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SEPARATION AXIOMS AND SUBCATEGORIES OF TOP

(Dedicated to Professor K. Morita on his 60th birthday)

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Abstract

(Point, closed subset)-separation axioms and closed subsets separation axioms for topological spaces will be uniformly defined. Then it is shown that a subcategory $\mathscr{A}$ of TOP is bireflective in TOP if and only if Ob $\mathscr{A}$ consists of all separated spaces for some (point, closed subset)-separation axiom. A characterization theorem on subcategories of all separated spaces for closed subsets separation axioms is also given by using the category SEP of all separation spaces and the embedding functor G: TOP$\rightarrow$SEP. As an application we have that a T_1-space is normal if and only if it is embedded in a product space of the unit intervals in SEP.

There are three basic types of separation axioms depending on whether they involve separation of: I. pairs of points; II. pairs consisting of a point and a subset; or III. pairs of subsets. Wyler (1973) gave a characterization of those full subcategories of the category TOP of topological spaces which consist of all spaces satisfying given axioms of type I.

In this paper, we vastly generalize Wyler's result to one involving the topological functors of Herrlich (1974). In particular, we obtain characterization theorems for separation axioms of types II and III. For type II, we take $\mathscr{X}$ to be the category TOP, and $\mathscr{Y}$ to be the category CLS of so-called 'closure spaces'; for type III, we take $\mathscr{X}$ to be the category TOP, and $\mathscr{Y}$ to be the category SEP of 'separation spaces' in the sense of Wallace (1941). In each case there is a distinguished functor $G:\mathscr{X}\rightarrow\mathscr{Y}$. It is seen that to give axioms of the particular type is to give a functor $\Sigma:\mathscr{X}\rightarrow\mathscr{Y}$ together with a comparison natural transformation $\eta: G \Rightarrow \Sigma$ whose components become isomorphisms in ENS. A space X is considered to satisfy a given separation axiom (Σ, η) if η_X is an isomorphism. Our main results can be stated as follows. A subcategory of TOP consists of all spaces satisfying a separation axiom of type II if and only if it is bireflective in TOP (Theorem 3.1). A subcategory of the category R_0-TOP of R_0-spaces in the sense of Davis (1961) consists of all R_0-spaces satisfying a separation axiom of type III if and only if it is an intersection of a bireflective subcategory and the subcategory R_0-TOP in the category SEP (Theorem 2.9). Examples for axioms of types II and III and of other types will be given (§4).

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Terminology not explained here is from Herrlich (1974) and Herrlich and Strecker (1973). Subcategories are assumed to be full and replete (= isomorphism closed).

1. Separations for topological functors

We shall recall the definition of topological functors defined by Herrlich (1974).

Let $\mathscr{X}$ be a category. A *source* in $\mathscr{X}$ is a pair $(X, f_i)_I$ consisting of an $\mathscr{X}$-object X and a family of $\mathscr{X}$-morphisms $f_i: X \to X_i$ indexed by a class I. Let E be a class of epimorphisms in $\mathscr{X}$ closed under composition with isomorphisms and M be a class of sources in $\mathscr{X}$ closed under composition with isomorphisms. $\mathscr{X}$ is *(E, M)-factorizable* if and only if for every source $(X, f_i)_I$ in $\mathscr{X}$ there exists $e: X \to Y$ in E and $(Y, m_i)_I$ in M such that $f_i = m_i \cdot e$ for each $i \in I$. $\mathscr{X}$ has the *(E, M)-diagonalization property* provided that whenever f and e are morphisms and $(Y, m_i)_I$ and $(Z, f_i)_I$ are sources in $\mathscr{X}$ such that $e \in E$, $(Y, m_i)_I \in M$ and $f_i \cdot e = m_i \cdot f$ for each $i \in I$, then there exists a morphism $g: Z \to Y$ such that $f = g \cdot e$ and $f_i = m_i \cdot g$ for each $i \in I$. $\mathscr{X}$ is called an (E, M)-category if and only if it is (E, M)-factorizable and has the (E, M)-diagonalization property.

