

## FINITELY PRESENTED GROUPS OF FIBONACCI TYPE. PART II

COLIN M. CAMPBELL and EDMUND F. ROBERTSON

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### Abstract

A class of cyclically presented groups with  $n$  generators and  $n$  Fibonacci type relations is discussed. Conditions are given for these groups to be finite and metacyclic. With these conditions the presentations are reduced to the standard form for metacyclic groups with trivial Schur multiplier. This enables certain isomorphisms between the groups to be found.

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### 1. Introduction

Groups with symmetrical presentations, called cyclically presented groups, have been the subject of investigation in recent years. In this paper we continue work on cyclically presented groups which was begun in Campbell and Robertson (1975b).

A group is said to be *cyclically presented* if it has a presentation on  $n$  generators  $a_1, a_2, \dots, a_n$  with  $n$  relations obtained from a single word  $w = w(a_1, a_2, \dots, a_n)$  by permuting the subscripts modulo  $n$  according to the powers of the permutation  $(1\ 2 \dots n)$ . The resulting group is denoted by  $G_n(w)$ . The interest in cyclically presented groups most relevant to this paper arises from the study of *Fibonacci groups*, these being the groups  $F(r, n) = G_n(w)$  where  $w = a_1 a_2 \dots a_r a_{r+1}^{-1}$ . A survey of results on Fibonacci groups and other cyclically presented groups is contained in Johnson, Wamsley and Wright (1974) and Chalk and Johnson (1977), see also Johnson (1976), Chapter 16 and Lyndon and Schupp (1977), Chapter 2. In Johnson, Wamsley and Wright (1974) the groups  $F(r, n)$ , with  $r \equiv 1 \pmod n$ , are shown to be finite metacyclic groups and one of the problems posed in that article is to find standard two generator presentations for these metacyclic groups. This problem

is solved in Campbell and Robertson (1975a). For integers  $r, n, k$  and  $s$  with  $r > s \geq 1; n \geq 1, k \geq 0$  define the class of groups  $F(r, n, k, s)$  by  $F(r, n, k, s) = G_n(w)$  where  $w = a_1 a_2 \dots a_r (a_{r+k} a_{r+k+1} \dots a_{r+k+s-1})^{-1}$ .

In this paper we examine metacyclic groups in the class  $F(r, n, k, s)$  giving presentations in standard two generator form. As a corollary the presentation problem is solved for the class  $F(r, n, 1, s)$  studied in Campbell and Robertson (1975b). We shall, where possible, use the notation and results of Campbell and Robertson (1975b). The authors would like to thank the referee for encouraging them to produce a more readable paper.

### 2. A two generator two relation presentation

All finite metacyclic groups may be presented in the form

$$(2.1) \quad \langle a, b \mid a^M = 1, b^{-1}ab = a^R, b^N = a^T \rangle$$

where  $R^N \equiv 1 \pmod M$  and  $T = LM / (M, R - 1)$  for some non-zero integer  $L$ , see Zassenhaus (1958), p. 129. Beyl (1973) shows that the finite metacyclic groups with trivial Schur multiplier are precisely those which may be given by the presentation (2.1) with  $L = 1$ . Now  $F(r, n, k, s)$ , when finite, has deficiency zero and so has trivial Schur multiplier. Hence, in order to apply Beyl's result, we aim to reduce the presentation for  $F(r, n, k, s)$  to the form (2.1) when  $r, n, k$  and  $s$  satisfy certain conditions.

We begin by obtaining two generator two relation presentations for the groups  $F(r, n, k, s)$  under slightly more general conditions than we shall need for our main result. These more general conditions are  $r \equiv s \pmod n, (r, n) = 1, (r + k - 1, n) = 1$  and we shall assume that these hold for the rest of this section. Our first step towards producing a two generator presentation of  $F(r, n, k, s)$  is to obtain another  $n$ -generator presentation and to do this we require the following notation.

