Bull. Aust. Math. Soc. **95** (2017), 495–499
doi:10.1017/S0004972716001271

A DDVV INEQUALITY FOR SUBMANIFOLDS OF WARPED PRODUCTS

JULIEN ROTH

(Received 3 October 2016; accepted 3 November 2016; first published online 5 January 2017)

Abstract

We prove a DDVV inequality for submanifolds of warped products of the form $I \times_a \mathbb{M}^n(c)$, where I is an interval and $\mathbb{M}^n(c)$ is a real space form of curvature c. As an application, we give a rigidity result for submanifolds of $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$.

2010 *Mathematics subject classification*: primary 53C42; secondary 53C24.

Keywords and phrases: warped product, rigidity, immersion.

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold isometrically immersed into an $(n+p)$-dimensional Riemannian manifold $(N^{n+p}, \bar{g})$. When the ambient space is a real space form of constant sectional curvature c, we have the pointwise inequality

$$\|H\|^2 \geqslant \rho + \rho^\perp - c,$$

where

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} \langle R(e_i, e_j)e_j, e_i \rangle$$

is the normalised scalar curvature of (M, g) and

$$\rho^\perp = \frac{2}{n(n-1)} \Big(\sum_{i<j} \sum_{\alpha<\beta} \langle R^\perp(e_i, e_j)\xi_\alpha, \xi_\beta \rangle^2 \Big)^{1/2}$$

is the normalised normal curvature of the immersion. Here, $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots \xi_p\}$ are respectively orthonormal frames of TM and $T^\perp M$. This inequality, known as the DDVV conjecture, was conjectured by De Smet *et al.* in [2] and proved recently by Lu [6] and by Ge and Tang [4] independently. More recently, Chen and Cui [1] generalised the inequality in the setting of product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

In this note, we extend the result of Chen–Cui by proving a DDVV inequality for submanifolds of warped products $I \times_a \mathbb{M}^n(c)$, where $I \subset \mathbb{R}$ is an interval and $a : I \to \mathbb{R}$ is a nowhere-vanishing smooth function. Denote by $\partial_t = \partial/\partial t$ the unit vector field tangent to the factor I. We prove the following result.

THEOREM 1.1. *Let M^m be a submanifold of the warped product $I \times_a \mathbb{M}^n(c)$ with normalised scalar and normal scalar curvatures ρ and $\rho^\perp$ and mean curvature H. Then*

$$\|H\|^2 \geqslant \rho + \rho^\perp + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\left(1 - \frac{2}{m}\|T\|^2\right) + \frac{2a''}{ma}\|T\|^2,$$

where T is the part of ∂t tangent to M.

REMARK 1.2. Note that, of course, we recover the DDVV inequality of [1] for product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ as well as for $\mathbb{R}^{n+1}$ by taking $a = 1$, but we also recover the inequality for space forms. Indeed, $\mathbb{S}^n$ and $\mathbb{H}^n$ can be expressed in terms of warped products. Namely:

(1) $\mathbb{S}^n = [0, 2\pi] \times_a \mathbb{S}^{n-1}$ with $a(t) = \sin(t)$. In this case, the inequality of Theorem 1.1 becomes $\|H\|^2 \geqslant \rho + \rho^\perp - 1$;
(2) $\mathbb{H}^n = [0, +\infty[\times_a \mathbb{S}^{n-1}$ with $a(t) = \sinh(t)$ or $\mathbb{H}^n = \mathbb{R} \times_a \mathbb{R}^{n-1}$ with $a(t) = e^{-t}$. For both cases, the inequality of Theorem 1.1 becomes $\|H\|^2 \geqslant \rho + \rho^\perp + 1$.

