25
Gravity as a field theory

This chapter provides the briefest, tangential encounter with the Einsteinian gravity viewed as a field theory. Gravity is a huge topic, full of subtleties, and it deserves to be introduced as a systematic tower of thought, rather than as a gallery of sketchy assertions. The purpose of this chapter is therefore no more than to indicate, to those who already know the general theory of relativity, how gravity fits into the foregoing discussions, i.e. why the foregoing ideas are still valid in the presence of gravity, and how we generalize our notion of covariance to include the gravitational force.

25.1 Newtonian gravity

Newtonian gravity plays virtually no role in field theory, for the simple reason that gravity barely couples to any of the fields. Gravity is such a weak force at the scale of elementary particles that it is almost completely negligible. There are occasions, however, when we use field theory outside of the realm of the elementary physics. For instance, fluid dynamics is a field theory where gravity plays an often significant role.

In order to include gravity in terrestrial systems, we do not need to think about Einstein or relativity. Gravity is simply an effective potential

$$V = mgx + \text{const.}, \tag{25.1}$$

where x is the height above the centre of gravity. In this effective theory of gravity, planets and large objects are considered to be point particles, located at the centre of gravity of the system. Eqn. (25.1) expresses a linear, flat-Earth geometry, in which the potential is usually measured from the ground up (for small distances of a few hundred metres). The arbitrary constant in the potential is analogous to the arbitrariness in the electromagnetic potential A_μ. Instead of gauge invariance, we have a corresponding arbitrariness in the origin of the gravitational potential.

https://doi.org/10.1017/9781009289887.029 Published online by Cambridge University Press

25.2 Curvature

On astrophysical scales, gravity is the dominant force, and we need to consider the subtleties of general relativity. There are two motivations for wanting to do this

- Einsteinian gravity can be formulated as a field theory, in which the metric tensor (spacetime itself) is also a dynamical field. This enables us to understand gravity and spacetime as a dynamical system, leaning on all of the lessons we have learned from electromagnetism etc.
- In the early universe, there was an important coupling between gravity and other fundamental fields. Thus, relativistic, covariant formulations of fields which include gravity are important models to consider.

Gravity therefore means Einsteinian gravity here, and this, we know, has a natural expression in terms of the intrinsic curvature of spacetime. For the reasons discussed in the previous section, it makes no sense to look at non-relativistic theories in the presence of a relativistically generalized gravitational potential; such combinations would not be consistently compatible. We therefore dispense with the non-relativistic theories for the remainder of this short chapter.

25.3 Particles in a gravitational field

The essence of general relativity is that gravitational effects can be considered as physics in non-inertial frames. A non-inertial frame is a coordinate basis which is either accelerating or which contains a gravitational field. These two situations are indistinguishable, according to the equivalence principle, and so this is a kind of tautology. Indeed, we could go on to refer to the gravitational field as an acceleration field.

How shall we describe physics in such frames? Non-linear coordinate transformations can always map us from a locally inertial frame,[1] so covariance will help us to formulate theories optimally. The discussion which follows is based on the conventions and notations of Weinberg [133]. Readers who are unfamiliar with gravity could do worse than to consult his book, since there is no room for more than a cursory sketch here.

Let us denote the coordinates and derivatives and metric in a locally inertial Cartesian frame by ξ^μ, $\overset{\xi}{\partial}_\mu$, $\eta_{\mu\nu}$, and the corresponding quantities in any other coordinate system (flat, curvilinear, curved, accelerating etc.) by x^μ, ∂_μ, $g_{\mu\nu}$. The transformation which relates the two metrics is written according to the

[1] Suppose you are in a fighter plane and are suffering from the effects of strong acceleration G forces: to transform to a locally inertial frame, simply press the ejector seat button and you will soon be in a freely falling coordinate system.

