RET. If the former alternative held, then $[E_i] \leqslant [\bigcup_{j=i+1}^{\infty}\langle j, E_j\rangle]$, contradicting Proposition 1.1(c). And the latter alternative certainly cannot hold, since $[B_i]$ is indecomposable.

Therefore, the original assumption is always contradicted, and the proposition is proved.

2. The main manifold construction

Recall the classical result of Dekker and Myhill that was given earlier as Proposition 0.1. We restate it here.

2.1. Proposition. *For any* RET's $[C]$ *and* $[D]$, $[C] = [C] + [D]$ *if and only if* $[C] \geqslant \infty \cdot [D]$.

In the setting of recursive manifolds, this proposition will not carry over in general, at least, not if we hope to find, for each RET $[D]$ on a manifold, a corresponding RET $\infty \cdot [D]$ that behaves as in Proposition 2.1. Here is a counterexample from [2].

2.2. Example. Consider the manifold with base set N^2 described in Section 0. On this manifold, let $D = \{(0,0)\}$, $C_1 = \{0\} \times N$ (i.e., the patch A_0), and $C_2 = N \times \{0\}$. Then $[C_1] + [D] = [C_1]$, and $[C_2] + [D] = [C_2]$. Assume that for some $T \subseteq N^2$, we have $[C] + [D] = [C]$ if and only if $[C] \geqslant [T]$. Then $[C_1] \geqslant [T]$ and $[C_2] \geqslant [T]$; it easily follows that T must be finite. But now we have $[T] \geqslant [T]$, and yet $[T] + [D] \neq [T]$, contradicting the assumption. Thus for $D = \{(0,0)\}$, there is no $[T]$ such that $[C] + [D] = [C]$ if and only if $[C] \geqslant [T]$.

The crux of Example 2.2 lies in the fact that we have two *different* RET's, $[C_1]$ and $[C_2]$, each of which is qualified, in a natural way, to be called "$\infty \cdot [D]$". The set C_1 consists of infinitely many copies of D, all placed within D's patch A_0. On the other hand, a single copy of D in each patch gives rise to C_2. In other words, we could call $[C_1]$ an "in-depth $\infty \cdot [D]$" and $[C_2]$ an "in-breadth $\infty \cdot [D]$".

On our manifold, with its infinitely many patches, this "in-depth vs. in-breadth" phenomenon must be taken into account. It thus becomes reasonable to investigate a modified version of Proposition 2.1 which allows for that phenomenon. If such a modified Proposition 2.1 held, the manifold's RET's might then be seen to behave similarly to RET's on N, especially with regards to a Schröder–Bernstein type of property. However, as we shall see, the differences between the two kinds of RET's run deeper than can be reconciled so easily. We first need some preliminaries in order to be able to state precisely Proposition 2.1's modification, which will appear in conjectural form as Question 2.6.

2.3. Definition. For any $D \subseteq N^2$, define
(i) $\infty_1 \cdot D$ to be $\bigcup_{p \in N} \alpha_p(\langle \alpha_p^{-1}(D), N \rangle)$, and
(ii) $\infty_2 \cdot D$ to be $\bigcup_{p,q \in N} \alpha_{\langle p,q \rangle}(\alpha_p^{-1}(D))$.

Figure 1 gives a rough indication of how $\infty_1 \cdot$ and $\infty_2 \cdot$ behave. Both of these operations preserve recursive equivalence, as we now prove.

2.4. Proposition. *Let* B, C *be subsets of* N^2 *with* $B \simeq C$. *Then*
(i) $\infty_1 \cdot B \simeq \infty_1 \cdot C$,
(ii) $\infty_2 \cdot B \simeq \infty_2 \cdot C$.

PROOF. Let f be a function with dom $f \supseteq B$ such that $B \simeq C$ via f. Define the function g on $\infty_1 \cdot \operatorname{dom} f$ such that if $f(\alpha_p(m)) = \alpha_q(n)$, then $g(\alpha_p(\langle m, k\rangle)) = \alpha_q(\langle n, k\rangle)$. It is then readily shown that $\infty_1 \cdot B \simeq \infty_1 \cdot C$ via g. Similarly, $\infty_2 \cdot B \simeq \infty_2 \cdot C$ via h, where h is the function defined on $\infty_2 \cdot \operatorname{dom} f$ such that if $f(\alpha_p(m)) = \alpha_q(n)$, then $f(\alpha_{\langle p,k\rangle}(m)) = \alpha_{\langle q,k\rangle}(n)$.