Let  $[i]$  denote the integer  $i$  reduced modulo  $n$  so that  $0 \leq [i] < n$  and define  $t$  and  $u$  by  $t = (r - [r])/n, u = (s - [s])/n$ . For any integer  $i$  not congruent to 1 modulo  $n$  let  $b_i = a_1 a_2 \dots a_{[i-1]}$ ; also define  $b_1 = 1$  and  $x = a_1 a_2 \dots a_n$ . Further define  $\varepsilon_i$ , for any integer  $i$ , by

$$\varepsilon_i = \begin{cases} 1 & \text{if } [i-1] + [r] \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Now the  $j$ th relation

$$a_j a_{j+1} \dots a_{j+r-1} = a_{j+r+k-1} a_{j+r+k} \dots a_{j+r+k+s-2}$$

becomes

$$(2.2) \quad b_j^{-1} x^{t+\varepsilon_j} b_{j+r} = b_{j+r+k-1}^{-1} x^{u+\varepsilon_{j+r+k-1}} b_{j+2r+k-1}$$

We now show that  $x$  and  $y = b_{r+k}$  generate  $F(r, n, k, s)$  and then find a presentation using these generators. Given any  $j$  there is an  $i$  with  $j \equiv ir + 1 \pmod n$  so we may define  $c_j$  by

$$(2.3) \quad c_{ir+1} = b_{(i+1)r+k} b_{ir+1}^{-1}.$$

Now (2.2) with  $j = r + 1$  gives

$$(2.4) \quad c_{ir+1} = x^{u+\varepsilon(i+1)r+k} c_{(i+1)r+1} x^{-(t+\varepsilon ir+1)}.$$

Since  $(r, n) = 1$  and  $c_1 = b_{r+k} b_1^{-1} = y$  we see, from (2.4), that  $c_j \in \langle x, y \rangle$  for all  $j$ . Now, since  $(r+k-1, n) = 1$ , (2.3) shows that  $b_j \in \langle x, y \rangle$  for all  $j$  and hence  $\langle x, y \rangle = F(r, n, k, s)$ . Again using  $c_1 = y$  together with repeated use of (2.4) we obtain the relation

$$y = x^{-p} y x^q$$

where  $p = \sum_{i=1}^n (t + \varepsilon_{ir+1})$  and  $q = \sum_{i=1}^n (u + \varepsilon_{(i+1)r+k})$ . Since  $(r, n) = 1$  this gives

$$(2.5) \quad y^{-1} x^q y = x^p.$$

Only one further non-trivial relation in  $x$  and  $y$  is obtained for  $F(r, n, k, s)$ . Substituting for the  $c$ 's in  $\prod_{i=1}^n c_{i(1-r-k)+1}$  their expressions given by (2.3) shows that

$$(2.6) \quad \prod_{i=1}^n c_{i(1-r-k)+1} = 1.$$

We introduce further notation to enable (2.6) to be written as a relation in  $x$  and  $y$ . Define  $\lambda$  to be  $[r^{-1}(r+k-1)]$ . Let  $p_1 = \varepsilon_1 + \varepsilon_{1+r-1} + \varepsilon_{1+2r-1} + \dots + \varepsilon_k$  and, for  $1 < i \leq n$ , define

$$p_i = \begin{cases} p_1 - [r] & \text{if there exists an integer } j \text{ with } (i-1)\lambda < jn \leq i\lambda, \\ p_1 & \text{otherwise.} \end{cases}$$

Also let  $\beta(i) = [(n-i+1)\lambda]t - [(n-i)\lambda]u + p_{n-i+1}$ . Now  $c_{i(r+k-1)+1} = c_{i\lambda r+1}$ . Therefore (2.4) together with  $c_1 = y$  gives

$$(2.7) \quad c_{i(r+k-1)+1} = x^{-a(i)} y x^{b(i)}$$

where

$$a(i) = [i\lambda]u + \sum_{j=1}^{[i\lambda]} \varepsilon_{jr+k} \quad \text{and} \quad b(i) = [i\lambda]t + \sum_{j=1}^{[i\lambda]} \varepsilon_{1+(j-1)r}.$$

Substituting (2.7) into (2.6) gives, after some elementary calculation,

$$(2.8) \quad \prod_{i=1}^n (x^{\beta(i)} y) = 1.$$

We have proved the following theorem.

**THEOREM 1.** *Suppose that*

$$(i) r \equiv s \pmod n, \quad (ii) (r, n) = 1, \quad (iii) (r+k-1, n) = 1.$$

*Then  $F(r, n, k, s)$  has the two generator two relation presentation*

$$\left\langle x, y \mid y^{-1}x^s y = x^r, \prod_{i=1}^n (x^{\beta^{(i)}} y) = 1 \right\rangle.$$

**COROLLARY.**

$$F(r, n, 1, s) = \left\langle x, y \mid y^{-1}x^s y = x^r, x^r = \prod_{i=1}^n (x^{(n-i+1)t - (n-i)u} y) \right\rangle$$

where  $r \equiv s \pmod n$  and  $(r, n) = 1$ .