2. Preliminaries

Let $\mathbb{M}^n(c)$ be the simply connected real space form of dimension n and constant curvature c. Let $I \subset \mathbb{R}$ be an interval and $a : I \longrightarrow \mathbb{R}$ be a nowhere-vanishing smooth function. We consider the warped product $\widetilde{P}^{n+1} = I \times_a \mathbb{M}^n(c)$, formed from the product $I \times \mathbb{M}^n(c)$ endowed with the metric $\widetilde{g} = dt^2 + a(t)^2 g_{\mathbb{M}^n(c)}$. Denote by $\partial_t = \partial/\partial t$ the unit vector field tangent to the factor I. Recall (see, for example, [5]) that the curvature tensor of $(\widetilde{P}^{n+1}, \widetilde{g})$ is given by

$$\begin{aligned}\widetilde{R}(X, Y)Z = &\left(\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)(\langle X, Z\rangle Y - \langle Y, Z\rangle X)\\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2}\right)(\langle X, Z\rangle\langle Y, \partial_t\rangle\partial_t - \langle Y, Z\rangle\langle X, \partial_t\rangle\partial_t\\ &- \langle Y, \partial_t\rangle\langle Z, \partial_t\rangle X + \langle X, \partial_t\rangle\langle Z, \partial_t\rangle Y).\end{aligned}$$

Let (M^m, g) be a Riemannian manifold isometrically immersed into $\widetilde{P}$. We denote by B its second fundamental form and by A the shape operator defined for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$ by $\langle B(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle$. Moreover, ∂_t can be written as

$$\partial_t = T + \sum_{\alpha=1}^{p} f_\alpha \xi_\alpha,$$

where T is a vector field tangent to M, $\{\xi_1, \ldots, \xi_p\}$ is a local orthonormal frame of $T^\perp M$ and $f_1, \ldots, f_p$ are smooth functions over M. We will denote A_{ξ_α} simply by A_α.

From the expression of the curvature tensor of $\widetilde{P}$, we get immediately the Gauss, Codazzi and Ricci equations for a submanifold of $\widetilde{P}$. Namely, if we denote by R and $R^\perp$ the curvature tensor of (M, g) and the normal curvature, respectively, we have the following proposition. The proof is straightforward from the expression for $\widetilde{R}$.

https://doi.org/10.1017/S0004972716001271 Published online by Cambridge University Press

PROPOSITION 2.1. *The Gauss, Codazzi and Ricci equations of the immersion of M into $\widetilde{P}$ are respectively*

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \langle B(Y,Z),B(X,W)\rangle - \langle B(Y,W),B(X,Z)\rangle\\ &\quad + \Big(\frac{(a')^2}{a^2}-\frac{c}{a^2}\Big)(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle)\\ &\quad + \Big(\frac{a''}{a}-\frac{(a')^2}{a^2}+\frac{c}{a^2}\Big)(\langle X,Z\rangle\langle Y,T\rangle\langle W,T\rangle - \langle Y,Z\rangle\langle X,T\rangle\langle W,T\rangle\\ &\quad - \langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle + \langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle),\\ \langle(\widetilde{\nabla}_X B)(Y,Z),\xi_\alpha\rangle &= \Big(\frac{a''}{a}-\frac{(a')^2}{a^2}+\frac{c}{a^2}\Big)f_\alpha(\langle Y,T\rangle\langle X,Z\rangle - \langle X,T\rangle\langle Y,Z\rangle),\\ \langle R^\perp(X,Y)\nu,\xi\rangle &= \langle[A_\nu,A_\xi]X,Y\rangle.\end{aligned}$$

Finally, we recall that the DDVV conjecture can be reduced to the following algebraic result (see [3]) proved by Lu.

THEOREM 2.2 [6]. *Let $n,p\geqslant 2$ be two integers and $M_1, M_2,\dots,M_p$ be some $n\times n$ real symmetric and trace-free matrices. Then*

$$\sum_{\alpha,\beta=1}^{p}\|[M_\alpha,M_\beta]\|^2 \leqslant \Big(\sum_{\alpha=1}^{p}\|M_\alpha\|^2\Big)^2.$$