Proposition 2.4 allows us to make the following definition.

2.5. DEFINITION. For any $D \subseteq N^2$, define
(i) $\infty_1 \cdot [D]$ to be $[\infty_1 \cdot D]$, and
(ii) $\infty_2 \cdot [D]$ to be $[\infty_2 \cdot D]$.
For example, in Example 2.2, $[C_1]$ equalled $\infty_1 \cdot [D]$, and $[C_2]$ equalled $\infty_2 \cdot [D]$.

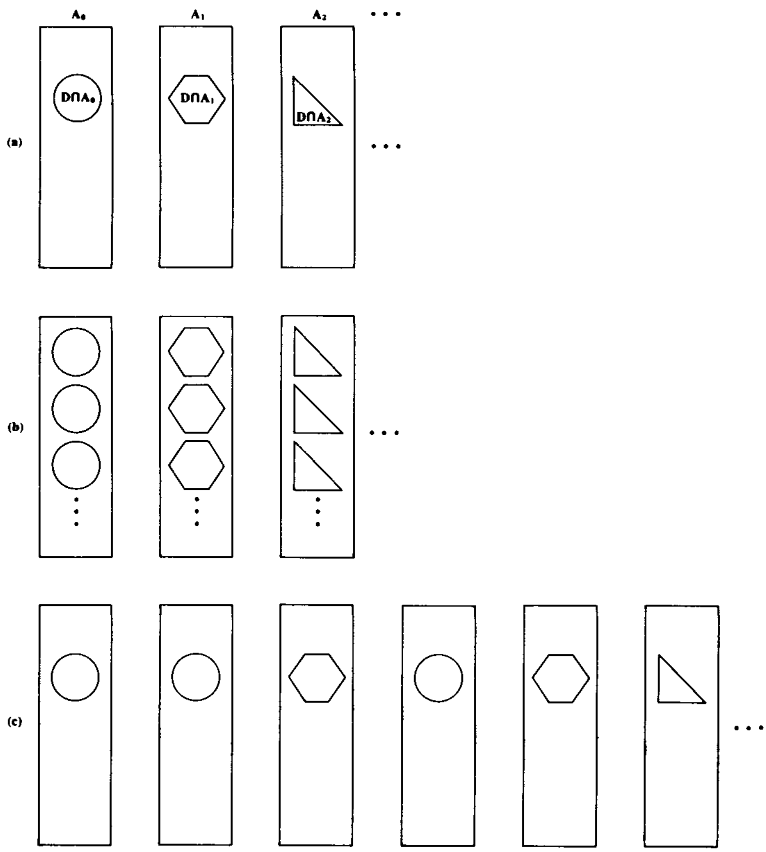

FIGURE 1

(a) A set $D \subseteq N^2$ (b) $\infty_1 \cdot D$ (c) $\infty_2 \cdot D$

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We may now state conjecturally the modification to Proposition 2.1 suggested by Example 2.2.

2.6. QUESTION. Let $C, D \subseteq N^2$ with $[C] + [D] = [C]$. Must $[D]$ have a decomposition $[D] = [E] + [F]$ such that $[C] \geqslant \infty_1 \cdot [E] + \infty_2 \cdot [F]$?

When D is as in Example 2.2, the answer is affirmative. It can be easily shown that any $[C]$ which additively absorbs $[D]$ satisfies either $[C] \geqslant \infty_1 \cdot [D]$ $(= \infty_1 \cdot [D] + \infty_2 \cdot [\varnothing])$ or $[C] \geqslant \infty_2 \cdot [D]$ $(= \infty_1 \cdot [\varnothing] + \infty_2 \cdot [D])$. In general, however, the answer is negative. The remainder of this paper is devoted to constructing sets C and D which demonstrate this. We start with some basic properties of ∞_1 and ∞_2.