Notice that with  $a = x, b = yx^t$  we obtain from this corollary the presentation for  $F(r, n, 1, s)$  given in Theorem 2 of Campbell and Robertson (1975b).

### 3. A standard metacyclic Schur group presentation

In this section we shall assume that the conditions  $r \equiv s \pmod n, (r, s) = 1$  and  $(r+k-1, n) = 1$  hold. Note that these conditions imply  $(r, n) = 1$  and so the results of Section 2 are valid. We shall now show how, under these conditions, the group  $F(r, n, k, s)$  may be given by the presentation (2.1) and so, by the comments at the beginning of Section 2, as a standard presentation for a metacyclic Schur group. (A Schur group is a group with trivial Schur multiplier.) Put  $\sigma = r^n - s^n$ . Since  $r$  and  $s$  are coprime  $(\sigma, s) = 1$  and we may define  $\alpha$  to be the inverse of  $s$  modulo  $\sigma$ . Although the next two lemmas are not stated explicitly in Campbell and Robertson (1975b) they can readily be deduced from the proof of Theorem 3 of that paper.

**LEMMA 1.** *The relations*

$$y^{-1}x^s y = x^r \quad \text{and} \quad x^r = \prod_{i=1}^n (x^{(n-i+1)t - (n-i)u} y)$$

*together imply*

$$y^{-1}xy = x^{\alpha r} \quad \text{and} \quad y^n = x^\eta,$$

where  $\eta$  is such that

$$(1 - \alpha r)\eta \equiv \left(\frac{s-r}{n}\right) \left(\frac{1 - (\alpha r)^n}{1 - \alpha r}\right) \pmod{\sigma}.$$

LEMMA 2.

$$\left( \sigma, \left( \frac{r-s}{n} \right) \left( \frac{1-(\alpha r)^n}{1-\alpha r} \right) \right) = \frac{\sigma}{n}.$$

It turns out that (2.8) can be expressed in a more suitable form. Indeed the next three results are concerned with doing just this.

LEMMA 3. *The relation  $\prod_{i=1}^n (x^{\beta(i)} y) = 1$  may be written in the form*

$$x^{\lambda r} = \left( \prod_{i=1}^n (x^{(n-i+1)\lambda t - (n-i)\lambda u} y) \right) x^{\mu},$$

where  $\mu \equiv (p_1(1 - (\alpha r)^n) / (1 - \alpha r)) \pmod{\sigma}$ .

PROOF. Let  $\gamma_i = (i\lambda - [i\lambda]) / n$ . Then

$$\begin{aligned} \beta(n-i+1) &= i\lambda t - \gamma_i n t - (i-1)\lambda u + \gamma_{i-1} n u + p_i \\ &= (i\lambda t - (i-1)\lambda u) - \gamma_i r + \gamma_{i-1} s + p_1. \end{aligned}$$

Now  $\gamma_1 = \gamma_0 = 0$  and  $\gamma_{n-1} = \lambda - 1$  and so (2.8) becomes

$$(3.1) \quad (x^{\lambda u - n + p_1 - [r]} y) \left( \prod_{i=2}^n (x^{(n-i+1)\lambda t - (n-i)\lambda u} x^{d(i)} y) \right) = 1,$$

where  $d(i) = -\gamma_{n-i+1} r + \gamma_{n-i} s + p_1$ . If we take each of the expressions  $x^{d(i)}$  to the right hand side of (3.1) using the relation  $y^{-1} x y = x^{\alpha r}$  we obtain

$$(3.2) \quad x^{\lambda r} = \left( \prod_{i=1}^n (x^{(n-i+1)\lambda t - (n-i)\lambda u} y) \right) x^{\mu},$$

where  $\mu = (\sum_{i=1}^{n-1} (-\gamma_i r + \gamma_{i-1} s + p_1) (\alpha r)^i) + (\lambda - 1) s + p_1$ . To complete the proof we simplify  $\mu$  working modulo  $\sigma$ . Since  $\alpha s \equiv 1$  we have  $\gamma_{i-1} \alpha s r \equiv \gamma_{i-1} r$  and so

$$\mu \equiv -\gamma_{n-1} r (\alpha r)^{n-1} + p_1 \sum_{i=0}^{n-1} (\alpha r)^i + (\lambda - 1) s.$$

But  $(\lambda - 1) s \equiv (\lambda - 1) r (\alpha r)^{n-1}$  since both  $(\alpha r)^n \equiv 1$  and  $\alpha s \equiv 1$ . Therefore

$$\mu \equiv p_1 \sum_{i=0}^{n-1} (\alpha r)^i$$

as required.