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usual tensor rules,

$$\begin{aligned}\eta^{\alpha\beta} &= g^{\mu\nu}\, L_\mu^{\ \alpha} L_\nu^{\ \beta} \\ &= g^{\mu\nu}\, (\partial_\mu \xi^\alpha)(\partial_\nu \xi^\beta).\end{aligned} \tag{25.2}$$

In its locally inertial, or freely falling, coordinate frame, a moving particle seems to be following a straight-line path (although, since the frame is only inertial locally, we should not extrapolate too far from our position of observation). The equation of motion of such a particle would then be

$$m \frac{\mathrm{d}^2 \xi^\alpha}{\mathrm{d}\tau^2} = 0, \tag{25.3}$$

where the proper time τ is defined in the usual way by

$$-c^2 \mathrm{d}\tau^2 = \eta_{\alpha\beta} \mathrm{d}\xi^\alpha \mathrm{d}\xi^\beta. \tag{25.4}$$

Suppose now we transform into a general set of coordinates, using the Lorentz transformation $L_\mu^{\ \nu}$. We then have to transform ξ^α, so that eqn. (25.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\xi^\alpha(x)}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\xi^\alpha(x)}{\mathrm{d}x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right). \tag{25.5}$$

Thus, the equation of motion becomes

$$(\partial_\mu \xi^\alpha)\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + (\partial_\mu \partial_\nu \xi^\alpha)\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0. \tag{25.6}$$

This can be simplified by multiplying through by $\overset{\xi}{\partial}_\alpha\, x^\lambda$ and using the chain-rule $(\overset{\xi}{\partial}_\alpha\, x^\lambda)(\partial_\lambda \xi^\beta) = \delta_\alpha^\beta$ to give

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0, \tag{25.7}$$

which is the geodesic equation, where

$$\Gamma^\lambda_{\mu\nu} = (\partial_\mu \partial_\nu \xi^\alpha)(\overset{\xi}{\partial}_\alpha\, x^\lambda). \tag{25.8}$$

The presence of the affine connection $\Gamma^\lambda_{\mu\nu}$ signals the non-linear nature of the coordinates. The connection may also be expressed in terms of the metric tensor as

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left\{\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right\}. \tag{25.9}$$

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25.4 Geodesics

The geodesic equation can also be understood in a different way, from the action principle. The geodesic equation is, in a sense, the structure of empty space, so what if we take an empty action in a locally inertial rest frame of a general curved spacetime and vary it with respect to different paths, as follows:

$$x^\mu \to x^\mu(\lambda) + \delta x^\mu(\lambda)? \tag{25.10}$$

The action would then be

$$S = a \int \mathrm{d}\tau, \tag{25.11}$$

where τ is the proper time, defined in eqn. (3.38) and a is a constant with the dimensions of energy. Writing this in general coordinates, we have

$$S = a \int \sqrt{g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu}, \tag{25.12}$$

or – introducing a parameter λ,

$$S = a \int \mathrm{d}\lambda \, \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \int \mathrm{d}\lambda \sqrt{g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}. \tag{25.13}$$

This equation can now be varied with respect to x^μ to obtain the path of 'least action' in the coordinate system x. We already know that, in a locally inertial frame, the path of an object would be a straight line, and in a rest frame there is no motion. So the question is: how does this look to a different observer in possibly accelerating coordinates? The variation of the action is

$$\delta S = a \int \mathrm{d}\lambda \, \frac{1}{2} \frac{\mathrm{d}\lambda}{\mathrm{d}\tau} \left\{ \delta g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + 2 g_{\mu\nu} \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \right\} = 0. \tag{25.14}$$