2.7. PROPOSITION. *For $B, C \subseteq N^2$, we have*
(i) $\infty_1 \cdot ([B] + [C]) = \infty_1 \cdot [B] + \infty_1 \cdot [C]$,
(ii) $\infty_2 \cdot ([B] + [C]) = \infty_2 \cdot [B] + \infty_2 \cdot [C]$.

PROOF. (i) It suffices to find a function f such that $\infty_1 \cdot (B \oplus C) \simeq \infty_1 \cdot B \oplus \infty_1 \cdot C$ via f. First, define h: $N \to N$ by

$$h(\langle m, n\rangle) = \begin{cases} 2\langle m/2, n\rangle & \text{if } m \text{ is even,} \\ 2\langle (m-1)/2, n\rangle + 1 & \text{if } m \text{ is odd.} \end{cases}$$

Then define f: $N^2 \to N^2$ by $f(\alpha_p(k)) = \alpha_p(h(k))$; this is the desired function.

(ii) This part is even simpler, since $\infty_2 \cdot (B \oplus C)$ actually equals $\infty_2 \cdot B \oplus \infty_2 \cdot C$.

2.8. PROPOSITION. (i) *If $[B] \leqslant [C]$, then $\infty_1 \cdot [B] \leqslant \infty_1 \cdot [C]$.*
(ii) *If $[B] \leqslant [C]$, then $\infty_2 \cdot [B] \leqslant \infty_2 \cdot [C]$.*

PROOF. (i) If $[B] \leqslant [C]$, then for some D, we have $[C] = [B] + [D]$. By Proposition 2.7(i), we have $\infty_1 \cdot [C] = \infty_1 \cdot [B] + \infty_1 \cdot [D]$, and so $\infty_1 \cdot [B] \leqslant \infty_1 \cdot [D]$.

(ii) This is proved similarly.

2.9. DEFINITION. For $m \in N$, let $[m]$ denote the RET comprised of all m-element subsets of N^2.

2.10. PROPOSITION. *Let $\varnothing \neq B \subseteq N^2$, and let $m \in N$. Then*
(i) $\infty_1 \cdot [B] + \infty_1 \cdot [m] = \infty_1 \cdot [B]$,
(ii) $\infty_2 \cdot [B] + \infty_2 \cdot [m] = \infty_2 \cdot [B]$.

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PROOF. (i) First note that $\infty_1 \cdot [1] = [A_0]$ (where A_0 is, as always, $\{0\} \times N$) so that $\infty_1 \cdot [1] + \infty_1 \cdot [1] = [A_0] + [A_0] = [A_0 \oplus A_0]$. But $A_0 \oplus A_0 = A_0$, so $\infty_1 \cdot [1] + \infty_1 \cdot [1] = \infty_1 \cdot [1]$. It then follows inductively that $\infty_1 \cdot [1] + \infty_1 \cdot [m] = \infty_1 \cdot [1]$ for any $m \in N$. But now, if $B \subseteq N^2$ is nonempty, we may pick $b_0 \in B$ and write

$$\begin{aligned}
\infty_1 \cdot [B] + \infty_1 \cdot [m] &= \infty_1 \cdot \left([B \backslash \{b_0\}] + [\{b_0\}]\right) + \infty_1 \cdot [m] \\
&= \infty_1 \cdot [B \backslash \{b_0\}] + \infty_1 \cdot [\{b_0\}] + \infty_1 \cdot [m] \\
&= \infty_1 \cdot [B \backslash \{b_0\}] + \infty_1 \cdot [1] + \infty_1 \cdot [m] \\
&= \infty_1 \cdot [B \backslash \{b_0\}] + \infty_1 \cdot [1] \\
&= \infty_1 \cdot [B \backslash \{b_0\}] + \infty_1 \cdot [\{b_0\}] \\
&= \infty_1 \cdot [B].
\end{aligned}$$

(ii) Observe that $\infty_2 \cdot [1] = [\{2n \mid n \in N\} \times \{0\}] = [\{2n+1 \mid n \in N\} \times \{0\}]$, so that $\infty_2 \cdot [1] + \infty_2 \cdot [1] = [N \times \{0\}] = \infty_2 \cdot [1]$. Then reasoning as in (i) completes the proof.