3. Proof of Theorem 1.1

First, from the definition of ρ and using the Gauss equation,

$$\begin{aligned}\rho &= \frac{2}{m(m-1)}\sum_{i<j}\langle R(e_i,e_j)e_j,e_i\rangle = \frac{1}{m(m-1)}\sum_{i\neq j}\langle R(e_i,e_j)e_j,e_i\rangle\\ &= \frac{1}{m(m-1)}\sum_{i\neq j}\Big(\langle B(e_j,e_j),B(e_i,e_i)\rangle - \|B(e_i,e_j)\|^2 - \Big(\frac{(a')^2}{a^2}-\frac{c}{a^2}\Big)\\ &\qquad -\Big(\frac{a''}{a}-\frac{(a')^2}{a^2}+\frac{c}{a^2}\Big)(\langle T,e_i\rangle^2+\langle T,e_j\rangle^2)\Big)\\ &= -\Big(\frac{(a')^2}{a^2}-\frac{c}{a^2}\Big) + \frac{1}{m(m-1)}\Big(n^2\|H\|^2 - \|B\|^2 - 2(m-1)\Big(\frac{a''}{a}-\frac{(a')^2}{a^2}+\frac{c}{a^2}\Big)\|T\|^2\Big).\end{aligned}$$

Now, set $\tau = B - Hg$, the traceless part of the second fundamental form. Clearly, we have $\|\tau\|^2 = \|B\|^2 - n\|H\|^2$. Hence,

$$\rho = -\Big(\frac{(a')^2}{a^2}-\frac{c}{a^2}\Big)\Big(1-\frac{2}{m}\|T\|^2\Big) - \frac{2a''}{ma}\|T\|^2 + \|H\|^2 - \frac{1}{m(m-1)}\|\tau\|^2. \qquad (3.1)$$

For any $\alpha\in\{1,\dots,p\}$, define the operator $S_\alpha : TM\to TM$ by

$$\langle S_\alpha X,Y\rangle = \langle\tau(X,Y),\xi_\alpha\rangle.$$

https://doi.org/10.1017/S0004972716001271 Published online by Cambridge University Press

Obviously, $S_\alpha = A_\alpha - \langle H, \xi_\alpha \rangle \mathrm{Id}$ and $[A_\alpha, A_\beta] = [S_\alpha, S_\beta]$. From the Ricci equation, given in Proposition 2.1,

$$\rho^\perp = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha,\beta=1}^{p} \|[A_\alpha, A_\beta]\|^2} = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha,\beta=1}^{p} \|[S_\alpha, S_\beta]\|^2}.$$

Since the operators S_α are symmetric and trace-free, we can apply Theorem 2.2 at any point of M to get

$$\sum_{\alpha,\beta=1}^{p} \|[S_\alpha, S_\beta]\|^2 \leqslant \Big(\sum_{\alpha=1}^{p} \|S_\alpha\|^2\Big)^2.$$

Thus,

$$\rho^\perp \leqslant \frac{1}{m(m-1)} \sum_{\alpha=1}^{m} \|S_\alpha\|^2 = \frac{1}{m(m-1)} \|\tau\|^2.$$

Combining this with (3.1) gives

$$\|H\|^2 \geqslant \rho + \rho^\perp + \Big(\frac{(a')^2}{a^2} - \frac{c}{a^2}\Big)\Big(1 - \frac{2}{m}\|T\|^2\Big) + \frac{2a''}{ma}\|T\|^2,$$

which concludes the proof. □

4. An application to submanifolds of $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$

To finish this note, we apply Theorem 1.1 to submanifolds of the warped product of the type $\mathbb{R} \times_a \mathbb{H}^n(c)$, where a is the real function defined by $a(t) = e^{\lambda t}$ and λ is a real constant.

