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POSITIVE DEFINITE FUNCTIONS FOR THE CLASS $L_p(G)$

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1. Introduction. There are several notions of positive definiteness for functions on topological groups, the two of which are: Bochner type positive definite functions and integrally positive definite functions. The class $P(F)$ of positive definite functions for the class F can be defined more generally and it is interesting to observe that a change in F produces a different class $P(F)$ of positive definite functions. The purpose of this paper is to study the functions in $P(L_p(G))$ which are positive definite for the class $L_p(G)$ $(1 \leqq p < \infty)$, where G is a compact or locally compact group. The relevant information about the class $P(F)$ can be found in [**1**; **2**; **3** and **8**].

2. $P(L_p(G))$, when G is a compact group.

Definition 2.1. Let G be a Hausdorff locally compact group with the left Haar measure λ (normalized by $\lambda(G) = 1$ if G is compact). For brevity we shall write dx in place of $d\lambda(x)$ and $d(x, y)$ in place of $d(\lambda \times \lambda)$ (x, y). Let F be a set of complex-valued measurable functions on G. A complex-valued Borel measurable function ϕ on G is called positive definite for F if

$$\int_{G\times G} |\phi(y^{-1}x)\overline{f(y)}f(x)|d(x, y) < \infty,$$

and

$$\int_{G\times G} \varphi(y^{-1}x)\overline{f(y)}f(x)d(x, y) \geqq 0 \text{ for all } f \in F.$$

The class of functions which are positive definite for F will be denoted by $P(F)$. Clearly $F_1 \subset F_2$ implies that $P(F_1) \supset P(F_2)$.

We have the following:

THEOREM 2.2. *If G is a compact Hausdorff topological group, then for $1 \leqq p < \infty$ and $p^{-1} + q^{-1} = 1$,*

$$P(C_{00}) \cap L_p = P(L_q) \cap L_p,$$

where C_{00} denotes the set of all complex-valued continuous functions on G with compact support and q is defined ∞ if $p = 1$.

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The proof of this result is based on the following two lemmas.

LEMMA 1. *Let G be a compact topological group and* $1 \leqq p < \infty$. *If* $f_n \to f$ *in* L_p *and* $g_n \to g$ *in* L_p, *then* $f_n * g_n \to f * g$ *in* L_p, *where* $*$ *denotes the convolution.*

Proof. Since G is compact, $L_p(G)$ is, by Theorem 2.8. 46 [**5**], a Banach algebra. Hence

$$||f_n * g_n - f * g||_p \leqq ||f_n||_p||g_n - g||_p + ||f_n - f||_p||g||_p$$

gives the result.

LEMMA 2. *If the function* $(x, y) \to \phi(y^{-1}x)\,\overline{f(y)}\,f(x)$ *is in* $L_1(G \times G, \lambda \times \lambda)$, *then*

$$\int_{G\times G} \phi(y^{-1}x)\overline{f(y)}f(x)d(x, y) = \int_G f^{*}*f(x)\phi(x)dx,$$

where $f^*(x) = f(x^{-1})$ *and* $f \in L_1(G)$.

Proof. By Fubini's theorem, the left-side integral of the equation in the lemma can be re-written as

$$\int_G \int_G \phi(y^{-1}x)f(y)dx\overline{f(x)}dy = \int_G \int_G \phi(x)f(yx)dx\overline{f(y)}dy$$
$$= \int_G \int_G f(yx)\overline{f(y)}dy\,\phi(x)dx.$$

Since by Theorem 8, page 119 [**6**] every compact Hausdorff topological group is unimodular, by Theorem 20.10 [**4**], we have:

$$f^* * f(x) = \int_G \overline{f(y)}f(yx)dy.$$

Hence again by Fubini's theorem, $f^* * f$ is in $L_1(G)$ and

$$\int_{G\times G} \phi(y^{-1}x)\overline{f(y)}f(x)d(x, y) = \int_G f^* * f(x)\phi(x)dx.$$

Proof of Theorem 2.2. Since by Theorem 13.21 [**3**], $C_{00} \subset L_q(G)$, $1 < q < \infty$, we get $P(C_{00}) \supset P(L_q)$ and hence $P(C_{00}) \cap L_p \supset P(L_q) \cap L_p$.

