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PARACOMPACTNESS IN SMALL PRODUCTS

BY
S. WILLARD

All spaces in this note are regular, Hausdorff topological spaces.

At the topology conference in Pullman, Washington, in March 1970, E. Michael posed the following question: *if X is Lindelöf, Y is separable and metrizable and $X\times Y$ is paracompact, must $X\times Y$ be Lindelöf?*

With the aid of the following lemma, we can provide the affirmative answer, even in the case when Y fails to be metrizable. The lemma has some independent interest, being a generalization of the well-known result that a separable paracompact space is Lindelöf.

LEMMA. *A paracompact space with a dense Lindelöf subspace is Lindelöf.*

Proof. Suppose A is a dense Lindelöf subspace of the paracompact space X. Let $\mathscr{U}$ be any open cover of X and, simultaneously using regularity and paracompactness, let $\mathscr{V}$ be a locally finite open cover of X such that each $V\in\mathscr{V}$ has closure contained in some $U\in\mathscr{U}$. Since A is Lindelöf, a countable subcollection $\mathscr{V}_0$ of $\mathscr{V}$ covers A. Now A is dense in X, so $X=Cl\,[\bigcup\{V\mid V\in\mathscr{V}_0\}]$ and, since $\mathscr{V}$ (and hence $\mathscr{V}_0$) is locally finite, $X=\bigcup\{\overline{V}\mid V\in\mathscr{V}_0\}$. But each $\overline{V}$ is contained in some element of $\mathscr{U}$, so a countable subcollection from $\mathscr{U}$ covers X.

With the lemma, the theorem which settles Michael's question becomes easy.

THEOREM. *If X is Lindelöf and Y is separable, then $X\times Y$ is paracompact iff $X\times Y$ is Lindelöf.*

Proof. If D is countable dense set in Y, then $X\times D$ is a dense Lindelöf subspace of $X\times Y$. Lemma 1 now applies.

It is of passing interest that the theorem (as well as its proof) remains true whenever Y has a dense σ-compact subspace. For if D is this subspace, $X\times D$ will still be a dense Lindelöf subspace of $X\times Y$.

Finally, one easy consequence of the theorem is perhaps worth noting. It is a well-known result of Michael [1] that the product of a perfectly normal paracompact space with a metrizable space is paracompact. Using this together with Theorem 1 we obtain the

COROLLARY. *The product of a perfectly normal Lindelöf space with a separable metric space is Lindelöf.*

REFERENCE

1. E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.

UNIVERSITY OF ALBERTA,
EDMONTON, ALBERTA