Let $\mathscr{X}$ be an (E, M)-category and $T: \mathscr{A} \to \mathscr{X}$ be a functor. A source $(A, f_i: A \to A_i)_I$ in $\mathscr{A}$ is called *T-initial* if and only if for each source $(B, g_i: B \to A_i)_I$ in $\mathscr{A}$ and each morphism $f: TB \to TA$ in $\mathscr{X}$ such that $Tg_i = Tf_i \cdot f$ for each $i \in I$ there exists a unique morphism $\tilde{f}: B \to A$ in $\mathscr{A}$ such that $T\tilde{f} = f$ and $g_i = f_i \cdot \tilde{f}$ for each $i \in I$. A source $A, f_i: A \to A_i)_I$ in $\mathscr{A}$ *T-lifts* a source $(X, g_i: X \to TA_i)_I$ in $\mathscr{X}$ if and only if there exists an isomorphism $h: X \to TA$ in $\mathscr{X}$ with $g_i = Tf_i \cdot h$ for each $i \in I$. T is called *(E, M)-topological* if and only if for each family $(A_i)_I$ of $\mathscr{A}$-objects and each source $(X, m_i: X \to TA_i)_I$ in M there exists a T-initial source $(A, f_i: A \to A_i)_I$ in $\mathscr{A}$ which T-lifts $(X, m_i)_I$.

The following result is due to Herrlich (1974).

PROPOSITION 1.1 *If T is an embedding of a subcategory $\mathscr{A}$ of $\mathscr{X}$ into $\mathscr{X}$, then the following conditions are equivalent*:

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(a) *T is (E, M)-topological;*

(b) *if $(X, m_i : X \to A_i)_I$ belongs to M and all A_i belong to $\mathscr{A}$, then $(X, m_i)_I$ belongs to $\mathscr{A}$;*

(c) *$\mathscr{A}$ is an E-reflective subcategory of $\mathscr{X}$.*

Let $\mathscr{A}$ be a subcategory of $\mathscr{X}$. Then from this proposition we have a smallest E-reflective subcategory $\tilde{\mathscr{A}}$ of $\mathscr{X}$ which contains $\mathscr{A}$. In fact an $\mathscr{X}$-object X belongs to $\tilde{\mathscr{A}}$ if and only if there exists a source $(X, m_i : X \to A_i)_I$ in M with $\mathscr{A}$-object A_i for each $i \in I$.

Let $\mathscr{X}$ be an (E, M)-category and $S : \mathscr{A} \to \mathscr{X}$ be a functor. E_S denotes the class of all morphisms f in $\mathscr{A}$ with $Sf \in E$ and M_S the class of all S-initial sources $(A, f_i)_I$ in $\mathscr{A}$ with $(SA, Sf_i)_I \in M$. Herrlich (1974) shows that if S is (E, M)-topological, then $\mathscr{A}$ is an (E_S, M_S)-category. If $G : \mathscr{B} \to \mathscr{A}$ is (E_S, M_S)-topological, then SG is (E, M)-topological.

PROPOSITION 1.2 *Let $\mathscr{X}$ be an (E, M)-category and $S : \mathscr{A} \to \mathscr{X}$, $T : \mathscr{B} \to \mathscr{X}$, $F : \mathscr{B} \to \mathscr{A}$ and $G : \mathscr{A} \to \mathscr{B}$ be functors with $SF = T$ and $TG = S$. Suppose that T is (E, M)-topological and there is a natural equivalence $\alpha : 1 \Rightarrow FG$ such that $S\alpha = 1 : S \Rightarrow S$. Then S is (E, M)-topological.*

PROOF. Let $(A_i)_I$ be a family of $\mathscr{A}$-objects and $(X, m_i : X \to SA_i)_I$ be a source in M. Since $SA_i = TGA_i$ and T is (E, M)-topological, there exists a T-initial source $(B, n_i : B \to GA_i)_I$ and an isomorphism $h : X \to TB$ such that $m_i = Tn_i \cdot h$. Consider a source $(FB, \alpha_{A_i}^{-1} \cdot Fn_i)_I$ in $\mathscr{A}$. Since $SFB = TB$ and $S(\alpha_{A_i}^{-1} \cdot Fn_i) \cdot h = Tn_i \cdot h = m_i$, $(FB, \alpha_{A_i}^{-1} \cdot Fn_i)_I$ S-lifts $(X, m_i)_I$. Suppose that $(C, f_i : C \to A_i)_I$ is a source in $\mathscr{A}$ and $k : SC \to SFB$ is an $\mathscr{X}$-morphism with $Sf_i = Tn_i \cdot k$. By the assumption that $(B, n_i)_I$ is T-initial, we have a $\mathscr{B}$-morphism $\tilde{k} : GC \to B$ such that $Gf_i = n_i \cdot \tilde{k}$ and $T\tilde{k} = k$. Let $\hat{k} = F\tilde{k} \cdot \alpha_C$. Then $\alpha_{A_i}^{-1} \cdot Fn_i \cdot \hat{k} = f_i$ and $S\hat{k} = k$. Thus we have that $(FB, \alpha_{A_i}^{-1} \cdot Fn_i)_I$ is S-initial and that S is (E, M)-topological.