Since we are looking at a coordinate variation, we have

$$\delta g_{\mu\nu} = (\partial_\lambda g_{\mu\nu}) \, \delta x^\lambda; \tag{25.15}$$

see eqn. (4.85). Thus, integrating by parts and writing $\mathrm{d}\lambda \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}$ as $\mathrm{d}\lambda \frac{\mathrm{d}\lambda \mathrm{d}\tau}{\mathrm{d}\tau \mathrm{d}\tau}$,

$$\begin{aligned} \delta S = \frac{a}{2} \int \, \mathrm{d}\tau \Big\{ (\partial_\lambda g_{\mu\nu}) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - 2(\partial_\rho g_{\mu\nu}) \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} g_{\mu\lambda} \\ -2 g_{\mu\nu} \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} g_{\mu\lambda} \Big\} \delta x^\lambda = 0. \end{aligned} \tag{25.16}$$

Here we have assumed that the surface term

$$\Delta \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \delta x_\mu \right) = 0 \tag{25.17}$$

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vanishes for continuity. From eqn. (25.9), the result may be identified as

$$\delta S = a \int \left\{ -\Gamma^{\lambda}_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} - \frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tau^2} \right\} g_{\lambda\sigma} \delta x^{\sigma} \mathrm{d}\tau = 0. \tag{25.18}$$

We have used the symmetry on the lower indices of $\Gamma^{\lambda}_{\mu\nu}$. Thus we end up with the geodesic equation once again:

$$\frac{\mathrm{d}^2 x^{\lambda}}{\mathrm{d}\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{x^{\nu}}{\mathrm{d}\tau} = 0. \tag{25.19}$$

25.5 Curvature

The curvature of a vector field ξ^{σ} may be defined by the commutator of covariant derivatives, just as in the case of the electromagnetic field (see eqn. (10.45)). This defines a process of parallel transport of vectors and a tensor known as the Riemann curvature tensor:

$$[\nabla_{\mu}, \nabla_{\nu}]\xi^{\sigma} = -R^{\lambda}{}_{\sigma\mu\nu}\xi_{\lambda}. \tag{25.20}$$

Also analogous to electromagnetism is the expression of the curvature as a covariant curl:

$$R^{\lambda}{}_{\mu\nu\kappa} = \nabla_{\kappa}\Gamma^{\lambda}_{\mu\nu} - \nabla_{\nu}\Gamma^{\lambda}_{\mu\kappa}. \tag{25.21}$$

This may be compared with eqn. (2.24). The Riemann tensor has the following symmetry properties:

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \tag{25.22}$$

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = R_{\lambda\mu\kappa\nu} = R_{\mu\lambda\kappa\nu} \tag{25.23}$$

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0. \tag{25.24}$$

The Ricci tensor is defined as the contraction

$$R_{\mu\kappa} = R_{\lambda\mu\nu\kappa} g^{\lambda\nu} = R^{\nu}{}_{\mu\nu\kappa}, \tag{25.25}$$

and satisfies

$$R_{\mu\nu} = R_{\nu\mu}. \tag{25.26}$$

The scalar curvature is the total contraction

$$R = R^{\mu\nu}{}_{\mu\nu}. \tag{25.27}$$

The curvature satisfies Bianchi identities, just like the electromagnetic field:

$$\nabla_{\rho} R_{\lambda\mu\nu\kappa} + \nabla_{\kappa} R_{\lambda\mu\rho\nu} + \nabla_{\nu} R_{\lambda\mu\kappa\rho} = 0. \tag{25.28}$$

Contracting with $g^{\lambda\nu}$ gives

$$\nabla_{\mu} \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right] = 0. \tag{25.29}$$

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25.6 The action

The action for matter coupled to gravity is written

$$S = S_{\mathrm{M}} + S_{\mathrm{G}}, \tag{25.30}$$

where

$$S_{\mathrm{G}} = -\frac{c^4}{16\pi G}\int(\mathrm{d}x)\ \left[R - 2\Lambda\right]; \tag{25.31}$$

$(\mathrm{d}x) = \mathrm{d}t\mathrm{d}^n\mathbf{x}\sqrt{g}$ and $g = -\det g_{\mu}$. S_{M} is the action for matter fields. These act as the source of the gravitational field, i.e. they carry gravitational charge (mass/energy).