To prove the opposite inclusion, suppose ϕ is in $P(C_{00}) \cap L_p$ and let $f \in L_q \subset L_1$ by [**4**, Theorem 15.9]. The denseness of C_{00} in L_q ensures the existence of a sequence $\{f_n\}$ in C_{00} such that $||f_n - f||_q \to 0$ and hence $||f_n^* - f^*||_q \to 0$. Since $\phi \in P(C_{00})$, we have

$$\int_{G\times G} |\phi(y^{-1}x)\overline{f_n(y)}f_n(x)|d(x, y) < \infty$$

and

$$\int_{G\times G} \phi(y^{-1}x)\overline{f_n(y)}f_n(x)d(x, y) \geqq 0 \text{ for each } f_n \in C_{00}.$$

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By Lemma 2,

$$\int_{G\times G} \phi(x^{-1}x)\overline{f_n(y)}f_n(x)dxdy = \int_G f_n^* * f_n(x)\phi(x)dx \geqq 0$$

for each n.

Next we claim that if $f \in L_q \subset L_1$ and $\phi \in L_p$, then the integral

$$\int_{G\times G} |\phi(y^{-1}x)f(y)\overline{f(x)}|d(x, y) < \infty .$$

By Corollary 2.14 [**4**].

$$\tilde{f} * \phi(x) = \int_G \overline{f(y)}\phi(y^{-1}x)dy$$

exists and is finite for λ-almost all $x \in G$ and is a function in $L_p(G)$. Since $f \in L_q$, it follows by Hölder's inequality that the integral

$$\int_G \int_G |\overline{f(x)}||\phi(y^{-1}x)|dy|f(x)|dx < \infty .$$

Fubini's theorem implies that the integral

$$\int_{G\times G} |\phi(y^{-1}x)\overline{f(y)}f(x)|d(x, y)$$

exists and is finite for $\phi \in L_p$ and $f \in L_q$. Hence by Lemma 2,

$$\int_G \int_G \phi(y^{-1}x)\overline{f(y)}f(x)dxdy = \int_G f^* * f(x)\phi(x)dx.$$

It remains to show that the above integral is non-negative. To see this we appeal to Lemma 1. By Hölder's inequality,

$$\left| \int_G (f_n^* * f_n)\phi d\lambda - \int_G (f^* * f)\phi d\lambda \right|$$

$$\leqq \int_G |(f_n^* * f_n - f^* * f)\phi| d\lambda$$

$$\leqq ||f_n^* * f_n - f^* * f||_q ||\phi||_p \to 0 \text{ as } n \to \infty .$$

Hence

$$\int_G f^* * f(x)\phi(x)dx \geqq 0.$$

Consequently $\phi \in P(L_q) \cap L_p$ and this proves the theorem.

Remark 2.3. It may be noted that the above theorem remains valid if C_{00} is replaced by any dense subspace of L_q.

COROLLARY 2.4. *If* $1 \leqq p \leqq 2$ *and* $q = p/p - 1$, *then*

$$P(C_{00}) \cap L_2 = P(L_2) \cap L_q = P(L_p) \cap L_q.$$

Proof. G being compact with Haar measure λ, we have by Theorem 13.17 [**3**], $C_{00} \subset L_2 \subset L_p$ so that

$$P(C_{00}) \cap L_q \supset P(L_2) \cap L_q \supset P(L_p) \cap L_q.$$

Following the method of proof in Theorem 2.2, it can be shown that

$$P(C_{00}) \cap L_q \subset P(L_p) \cap L_q,$$

and hence the equality follows.

Remark 2.5. For $1 \leqq p < \infty$ and compact G, Theorem 2.2 provides another way of looking at the theorem of Weil [**7**, p. 1311], replacing $P(C_{00}) \cap L_p$ by the class $P(L_q) \cap L_p$, where $p^{-1} + q^{-1} = 1$.

The following theorem is proved in case p and q are not necessarily conjugate real numbers.

THEOREM 2.6. *Let G be a compact topological group with Haar measure* λ. *Let* $1 \leqq p \leqq 2$ *and* $q = p/2(p-1)$. *Then*

$$P(C_{00}) \cap L_q = P(L_2) \cap L_q = P(L_p) \cap L_q.$$

Proof. Consider the case $1 < p < 2$. By Theorems 13.17 [**3**] and 13.21 [**3**], we have $C_{00} \subset L_2 \subset L_p$. Hence

$$P(C_{00}) \cap L_q \supset P(L_2) \cap L_q \supset P(L_p) \cap L_q.$$

Let $\phi \in P(C_{00}) \cap L_q$ and let $f \in L_p$. If we let $p' = p$ in Theorem 20.18 [**4**], then