A *separation system* is a family $\mathfrak{g} = (\mathscr{A}, \mathscr{B}, \mathscr{X}, S, T, F, G, \alpha)$ consisting of an (E, M)-category $\mathscr{X}$, (E, M)-topological functors $S : \mathscr{A} \to \mathscr{X}$ and $T : \mathscr{B} \to \mathscr{X}$, functors $F : \mathscr{B} \to \mathscr{A}$ and $G : \mathscr{A} \to \mathscr{B}$ with $S = TG$ and $T = SF$ and a natural transformation $\alpha : 1 \Rightarrow FG$. A $\mathfrak{g}$-*separation* is a pair (Σ, η) of a functor $\Sigma : \mathscr{A} \to \mathscr{B}$ and a natural transformation $\eta : G \Rightarrow \Sigma$ such that $T\eta_A$ belongs to E for each $\mathscr{A}$-object A. For a $\mathfrak{g}$-separation (Σ, η) an $\mathscr{A}$-object A is called (Σ, η)-*separated* if and only if η_A is an isomorphism. A full subcategory of $\mathscr{A}$ consisting of all (Σ, η)-separated objects is denoted by $\mathscr{A}_{(\Sigma, \eta)}$. Then we have the following.

THEOREM 1.3 *Let $\mathfrak{g} = (\mathscr{A}, \mathscr{B}, \mathscr{X}, S, T, F, G, \alpha)$ be a separation system and $\mathscr{A}_\alpha$ be a subcategory of $\mathscr{A}$ whose objects X satisfy that α_X be isomorphisms. For a*

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$\mathfrak{g}$-separation (Σ, η), there exists an E_T-reflective subcategory $\mathscr{B}_{(\Sigma,\eta)}$ of $\mathscr{B}$ such that

$$G(\mathscr{A}_{(\Sigma,\eta)} \cap \mathscr{A}_\alpha) = \mathscr{B}_{(\Sigma,\eta)} \cap G(\mathscr{A}_\alpha).$$

Conversely, if a subcategory $\mathscr{A}'$ of $\mathscr{A}_\alpha$ satisfies

$$G(\mathscr{A}') = \mathscr{B}' \cap G(\mathscr{A}_\alpha)$$

for an E_T-reflective subcategory $\mathscr{B}'$ of $\mathscr{B}$, then there exists a $\mathfrak{g}$-separation (Σ, η) such that $\mathscr{A}' = \mathscr{A}_{(\Sigma,\eta)} \cap \mathscr{A}_\alpha$.

PROOF. For a $\mathfrak{g}$-separation (Σ, η), let $\mathscr{B}_{(\Sigma,\eta)}$ be a smallest E_T-reflective subcategory of $\mathscr{B}$ which contains $G(\mathscr{A}_{(\Sigma,\eta)} \cap \mathscr{A}_\alpha)$. We shall show that $G(\mathscr{A}_{(\Sigma,\eta)} \cap \mathscr{A}_\alpha) \supset \mathscr{B}_{(\Sigma,\eta)} \cap G(\mathscr{A}_\alpha)$. Let B be an object in $\mathscr{B}_{(\Sigma,\eta)}$. Then there exists a source $(B, f_i : B \to GA_i)_I$ in M_T with (Σ, η)-separated $\mathscr{A}_\alpha$-objects A_i, $i \in I$. Let $B = GA$ for an $\mathscr{A}_\alpha$-object A. It is sufficient to show that A is (Σ, η)-separated. Let $g_i = \alpha_{A_i}^{-1} \cdot Ff_i \cdot \alpha_A : A \to A_i$. Then $TGg_i = TG\alpha_{A_i}^{-1} \cdot Tf_i \cdot TG\alpha_A$. Since T is faithful (cf. Herrlich (1974) Th. 3.1), $Gg_i = G\alpha_{A_i}^{-1} \cdot f_i \cdot G\alpha_A$. Since $G\alpha_{A_i}^{-1}$ and $G\alpha_A$ are isomorphisms and M_T is closed under composition with isomorphisms, $(B, Gg_i)_I$ belongs to M_T. By the naturality of η, $\Sigma g_i \cdot \eta_A = \eta_{Ai} \cdot Gg_i$. From the assumption, each η_{A_i} is an isomorphism and hence $(B, \Sigma g_i \cdot \eta_A)_I$ belongs to M_T. On the other hand $\eta_A \in E_T$ and we have that η_A is an isomorphism, that is, A is (Σ, η)-separated.