Armed with the preceding results, we may now embark upon the construction of sets that answer Question 2.6. We start by constructing a function f from N^2 to N^2. The definition of f will be by cases.

2.11. DEFINITION. For all values of r, k, m, and n, set
$f(\alpha_0(2r)) = \alpha_0(2r+1)$.
$f(\alpha_{3k+3}(2\langle k, n\rangle)) = \alpha_{3k+3}(2\langle k, n\rangle + 1)$, and
$f(\alpha_{3k+3}(2x)) = \alpha_{3k}(2x)$ otherwise.
$f(\alpha_{3k}(2\langle k, n\rangle + 1)) = \alpha_{3k+1}(\langle 0, n\rangle)$, and
$f(\alpha_{3k}(2x+1)) = \alpha_{3k+3}(2x+1)$ otherwise.
$f(\alpha_{3k+1}(\langle m, n\rangle)) = \alpha_{3k+1}(\langle m+1, n\rangle)$.
$f(\alpha_{3k+2}(\langle 0, n\rangle)) = \alpha_{3k}(2\langle k, n\rangle)$.
$f(\alpha_{3k+2}(\langle m+1, n\rangle)) = \alpha_{3k+2}(\langle m, n\rangle)$.

From Definition 2.11, it is immediate that f is $\mathscr{A}$-$\mathscr{A}$-rec., $\mathscr{A}$-$\mathscr{A}$-bounded, and $\mathscr{A}$-$\mathscr{A}$-compact. Not quite so immediately, but still routinely, one may verify that f is also bijective. Thus f is an $\mathscr{A}$-$\mathscr{A}$-embedding.

The definition of f is based on a pattern which is a modification of that used in [3]. The purpose of using such a pattern is to ensure that the iterated action of f behaves as in the next proposition.

2.12. PROPOSITION. *Let $k \in N$ and $S \subseteq N$, and let f be as in Definition 2.11. Define T to be $\{2\langle k, s\rangle + 1 \mid s \in S\}$. Then*

$$\alpha_m^{-1}\left(\bigcup_{n=0}^{\infty} f^n(\alpha_0(T))\right) = \begin{cases} T & \text{if } m = 0, 3, 6, \ldots, 3k, \\ \langle N, S\rangle & \text{if } m = 3k+1, \\ \varnothing & \text{otherwise}. \end{cases}$$

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PROOF. By the definitions of f and T, we have

$$
\begin{aligned}
f^0(\alpha_0(T)) &= \alpha_0(T),\\
f^1(\alpha_0(T)) &= \alpha_3(T),\\
&\cdots\\
f^k(\alpha_0(T)) &= \alpha_{3k}(T),\\
f^{k+1}(\alpha_0(T)) &= \alpha_{3k+1}(\langle 0, S\rangle),\\
f^{k+2}(\alpha_0(T)) &= \alpha_{3k+1}(\langle 1, S\rangle),\\
&\cdots\\
f^{k+1+r}(\alpha_0(T)) &= \alpha_{3k+1}(\langle r, S\rangle),\\
&\cdots \quad .
\end{aligned}
$$

The proposition follows directly from these equations.

We are now in a position to define the sets C and D that will answer Question 2.6. Let $\{B_k\}_{k=0}^{\infty}$ be as in Proposition 1.2, and let B equal $\bigcup_{k=0}^{\infty} B_k$. Also, define D_k to be $\{\alpha_0(2b+1) \mid b \in B_k\}$ for each k. Then set D equal to $\bigcup_{k=0}^{\infty} D_k$ $(= \{\alpha_0(2b+1) \mid b \in B\})$, and set C equal to $\bigcup_{n=1}^{\infty} f^n(D)$, where f is as in Definition 2.11. In the rest of this paper the symbols B, C, D, B_k, D_k, and f will be reserved for these particular values.