Using the same method as in Lemma 1 we can reduce (2.8) to  $y^n = x^\nu$  for some  $\nu$ . Therefore  $y^n$  is central and, since  $y^{-n} x^{\beta^n} y^n = x^{r^n}$ ,  $x^\sigma = 1$ .

LEMMA 4.  $\nu \equiv (\lambda \eta - \mu) \pmod{\sigma}$ .

PROOF. By Lemma 1

$$x^{\lambda r} = \prod_{i=1}^n (x^{(n-i+1)\lambda l - (n-i)\lambda u} y)$$

may be written as

$$y^n = x^{\lambda \eta}.$$

Substituting into (3.2) gives  $y^n = x^{\lambda \eta - \mu}$  as required.

LEMMA 5.  $(1 - \alpha r)\mu \equiv 0 \pmod{\sigma}$ .

PROOF. This follows immediately using the expression for  $\mu$  given in Lemma 3 since  $1 - (\alpha r)^n \equiv 0 \pmod{\sigma}$ .

We are now in a position to find the standard form presentations.

THEOREM 2. *Suppose that*

$$(i) r \equiv s \pmod{n}, \quad (ii) (r, s) = 1, \quad (iii) (r + k - 1, n) = 1.$$

Then  $F(r, n, k, s)$  is a finite metacyclic Schur group with presentation

$$\langle a, b \mid a^{\sigma/n} = 1, b^{-1}ab = a^{\alpha r}, b^n = a^{\sigma/n(r-s)} \rangle.$$

PROOF. The order of the generator  $x$  is  $((1 - \alpha r)\nu, \sigma)$ . But

$$\begin{aligned} ((1 - \alpha r)\nu, \sigma) &= ((1 - \alpha r)\lambda\eta - (1 - \alpha r)\mu, \sigma) && \text{by Lemma 4,} \\ &= ((1 - \alpha r)\lambda\eta, \sigma) && \text{by Lemma 5,} \\ &= \left( \left( \frac{r-s}{n} \right) \left( \frac{1 - (\alpha r)^n}{1 - \alpha r} \right), \sigma \right) && \text{by Lemma 1,} \\ &= \sigma/n && \text{by Lemma 2.} \end{aligned}$$

Also  $\nu$  is divisible by  $\sigma/n(s-r)$ . For

$$\begin{aligned} \nu &\equiv (\lambda\eta - \mu) \pmod{\sigma/n(s-r)} && \text{by Lemma 4,} \\ &\equiv \lambda\eta \pmod{\sigma/n(s-r)} && \text{by Lemma 5,} \\ &\equiv 0 \pmod{\sigma/n(s-r)} && \text{by Lemma 1.} \end{aligned}$$

Hence relation (2.8) becomes

$$y^n = x^{\xi\sigma/n(s-r)}$$

for some integer  $\xi$ .

Therefore, when conditions (i), (ii) and (iii) are satisfied,  $F(r, n, k, s)$  has the presentation

$$\langle x, y \mid x^{\sigma/n} = 1, y^{-1}xy = x^{\alpha r}, y^n = x^{\xi\sigma/n(s-r)} \rangle.$$

The fact that  $(\alpha r)^n \equiv 1 \pmod{\sigma/n}$  now shows that the results of Beyl (1973) are applicable to  $F(r, n, k, s)$  when  $r, n, k$  and  $s$  satisfy conditions (i), (ii) and (iii) and so it may be presented in the form

$$\langle a, b \mid a^{\sigma/n} = 1, b^{-1}ab = a^{\alpha r}, b^n = a^{\sigma/n(r-s)} \rangle.$$

Finally, as a consequence of the  $k$  independence of the presentation given in Theorem 2, we have the following corollary.

**COROLLARY.** *Suppose that*

(i)  $r \equiv s \pmod{n}$ , (ii)  $(r, s) = 1$ , (iii)  $(r+k-1, n) = 1$ , (iv)  $(r+k'-1, n) = 1$ .

*Then  $F(r, n, k, s)$  is isomorphic to  $F(r, n, k', s)$ .*

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The Mathematical Institute  
University of St. Andrews  
St. Andrews KY16 9SS  
Scotland