Conversely, for a given E_T-reflective subcategory $\mathscr{B}'$ of $\mathscr{B}$, denote the embedding functor by $E : \mathscr{B}' \to \mathscr{B}$, the reflector by $R : \mathscr{B} \to \mathscr{B}'$ and the reflection of a $\mathscr{B}$-object B by $r_B : B \to RB$. Define a functor Σ by $\Sigma = \mathrm{ERG}$ and a natural transformation $\eta : G \Rightarrow \Sigma$ by $\eta_A = r_{GA}$ for each $\mathscr{A}$-object A. Then it is easily verified that (Σ, η) is a $\mathfrak{g}$-separation with $\mathscr{A}' = \mathscr{A}_{(\Sigma,\eta)} \cap \mathscr{A}_\alpha$.

COROLLARY 1.4 *Let $\mathscr{X}$ be an (E, M)-category, $S : \mathscr{A} \to \mathscr{X}$, $T : \mathscr{B} \to \mathscr{X}$, $F : \mathscr{B} \to \mathscr{A}$ and $G : \mathscr{A} \to \mathscr{B}$ be functors with $SF = T$ and $TG = S$. Suppose that T is (E, M)-topological and $\mathscr{A}$ is E_T-reflective in $\mathscr{B}$ with the embedding functor G and the reflector F. Then $\mathfrak{g} = (\mathscr{A}, \mathscr{B}, \mathscr{X}, S, T, F, G, 1)$ is a separation system and a subcategory of $\mathscr{A}$ consists of all (Σ, η)-separated objects for a $\mathfrak{g}$-separation (Σ, η) if and only if it is E_S-reflective in $\mathscr{A}$.*

For a separation system $\mathfrak{g}$, two $\mathfrak{g}$-separations (Σ_1, η_1) and (Σ_2, η_2) are called *equivalent* if and only if there is a natural equivalence $\nu : \Sigma_1 \Rightarrow \Sigma_2$ such that $\nu\eta_1 = \eta_2$. If (Σ_1, η_1) and (Σ_2, η_2) are equivalent, $\mathscr{A}_{(\Sigma_1,\eta_1)} = \mathscr{A}_{(\Sigma_2,\eta_2)}$. But the converse does not hold (cf. § 4 below).

2. Separations of pairs of subsets in TOP

Let X be a set and δ_X be a binary relation in the power set $P(X)$ of X. A system (X, δ_X) satisfying the following axioms is called an *s-space* (*separation space*).

(s1) If $A\delta_X B$, then $B\delta_X A$.

(s2) $A\delta_X(B \cup C)$ if and only if $A\delta_X B$ or $A\delta_X C$.

(s3) $\{x\}\delta_X\{x\}$ for any $x \in X$.

(s4) $\phi\bar{\delta}_X X$,

where ϕ denotes the empty set and $\bar{\delta}_X$ means 'not δ_X'.

For s-spaces (X, δ_X) and (Y, δ_Y) a mapping $f: X \to Y$ is called continuous with respect to δ_X and δ_Y provided that if $A\delta_X B$ for $A, B \subset X$, $fA\delta_Y fB$. Thus we have a category SEP consisting of all s-spaces and all continuous mappings.

s-spaces were defined and investigated by Wallace (1941) and many variations of the concept were considered, for example, by Császár (1960), Hammer (1963) and Pervin (1963). The following properties are known or can easily be obtained.

PROPOSITION 2.1 (1) *A morphism* $f:(X, \delta_X) \to (Y, \delta_Y)$ *in* SEP *is a monomorphism in* SEP *if and only if the mapping* $f: X \to Y$ *is one-to-one.*

(2) f *is an epimorphism in* SEP *if and only if it is 'onto'.*

(3) f *is an extremal monomorphism if and only if it is a monomorphism and for any* $A, B \subset X$, $fA\delta_Y fB$ *implies* $A\delta_X B$.

(4) f *is an extremal epimorphism if and only if it is an epimorphism and for any* $C, D \subset Y$, $C\delta_Y D$ *implies* $f^{-1}C\delta_X f^{-1}D$.