Λ is the cosmological constant, which is usually set to zero. The variation of the action with respect to the metric is

$$\begin{aligned}
\delta\sqrt{g} &= -\frac{1}{2}\sqrt{g}g^{\mu\nu}\,\delta^{\mu\nu}\\
\delta R &= \delta(g^{\mu\nu}R_{\mu\nu})\\
&= \delta g^{\mu\nu}\,R_{\mu\nu}.
\end{aligned} \tag{25.32}$$

Thus,

$$\delta S = -\frac{c^4}{16\pi G}\int(\mathrm{d}x)\left[-\frac{1}{2}g_{\mu\nu}[R - 2\Lambda/c^2] + R_{\mu\nu}\right]\delta g^{\mu\nu} + \frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = 0. \tag{25.33}$$

The last term is the conformal energy–momentum tensor

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{\Lambda}{c^2}g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu}. \tag{25.34}$$

This is Einstein's field equation for gravity. It is, of course, supplemented by the field equations for matter to complete the dynamical system. Notice that matter and energy (the energy–momentum tensor) is the source of gravitation. Matter, in other words, carries the gravitational charge: mass/energy.

The solution of these field equations is non-trivial and beyond the scope of this book.

25.7 Kaluza–Klein theory

Following Maxwell's treatise on the electromagnetic field, Theodore Kaluza was amongst the first to propose a scheme for unifying the forces of nature using a classical field theory, based in Einstein's equations. Kaluza's paper,

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communicated to Einstein, endured a long delay before its publication in 1921. His main idea, later refined by Oskar Klein, made the bold assertion that, if one postulated the existence of extra dimensions, then both of the known forces of nature (electromagnetism and gravity) could be unified, using Einstein's idea of spacetime curvature. In Kaluza–Klein theory, the line element is assumed to have the usual form

$$\mathrm{d}s^2 = \hat{g}_{\hat{\mu}\hat{\nu}}\, \mathrm{d}x^{\hat{\mu}}\mathrm{d}x^{\hat{\nu}} \tag{25.35}$$

where the careted indices run from $0, \ldots, 5$ and $x^{\mu} = (ct, x^1, x^2, x^3, y) = (x^{\mu}, y)$. Uncareted indices represent the usual $3+1$ dimensional vectors of general relativity. In order to account for the $U(1)$ symmetry, Klein proposed that the extra dimension should have the topology of a circle, with length L. The electromagnetic field plays the role of a vector field on the $3+1$ dimensional spacetime, seen as the projection of the curvature of the extra dimension:

$$\begin{aligned}\mathrm{d}s^2 &= \hat{g}_{\hat{\mu}\hat{\nu}}\, \mathrm{d}x^{\hat{\mu}}\mathrm{d}x^{\hat{\nu}} \\ &= g_{\mu\nu}\, \mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + (\mathrm{d}y + \kappa A_{\mu}(x)\mathrm{d}x^{\mu})^2,\end{aligned} \tag{25.36}$$

where κ is a constant. Covariance in the extra dimension determines the transformation rule for A_{μ} under coordinate transformations $y' = \theta(y, x^{\mu})$:

$$\mathrm{d}y' = \frac{\partial\theta}{\partial y}\, \mathrm{d}y + \partial_{\mu}\theta\, \mathrm{d}x^{\mu}. \tag{25.37}$$

For consistency with eqn. (25.36), one requires $\partial\theta/\partial y = 1$, so that under a change of y only,

$$\begin{aligned}\mathrm{d}y + \kappa A_{\mu}\mathrm{d}x^{\mu} \to\ & \mathrm{d}y' + \kappa A'_{\mu}\mathrm{d}x^{\mu} \\ &= \left(\mathrm{d}y + \partial_{\mu}\theta\mathrm{d}x^{\mu}\right) + \kappa A'\mathrm{d}x^{\mu} \\ &= \mathrm{d}y + \kappa\left(A'_{\mu}(x) + \kappa^{-1}\partial_{\mu}\theta\right)\mathrm{d}x^{\mu}.\end{aligned} \tag{25.38}$$