$$1/r = 2/p - 1 = 1 - (2 - 2/p) = 1 - \frac{2(p-1)}{p}.$$

Hence $1/r = 1 - 1/q$, i.e., r and q are conjugate real numbers greater than 1. Theorem 20.2 [**4**] implies that if G is compact and $g \in L_p(G)$, then $g^* \in L_p(G)$. Now letting $g = f^*$, we conclude by Theorem 20.18 [**4**] that the function $f * f^*$ exists, is finite and belongs to $L_r(G)$. Since $\phi \in L_q(G)$, it follows by Hölder's inequality that the integral

$$\int_G f * f^*(x)\phi(x)dx$$

exists and is finite for λ-almost all $x \in G$ and for all $f \in L_p(G)$. As in Theorem 2.2

$$\int_G \int_G \phi(y^{-1}x)\overline{f(y)}f(x)dxdy = \int_G f * f^*(x)\phi(x)dx$$

for all $f \in L_p(G)$.

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It remains to show that the above integral is non-negative. To see this let $\{f_n\}$ be a sequence in C_{00} such that

$$\lim_{n\to\infty} \left|\left| f_n - f \right|\right|_p = 0. \quad \text{Also} \lim_{n\to\infty} \left|\left| f_n^* - f^* \right|\right|_p = 0.$$

Since $\phi \in P(C_{00}) \cap L_q$, it follows as before that

$$\text{(a)} \int_G \int_G \phi(y^{-1}x)\overline{f_n(y)}f_n(x)dxdy$$
$$= \int_G f_n * f_n^*(x)\phi(x)dx \geqq 0 \quad \text{for each } f_n \in C_{00}.$$

Theorem 20.18 [**4**] says that $f_n * f_n^*$ is in $L_r(G)$ and by Minkowski's inequality

$$\text{(b)} \quad ||f_n * f_n^* - f * f^*||_r \leqq ||f_n - f||_p||f^*_n||_p + ||f||_p||f_n^* - f^*||_p.$$

Since the sequence $\{||f_n^*||_p\}$ is bounded, the limit of the last line in (b) is zero. By Hölder's inequality

$$\left| \int_G (f_n * f_n^*)\phi d\lambda - \int_G (f * f^*)\phi d\lambda \right| \leqq ||f_n * f_n^* - f * f^*||_r||\phi||_q.$$

By (a) and (b) it follows that $\int_G f * f^*(x)\phi(x)\, dx$ is non-negative for all $f \in L_p(G)$ which implies $\phi \in P(L_p) \cap L_q$. This proves the result.

Remark. For $p = 1, q = \infty$ and $p = 2, q = 1$, the result follows by Theorem 2.2 and Corollary 20.14 [**4**].

THEOREM 2.7. *For $1 \leqq p < \infty$ and $p^{-1} + q^{-1} = 1$, $P(L_p) \cap L_q$ is a w^*-closed set in $L_q(G)$, where G is a compact group.*

Proof. Let $\phi \in P(L_p) \cap L_q$. Define $T : L_q(G) \to R$ by

$$T(\phi) = \int_G f^* * f(x)\phi(x)dx.$$

Clearly T is a linear functional on $L_q(G)$. It is easy to see that T is continuous in the weak$*$-topology and hence $P(L_p) \cap L_q$ is $w*$-closed in $L_q(G)$.

Remark. In [**8**] a similar result is proved for $p = 1$, $q = \infty$ with locally compact G.

COROLLARY 2.8. *The set of all normalized functions in $L_q(G)$ which are positive definite for the class $L_p(G)$ is a $w*$-compact subset of $L_q(G)$.*

Proof. By Alaoglu's theorem, the unit ball $B = \{\phi \in L_q: ||\phi||_q = 1\}$ is compact in the $w*$-topology of L_q. Denoting $A = P(L_p) \cap L_q$, we observe that $A \cap B$ is a $w*$-closed subset of the compact set B and hence $A \cap B$ is compact in the $w*$-topology of L_q.

3. $P(L_p(G))$, where G is a locally compact group. We shall throughout in this section assume that $\phi \in P(F)$. We wish to prove the following:

THEOREM 3.1. *Let G be a locally compact group and let $\Delta^{-1/2}\phi \in L_1(G)$, where Δ is the modular function for G. If ϕ is positive definite for the class $L_1(G) \cap L_2(G)$, then it is positive definite for the class $L_2(G)$.*

Proof. It is well known that for $1 < p < \infty$, $L_1 \cap L_p$ is a dense subset of L_p. Hence $L_1 \cap L_2 \subset L_2$ implies that

$$P(L_1 \cap L_2) \supset P(L_2).$$

Let $\phi \in P(L_1 \cap L_2)$ and suppose $f \in L_2$. There exists a sequence $\{f_n\}$ in $L_1 \cap L_2$ such that

$$\text{(i)} \lim_{n\to\infty} ||f_n - f||_2 = 0.$$

Also

$$\int_{G\times G} |\phi(y^{-1}x)\overline{f_n(y)}f_n(x)|\, d(x, y) < \infty,$$

and

$$\text{(ii)} \int_G \phi(y^{-1}x)\overline{f_n(y)}f_n(x)d(x, y) \geqq 0$$

for each $f_n \in L_1 \cap L_2$.