(5) *Let* $(X_\lambda, \delta_\lambda)$ *be an s-space for each element* λ *of a set* Λ *and* $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ *be the coproduct in* ENS *with the injection* $i_\lambda: X_\lambda \to X$. *Define a relation* δ_X *as follows:* $A\delta_X B$ *if and only if there exists an element* λ *such that* $i_\lambda^{-1}A\ \delta_\lambda\ i_\lambda^{-1}B$. *Then* (X, δ_X) *is an s-space which is the coproduct in* SEP *of* $(X_\lambda, \delta_\lambda)$, $\lambda \in \Lambda$.

(6) *Let* $(X_\lambda, \delta_\lambda)$ *be an s-space and* $X = \Pi_{\lambda\in\Lambda} X_\lambda$ *be the product in* ENS *with the projection* $p_\lambda: X \to X_\lambda$. *Define a relation* δ_X *as follows:* $A\delta_X B$ *if and only if for any finite coverings* $A = \cup A_i$, $B = \cup B_j$, *there exist numbers* i_0, j_0 *such that* $p_\lambda A_{i_0}\delta_\lambda p_\lambda B_{j_0}$ *for any* $\lambda \in \Lambda$. *Then* (X, δ_X) *is an s-space which is the product in* SEP *of* $(X_\lambda, \delta_\lambda)$, $\lambda \in \Lambda$.

Next we recall the definition of closure spaces (cf. Kannan (1972)). A set X with a mapping $u_X: P(X) \to P(X)$ is called a *closure space* if the following conditions are satisfied. (c1) $u_X A \supset A$. (c2) $u_X(A \cup B) = u_X A \cup u_X B$. (c3) $u_X\phi = \phi$. For closure spaces (X, u_X) and (Y, u_Y) a mapping $f: X \to Y$ is called

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continuous with respect to u_X and u_Y if $fu_XA \subset u_YfA$ for any $A \subset X$. Thus we have a category CLS consisting of all closure spaces and all continuous mappings.

The category TOP of all topological spaces and all continuous mappings is considered as a full subcategory of CLS and moreover it is bireflective in CLS (cf. Kannan (1972)). We shall denote the reflector and the embedding functor by $C:\text{CLS}\to\text{TOP}$ and $D:\text{TOP}\to\text{CLS}$ respectively.

PROPOSITION 2.2 *The forgetful functors* $T_S:\text{SEP}\to\text{ENS}$, $T_C:\text{CLS}\to\text{ENS}$ *and* $T_T:\text{TOP}\to\text{ENS}$ *are* (E, M)*-topological with the class* E *of all isomorphisms in* ENS *and the class* M *of all sources in* ENS. *Each class* E_{T_S}, E_{T_C} *or* E_{T_T} *consists of all bimorphisms in each category and each class* M_{T_S}, M_{T_C} *or* M_{T_T} *consists of all sources* $((X, *_X), f_i:(X, *_X)\to(X_i, *_i))_I$ *for which there exists a subset* K *of* I *such that the induced morphism* $f:(X, *_X)\to(Y, *_Y) = \Pi_K(X_i, *_i)$ *satisfies one of the following conditions respectively*:

(M_{T_S}) *for any* A, $B \subset X$, $fA\delta_YfB$ *implies* $A\delta_XB$,

(M_{T_C}) *for any* A, $B \subset X$, $u_XA = f^{-1}u_YfA$,

(M_{T_T}) = (M_{T_C}).

The proof is easy and so omitted.

A topological space (X, u_X) is called an R_0*-topological space* if it satisfies the following axiom.

(R_0) If $x \in u_X\{y\}$, $x, y \in X$, then $y \in u_X\{x\}$.

The full subcategory R_0-TOP of TOP consisting of all R_0-topological spaces is bireflective in TOP (cf. § 4 below). The forgetful functor $T_R: R_0\text{-TOP}\to\text{ENS}$ is also (E, M)-topological with $E_{T_R} = E_{T_T} \cap R_0$-TOP and $M_{T_R} = M_{T_T} \cap R_0$-TOP.

We shall denote the classes E_{T_S}, E_{T_C}, E_{T_T} and E_{T_R} by the same letter E_0 and the classes M_{T_S}, M_{T_C}, M_{T_T} and M_{T_R} by M_0.