Invariance of $\mathrm{d}s^2$ therefore requires

$$A'_{\mu}(x) = A_{\mu}(x) - \kappa^{-1}\partial_{\mu}\theta, \tag{25.39}$$

which is the electromagnetic gauge transformation. From the line element, the metric is

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + \kappa A_{\mu}A_{\nu} & \kappa A_{\mu} \\ \kappa A_{\nu} & 1 \end{pmatrix}; \tag{25.40}$$

however, by changing coordinates to the so-called horizontal lift basis, with 1-forms:

$$\begin{aligned}\tilde{\omega}^{\mu} &= \mathrm{d}x^{\mu} \\ \tilde{\omega}^{5} &= \mathrm{d}y + \kappa A_{\mu}(x)\mathrm{d}x^{\mu},\end{aligned} \tag{25.41}$$

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the metric may be diagonalized, at the expense of non-Cartesian coordinates:

$$\hat{g}'_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \tag{25.42}$$

The basis vectors conjugate to the 1-forms are $\tilde{\omega}^{\hat{\mu}}\hat{e}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\ \hat{\nu}}$, i.e.

$$\begin{aligned} \hat{e}_\mu &= \partial_\mu - \kappa A_\mu(x)\partial_y \\ \hat{E}_5 &= \partial_y. \end{aligned} \tag{25.43}$$

In this anholonomic basis, there is one non-zero commutator:

$$[\hat{e}_\mu, \hat{e}_\nu] = -\kappa F_{\mu\nu}(x)\ \partial_y, \tag{25.44}$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which gives the Lie algebra relation

$$[\hat{e}_{\hat{\mu}}, \hat{e}_{\hat{\mu}}] = C_{\hat{\mu}\hat{\nu}}^{\ \ \hat{\rho}}\ \hat{e}_{\hat{\rho}}. \tag{25.45}$$

The affine connection, in a non-holonomic basis, is

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}\left[\hat{e}_\lambda\, g_{\mu\nu} + \hat{e}_\nu\, g_{\mu\lambda} - \hat{e}_\mu\, g_{\lambda\nu} + C_{\mu\nu\lambda} + C_{\mu\lambda\nu} + C_{\lambda\nu\mu}\right], \tag{25.46}$$

so that we have non-zero components

$$\begin{aligned} \hat{\Gamma}_{\mu\nu5} = \hat{\Gamma}_{\mu5\nu} = -\hat{\Gamma}_{5\mu\nu} &= -\frac{1}{2}\kappa F_{\mu\nu} \\ \hat{\Gamma}_{555} = 0 \quad , \quad \hat{\Gamma}_{\mu\nu\lambda} &= \Gamma_{\mu\nu\lambda}. \end{aligned} \tag{25.47}$$

From these, one may calculate the scalar curvature for the Einstein action,

$$\begin{aligned} \hat{R} &= \hat{R}^{\mu\nu}_{\ \ \mu\nu} + 2\hat{R}^{\mu5}_{\ \ \mu5} \\ &= R + \frac{\kappa^2}{4}F^{\mu\nu}F_{\mu\nu}. \end{aligned} \tag{25.48}$$

Thus, the Einstein action, in five dimensions, automatically incorporates and extrapolates the Maxwell action:

$$S = -\frac{c^4}{16\pi GL}\int \mathrm{d}^4x\mathrm{d}y\sqrt{\hat{g}}\left[\hat{R} - 2\Lambda]\right]. \tag{25.49}$$

Kaluza–Klein theory came into trouble when it attempted to incorporate the newly discovered nuclear forces in a common framework, and was eventually abandoned in its original form. However, the essence of Kaluza–Klein theory lives on, in a more sophisticated guise, in super-string theory.

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