We claim that the integral

$$\int_{G\times G} |\phi(y^{-1}x)\overline{f(y)}f(x)|\, d(x, y) < \infty$$

for all $f \in L_2$. By Corollary 20.14 [**4**], the function

$$\bar{f} * \phi(x) = \int_G \bar{f}(y)\phi(y^{-1}x)dy$$

exists, is finite for λ-almost all $x \in G$ and is a function in $L_2(G)$ for which

$$\text{(iii)} \ ||f*\phi||_2 \leqq ||f||_2||\Delta^{-1/2}\phi||_1.$$

By Hölder's inequality,

$$\int_G \int_G |\overline{f(y)}||\phi(y^{-1}x)|dy|f(x)|dx < \infty.$$

By Fubini's theorem the integral

$$\int_G \int_G \phi(y^{-1}x)\overline{f(y)}f(x)dxdy$$

exists and is finite for all $f \in L_2$. Hence we can write

$$\int_{G\times G} \phi(y^{-1}x)\overline{f(y)}f(x)d(x,y) = \int_G (\bar{f} * \phi)fd\lambda.$$

It remains to show that the above integral is non-negative. To see this, we have, by (iii) and Hölder's inequality,

$$\left| \int_G (\bar{f}_n * \phi)f_n d\lambda - \int_G (\bar{f} * \phi)fd\lambda \right| \leqq (||f_n||_2||f_n - f||_2 + ||f_n - f||_2||f||_2)||\Delta^{-1/2}\phi||_1.$$

By (i) and (ii) on taking limit as $n \to \infty$, $\int_G(\bar{f} * \phi)fd\lambda$ is the limit of the non-negative sequence $\int_G(\bar{f}_n * \phi)fd\lambda$. Hence we have shown that $\int_G(\bar{f} * \phi)fd\lambda \geqq 0$ for all $f \in L_2$ and this implies $\phi \in P(L_2)$ which proves the theorem.

COROLLARY 3.2. *Let G be a locally compact group, p a real number greater than 1 and q conjugate to p. Let $\phi \in P(L_1 \cap L_p \cap L_q)$ be such that $\Delta^{-1/2}\phi \in L_1(G)$. Then $\phi \in P(L_p \cap L_q)$.*

Proof. Let $f \in L_p \cap L_q$. There always exists a sequence $\{f_n\}$ of functions in $L_1 \cap L_p \cap L_q$ such that

$$\lim_{n\to\infty} ||f_n - f||_p = \lim_{n\to\infty} ||f_n - f||_q = 0.$$

For example we may choose $\{f_n\}$ to be a sequence of simple functions which are dense in L_p. Now essentially the same proof of Theorem 3.1 goes for this corollary.

COROLLARY 3.3. *If $\phi \in P(C_{00})$ satisfies the condition $\Delta^{-1/2}\phi \in L_1(G)$, then $\phi \in P(L_p \cap L_q)$.*

Proof. Let $f \in L_p \cap L_q$. We can find a sequence $\{f_n\}$ in $C_{00} \subset L_p \cap L_q$ such that

$$\lim_{n\to\infty} ||f_n - f||_p = \lim_{n\to\infty} ||f_n - f||_q = 0.$$

The proof now follows as before.

COROLLARY 3.4. *If G is a compact group, then*

$$P(C_{00}) \cap L_1 = P(L_p \cap L_q) \cap L_1.$$

In particular, $P(C_{00}) \cap L_1 = P(L_2) \cap L_1$.

Proof. Immediate.

COROLLARY 3.5. *Suppose $\Delta^{-1/2}\phi \in L_1(G)$ and G a locally compact group. If $1 \leqq p < 2 < q$, where p and q are conjugate real numbers, then $\phi \in P(C_{00})$ implies $\phi \in P(L_2)$.*

Proof. By Theorem 13.19 [**3**], we have $C_{00} \subset L_p \cap L_q \subset L_2$. As Theorem 3.1, we prove the corollary.

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