Corollary 4.1. *Let M^m be a submanifold of the warped product $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ with normalised scalar and normal scalar curvatures ρ and $\rho^\perp$ and mean curvature H. Then*

$$\|H\|^2 \geqslant \rho + \rho^\perp + \lambda^2 - ce^{-2\lambda t}\Big(1 - \frac{2}{m}\|T\|^2\Big).$$

Proof. This comes directly from Theorem 1.1, using the facts that

$$\frac{(a')^2}{a^2} - \frac{c}{a^2} = \lambda^2 - ce^{-2\lambda t}, \quad \frac{a''}{a^2} = \lambda^2.$$

Hence, the terms

$$\Big(\frac{(a')^2}{a^2} - \frac{c}{a^2}\Big)\Big(1 - \frac{2}{m}\|T\|^2\Big) + \frac{2a''}{ma}\|T\|^2 = \lambda^2 - ce^{-2\lambda t}\Big(1 - \frac{2}{m}\|T\|^2\Big).$$ □

Comparing $\|H\|^2$ with ρ is a natural question which leads to rigidity results. Indeed, by the Gauss formula, we know that, for hypersurfaces of space forms, ρ is up to a constant (which is the sectional curvature k of the ambient space form) the second mean curvature H_2, that is, the second elementary symmetric polynomial in the principal curvatures. Moreover, it is a classical fact that $H^2 \geqslant H_2$ with equality at umbilical points. Hence, assuming $H^2 \leqslant \rho - k$ implies that M is a hypersphere. In this spirit, and using the above DDVV inequality, we give the following rigidity result.

https://doi.org/10.1017/S0004972716001271 Published online by Cambridge University Press

Corollary 4.2. *Let M^m be a complete submanifold without boundary of the warped product $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ with normalised scalar and normal scalar curvatures ρ and $\rho^\perp$ and mean curvature H. If $\|H\|^2 \leqslant \rho + \lambda^2$, then*

$$\|H\|^2 = \rho + \lambda^2, \quad \rho^\perp = 0, \quad m = 2 \quad \text{and} \quad \|T\| = 1.$$

Hence, M is a surface of the type $\mathbb{R} \times_{e^{\lambda t}} \gamma$, where γ is a curve in $\mathbb{H}^n(c)$.

Proof. First note that since $n \geqslant 2$, $\|T\|^2 \leqslant 1$ and $c < 0$,

$$ce^{-2\lambda t}\Big(1 - \frac{2}{n}\|T\|^2\Big) \leqslant 0.$$

By definition, $\rho^\perp \geqslant 0$. Hence, from Corollary 4.1, $\|H\|^2 \leqslant \rho + \lambda^2$ is possible if and only if $\|H\|^2 = \rho + \lambda^2$, $\rho^\perp = 0$, $m = 2$ and $\|T\| = 1$. Since $n = 2$, M is a surface and the fact that $\|T\| = 1$ implies that $T = \partial_t$ and so M is of the type $I \times_{e^{\lambda t}} \gamma$, where γ is a curve in $\mathbb{H}^n(c)$. Since we assume that M is complete and without boundary, $I = \mathbb{R}$. This concludes the proof. □

References

[1] Q. Chen and Q. Cui, ‘Normal scalar curvature and a pinching theorem in $\mathbb{S}^m \times \mathbb{R}$ and $\mathbb{H}^m \times \mathbb{R}$’, *Sci. China Math.* **54**(9) (2011), 1977–1984.

[2] P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, ‘A pointwise inequality in submanifold theory’, *Arch. Math.* **35**(1) (1999), 115–128.

[3] F. Dillen, J. Fastenakels and J. Van der Veken, ‘Remarks on an inequality involving the normal scalar curvature’, in: *Pure and Applied Differential Geometry* (Shaker, Aachen, 2007), 83–92.

[4] J. Q. Ge and Z. Z. Tang, ‘A proof of the DDVV conjecture and its equality case’, *Pacific J. Math.* **237** (2008), 87–95.

[5] M.-A. Lawn and M. Ortega, ‘A fundamental theorem for hypersurfaces in semi-Riemannian warped products’, *J. Geom. Phys.* **90** (2015), 55–70.

[6] Z. Q. Lu, ‘Normal scalar curvature conjecture and its applications’, *J. Funct. Anal.* **261** (2011), 1284–1308.

JULIEN ROTH, LAMA, UPEM-UPEC-CNRS, Cité Descartes,
Champs sur Marne, 77454 Marne-la-Vallée cedex 2, France
e-mail: julien.roth@u-pem.fr

https://doi.org/10.1017/S0004972716001271 Published online by Cambridge University Press