Because of the behavior of f (see the proof of Proposition 2.12), we have $C \subseteq N^2 \setminus A_0$. But since $D \subseteq A_0$, C and D are separable; so $[C] + [D] = [C \cup D]$. On the other hand, $f(C \cup D) = C$, which implies that $[C \cup D] = [C]$. Thus $[C] + [D] = [C]$.

The next proposition goes about halfway towards settling Question 2.6.

2.13. PROPOSITION. *Assume that $D = E \cup F$, where E and F are separable and $[C] \geqslant \infty_1 \cdot [E] + \infty_2 \cdot [F]$. Then $D_k \cap F$ is finite for each k.*

PROOF. Under the assumptions of this proposition, we have

$$(2.14) \qquad [D_k] = [(D_k \cap E) \cup (D_k \cap F)] = [D_k \cap E] + [D_k \cap F].$$

Applying the isomorphism φ described in Section 0, $\varphi([D_k]) = \varphi([D_k \cap E]) + \varphi([D_k \cap F])$. But $\varphi([D_k]) = [B_k]$, which is indecomposable. This implies that either $D_k \cap E$ or $D_k \cap F$ is finite. We prove the proposition by assuming that $D_k \cap E$ is finite and deriving, from that assumption, a contradiction to Proposition 1.2.

If $D_k \cap E$ is finite, of cardinality n, say, then we have, from (2.14), that $[D_k] = [D_k \cap F] + [n]$. Thus

$$
\begin{aligned}
\infty_2 \cdot [D_k] &= \infty_2 \cdot [D_k \cap F] + \infty_2 \cdot [n] \quad \text{(by Proposition 2.7(ii))}\\
&= \infty_2 \cdot [D_k \cap F] \quad \text{(by Proposition 2.10(ii))}.
\end{aligned}
$$

On the other hand, since $D_k \cap F$ and $F \setminus D_k$ are separable, $[F] = [D_k \cap F] + [F \setminus D_k]$, so $[D_k \cap F] \leqslant [F]$. Therefore,

$$\begin{aligned}\infty_2 \cdot [D_k \cap F] &\leqslant \infty_2 \cdot [F] \quad \text{(by Proposition 2.8(ii))}\\ &\leqslant [C] \quad \text{(by the assumptions of this proposition)}\\ &= [C \cup D].\end{aligned}$$

Now the two preceding paragraphs imply that $\infty_2 \cdot [D_k] \leqslant [C \cup D]$, and thus, for some $Q^* \subseteq N^2$, we have $\infty_2 \cdot [D_k] + [Q^*] = [C \cup D]$. Define Q to be $\{\alpha_m(2n) \mid \alpha_m(n) \in Q^*\}$; then $[Q] = [Q^*]$, and $\infty_2 \cdot D_k$ and Q are separable. Thus $[\infty_2 \cdot D_k \cup Q] = [C \cup D]$, and there is an $\mathscr{A}$-$\mathscr{A}$-embedding g whose domain contains $\infty_2 \cdot D_k \cup Q$, and which maps $\infty_2 \cdot D_k \cup Q$ onto $C \cup D$.

Since g is $\mathscr{A}$-$\mathscr{A}$-compact, $g^{-1}(\bigcup_{r=0}^{3k+2} A_r)$ is contained in a finite union of A_j's; hence $g^{-1}(\bigcup_{r=0}^{3k+2} A_r) \subseteq \bigcup_{j=0}^{M} A_j$ for some M. For the rest of this proof, let n be fixed at a value large enough so that $\langle 0, n\rangle$ exceeds M. Then $g(A_{\langle 0,n\rangle}) \subseteq \bigcup_{r=3k+3}^{\infty} A_r$. But because g is $\mathscr{A}$-$\mathscr{A}$-bounded, $g(A_{\langle 0,n\rangle})$ is contained in the union of only finitely many of those A_r's with $r \geqslant 3k+3$. Thus there exists an $L \geqslant 3k+3$ with $g(A_{\langle 0,n\rangle}) \subseteq \bigcup_{r=3k+3}^{L} A_r$. Denote $\bigcup_{r=3k+3}^{L} A_r$ by S.

As $\infty_2 \cdot D_k$ and Q are separable, so also are $\infty_2 \cdot D_k \cap g^{-1}(S)$ and $Q \cap g^{-1}(S)$. Therefore,

$$\begin{aligned}\left[\infty_2 \cdot D_k \cap g^{-1}(S)\right] + \left[Q \cap g^{-1}(S)\right] &= \left[(\infty_2 \cdot D_k \cup Q) \cap g^{-1}(S)\right]\\ &= \left[g\left((\infty_2 \cdot D_k \cup Q) \cap g^{-1}(S)\right)\right] = \left[(C \cup D) \cap S\right].\end{aligned}$$

Hence

$$\left[\infty_2 \cdot D_k \cap g^{-1}(S)\right] \leqslant \left[(C \cup D) \cap S\right]. \tag{2.15}$$

But $\infty_2 \cdot D_k \cap g^{-1}(S)$ may be written as the union of the separable sets $\infty_2 \cdot D_k \cap g^{-1}(S) \cap (N^2 \setminus A_{\langle 0,n\rangle})$ and $\infty_2 \cdot D_k \cap g^{-1}(S) \cap A_{\langle 0,n\rangle} = \infty_2 \cdot D_k \cap A_{\langle 0,n\rangle}$. So

$$\left[\infty_2 \cdot D_k \cap A_{\langle 0,n\rangle}\right] \leqslant \left[\infty_2 \cdot D_k \cap g^{-1}(S)\right]. \tag{2.16}$$

Furthermore,

$$[D_k] = \left[\infty_2 \cdot D_k \cap A_{\langle 0,n\rangle}\right]. \tag{2.17}$$

Now statements (2.15), (2.16), and (2.17) imply that $[D_k] \leqslant [(C \cup D) \cap S]$, and on applying the isomorphism φ, we have

$$\varphi([D_k]) \leqslant \varphi([(C \cup D) \cap S]). \tag{2.18}$$

But

$$\varphi([D_k]) = [B_k], \tag{2.19}$$

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as remarked earlier. On the other hand, the definitions of C, D, and S imply that

$$\begin{aligned}
\varphi([(C\cup D)\cap S]) &= \varphi\left(\left[(C\cup D)\cap \bigcup_{r=3k+3}^{L} A_r\right]\right)\\
&= \sum_{r=3k+3}^{L}\left[\alpha_r^{-1}(C\cup D)\right] = \sum_{r=3k+3}^{L}\left[\alpha_r^{-1}\left(\bigcup_{n=0}^{\infty} f^n(D)\right)\right]\\
&= \sum_{r=3k+3}^{L}\left[\alpha_r^{-1}\left(\bigcup_{n=0}^{\infty} f^n\left(\bigcup_{z=0}^{\infty} D_z\right)\right)\right] = \sum_{r=3k+3}^{L}\left[\bigcup_{z=0}^{\infty}\alpha_r^{-1}\left(\bigcup_{n=0}^{\infty} f^n(D_z)\right)\right].
\end{aligned}$$

By Proposition 2.12, if $r \equiv 2 \pmod 3$, then $\alpha_r^{-1}(\bigcup_{n=0}^{\infty} f^n(D_z)) = \varnothing$ for all z. By the same proposition, if $r \equiv 1 \pmod 3$, then only for $z = (r-1)/3$ is $\alpha_r^{-1}(\bigcup_{n=0}^{\infty} f^n(D_z))$ nonempty, and for that value of z, we have

$$\left[\alpha_r^{-1}\left(\bigcup_{n=0}^{\infty} f^n\left(D_{(r-1)/3}\right)\right)\right] = \left[\langle N, B_{(r-1)/3}\rangle\right].$$

Once more, Proposition 2.12 implies that if $r \equiv 0 \pmod 3$, then α_r^{-1} $(\bigcup_{n=0}^{\infty} f^n(D_z))$ is $\varnothing$ for $z < r/3$ and is $\{2b+1 \mid b \in B_z\}$ for $z \geqslant r/3$. Thus

$$\left[\bigcup_{z=0}^{\infty}\alpha_r^{-1}\left(\bigcup_{n=0}^{\infty} f^n(D_z)\right)\right] = \begin{cases} [\varnothing] & \text{for } r \equiv 2 \pmod 3\\ \left[\langle N, B_{(r-1)/3}\rangle\right] & \text{for } r \equiv 1 \pmod 3\\ \left[\bigcup_{z=r/3}^{\infty}\{2b+1 \mid b \in B_z\}\right] = \left[\bigcup_{z=r/3}^{\infty} B_z\right] & \\ \qquad\qquad\qquad\qquad \text{for } r \equiv 0 \pmod 3. \end{cases}$$

This then implies that

$$\begin{aligned}
\varphi([(C\cup D)\cap S]) &= \sum_{r=3k+3}^{L}\left[\bigcup_{z=0}^{\infty}\alpha_r^{-1}\left(\bigcup_{n=0}^{\infty} f^n(D_z)\right)\right]\\
&= \sum_{\substack{r=3k+3\\ r\equiv 1 (\mathrm{mod}\ 3)}}^{L}\left[\langle N, B_{(r-1)/3}\rangle\right] + \sum_{\substack{r=3k+3\\ r\equiv 0\ (\mathrm{mod}\ 3)}}^{L}\left[\bigcup_{z=r/3}^{\infty} B_z\right]\\
&\leqslant \sum_{i=k+1}^{\mathrm{int}((L-1)/3)}\left[\langle N, B_i\rangle\right] + \sum_{j=k+1}^{\mathrm{int}(L/3)}\left[\bigcup_{z=j}^{\infty} B_z\right]\\
&= \left[\left\langle N, \bigcup_{i=k+1}^{\mathrm{int}((L-1)/3)} B_i\right\rangle\right] + \sum_{j=k+1}^{\mathrm{int}(L/3)}\left[\bigcup_{z=j}^{\infty} B_z\right]\\
&\leqslant \left[\left\langle N, \bigcup_{i=k+1}^{\mathrm{int}((L-1)/3)} B_i\right\rangle\right] + \sum_{j=k+1}^{\mathrm{int}(L/3)}\left[\bigcup_{z=k+1}^{\infty} B_z\right]\\
&= \left[\left\langle N, \bigcup_{i=k+1}^{\mathrm{int}((L-1)/3)} B_i\right\rangle\right] + (\mathrm{int}(L/3)-k)\cdot\left[\bigcup_{z=k+1}^{\infty} B_z\right],
\end{aligned}$$

where int denotes the greatest integer function. And so we have

$$\text{(2.20)} \qquad \varphi([(C \cup D) \cap S]) \leqslant \left[\left\langle N, \bigcup_{i=k+1}^{\operatorname{int}((L-1)/3)} B_i \right\rangle\right] + (\operatorname{int}(L/3) - k) \cdot \left[\bigcup_{z=k+1}^{\infty} B_z\right].$$

The conjunction of statements (2.18), (2.19), and (2.20) produces the desired contradiction to Proposition 1.2, and Proposition 2.13 is proved.

We now settle Question 2.6 by an argument for ∞_1 that resembles the argument for ∞_2 used in Proposition 2.13.

2.21. THEOREM. *There exist subsets C and D of N^2 such that $[C] + [D] = [C]$ and such that for no E, F with $[E] + [F] = [D]$ will $[C] \geqslant \infty_1 \cdot [E] + \infty_2 \cdot [F]$ be true.*

PROOF. Let C and D be as in Proposition 2.13. We have already remarked that $[C] + [D] = [C]$. Now assume that E and F are such that $[E] + [F] = [D]$ and $[C] \geqslant \infty_1 \cdot [E] + \infty_2 \cdot [F]$. Under this assumption, we shall reach a contradiction to Proposition 1.2 and thereby prove the theorem.

First, notice that we may, without loss of generality, assume that E and F are separable and that $D = E \cup F$. (If the given sets E, F do not satisfy these properties, then appropriate sets, recursively equivalent to E, F, will.)

Since $\infty_1 \cdot [E] \leqslant [C] = [C \cup D]$, there exists Q^* with $\infty_1 \cdot [E] + [Q^*] = [C \cup D]$. Let $Q = \{\alpha_{m+1}(n) \mid \alpha_m(n) \in Q^*\}$. Then $\infty_1 \cdot E$ and Q are separable, since $\infty_1 \cdot E \subseteq A_0$ and $Q \subseteq N^2 \setminus A_0$. Thus $[\infty_1 \cdot E \cup Q] = [C \cup D]$, and there exists an $\mathscr{A}$-$\mathscr{A}$-embedding g whose domain contains $\infty_1 \cdot E \cup Q$, and which is such that $g(\infty_1 \cdot E \cup Q) = C \cup D$. Because g is $\mathscr{A}$-$\mathscr{A}$-bounded, $g(A_0) \subseteq \bigcup_{r=0}^{3M-2} A_r$ for some M. Denote $\bigcup_{r=0}^{3M-2} A_r$ by S.

The sets Q and $\infty_1 \cdot E$ are separable; thus, so are $Q \cap g^{-1}(S)$ and $\infty_1 \cdot E \cap g^{-1}(S) = \infty_1 \cdot E$. It follows that

$$\begin{aligned}
[\infty_1 \cdot E] + [Q \cap g^{-1}(S)] &= [(\infty_1 \cdot E \cap g^{-1}(S)) \cup (Q \cap g^{-1}(S))] \\
&= [(\infty_1 \cdot E \cup Q) \cap g^{-1}(S)] \\
&= [g((\infty_1 \cdot E \cup Q) \cap g^{-1}(S))] = [(C \cup D) \cap S].
\end{aligned}$$

Therefore,

$$\text{(2.22)} \qquad [\infty_1 \cdot E] \leqslant [(C \cup D) \cap S].$$

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Now define N_M to be the set $\{\alpha_0(\langle M, n\rangle) \mid n \in N\}$, with M as above. Then $E \cap N_M = (D \cap E) \cap N_M = (D \cap N_M) \cap E = D_M \cap E$. By Proposition 2.13, we have, for some $n \in N$, $[D_M \cap F] = [n]$, and thus $[D_M] = [D_M \cap E] + [n]$. Hence $[D_M] = [E \cap N_M] + [n]$. So

$$\text{(2.23)} \qquad \begin{aligned}[][\infty_1 \cdot D_M] &= \infty_1 \cdot [E \cap N_M] + \infty_1 \cdot [n] \quad \text{(Proposition 2.7(i))}\\ &= \infty_1 \cdot [E \cap N_M] \quad \text{(Proposition 2.10(i))}.\end{aligned}$$

On the other hand, $E \cap N_M$ is separable from $E \setminus N_M$, whence $[E \cap N_M] + [E \setminus N_M] = [(E \cap N_M) \cup (E \setminus N_M)] = [E]$. This implies that $[E \cap N_M] \leqslant [E]$, and so, by Proposition 2.8(i), we have

$$\text{(2.24)} \qquad \infty_1 \cdot [E \cap N_M] \leqslant \infty_1 \cdot [E].$$

Together, statements (2.22), (2.23), and (2.24) yield $[\infty_1 \cdot D_M] \leqslant [(C \cup D) \cap S]$. Applying the isomorphism φ and using the definitions of the sets involved, we get

$$[\langle N, B_M\rangle] \leqslant \left[\left\langle N, \bigcup_{i=0}^{M-1} B_i\right\rangle\right] + M \cdot \left[\bigcup_{r=M}^{\infty} B_r\right].$$

Since $[B_M]$ is an isol, we may cancel it (additively) M times from this inequality. But this cancellation leaves the left-hand side unchanged, so that

$$[\langle N, B_M\rangle] \leqslant \left[\left\langle N, \bigcup_{i=0}^{M-1} B_i\right\rangle\right] + M \cdot \left[\bigcup_{r=M+1}^{\infty} B_r\right].$$

In particular,

$$[B_M] \leqslant \left[\left\langle N, \bigcup_{i=0}^{M-1} B_i\right\rangle\right] + M \cdot \left[\bigcup_{r=M+1}^{\infty} B_r\right].$$

This contradicts Proposition 1.2 and therefore proves Theorem 2.21.

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