Let (X, δ_X) be an s-space. A function $u'_X: P(X)\to P(X)$ defined by $u'_XA = \{x \in X \mid \{x\}\delta_XA\}$, $A \subset X$ determines a closure space (X, u'_X). Let $f:(X, \delta_X)\to(Y, \delta_Y)$ be a morphism in SEP and let (X, u'_X), (Y, u'_Y) be closure spaces obtained from (X, δ_X), (Y, δ_Y) by the above method. Then the mapping $f: X\to Y$ is continuous with respect to u'_X and u'_Y. Thus by putting $F'(X, \delta_X) = (X, u'_X)$ and $F'(f) = f$, we obtain a functor $F':\text{SEP}\to\text{CLS}$. Define a functor $F:\text{SEP}\to\text{TOP}$ by $F = CF'$.

Let (X, u'_X) be a closure space. Define a relation δ_X as follows: $A\delta_XB$ if and only if $u'_XA \cap u'_XB \neq \varnothing$. Then (X, δ_X) is an s-space. Let $f:(X, u'_X)\to(Y, u'_Y)$ be a morphism in CLS and let (X, δ_X), (Y, δ_Y) be s-spaces obtained from (X, u'_X), (Y, u'_Y). Then the mapping $f: X\to Y$ is continuous with respect to δ_X and δ_Y. Thus by putting $G_1'(X, u_X') = (X, \delta_X)$ and $G_1'(f) = f$, we obtain a functor G_1': CLS $\to$ SEP. Define a functor G_1: TOP $\to$ SEP by $G_1 = G_1'D$.

Proposition 2.3 *G_1 preserves monomorphisms and epimorphisms. If f is a closed embedding in* TOP, *then $G_1(f)$ is an extremal monomorphism.*

Remark. The example in §4 below shows that G_1 need not preserve extremal monomorphisms and products.

Proof. It is obvious that G_1 preserves monomorphisms and epimorphisms. Suppose that $f:(X, u_X) \to (Y, u_Y)$ is a closed embedding. Then $G_1(f) = f:(X, \delta_X) \to (Y, \delta_Y)$ is a monomorphism. Let $fA\delta_Y fB$, $A, B \subset X$. Then $u_Y fA \cap u_Y fB \neq \emptyset$. Since f is a closed embedding, $fu_XA \cap fu_XB \neq \emptyset$ and hence $u_XA \cap u_XB \neq \emptyset$. This implies that $A\delta_XB$ and that f is an extremal monomorphism in SEP.

Let (X, u_X) be a topological space, $(X, \delta_X) = G_1(X, u_X)$ and $(X, v_X) = F(X, \delta_X)$. Then the identity mapping $1_X: X \to X$ induces a morphism $(\alpha_1)_X: (X, u_X) \to (X, v_X)$ and we have a natural transformation $\alpha_1: 1 \Rightarrow FG_1$.

Proposition 2.4 *If a topological space (X, u_X) satisfies the axiom* (R_0) *then $(\alpha_1)_X$ is an isomorphism in* TOP.

Proof. Let $(X, v_X') = F'G_1(X, u_X)$. Then $u_XA \subset v_X'A$ for any $A \subset X$. Let $x \in v_X'A$. Then $\{x\}\delta_XA$ and hence there exists an element $y \in u_X\{x\} \cap u_XA$. By the axiom (R_0), $x \in u_X\{y\} \subset u_Xu_XA = u_XA$. Hence $u_XA = v_X'A$ and this implies that $v_X = v_X' = u_X$.

Proposition 2.5 (1) *Let (X_λ, u_λ), $\lambda \in \Lambda$ and (X, u_X) be R_0-topological spaces such that $G_1(X, u_X) = \Pi_{\lambda\in\Lambda} G_1(X_\lambda, u_\lambda)$. Then $(X, u_X) = \Pi_{\lambda\in\Lambda}(X_\lambda, u_\lambda)$.*

(2) *Let (X, u_X) and (Y, u_Y) be R_0-topological spaces with an extremal monomorphism $f: G_1(X, u_X) \to G_1(Y, u_Y)$ in* SEP. *Then the mapping $f: X \to Y$ induces an extremal monomorphism $f:(X, u_X) \to (Y, u_Y)$ in* TOP.

This follows immediately from Proposition 2.4.

It is noted that an s-space (X, δ_X) belongs to $G_1(R_0\text{-TOP})$ if and only if the following are satisfied:

(1) if $\{x\}\delta_X\{x \in X \mid \{x\}\delta_XA\}$, $A \subset X$, then $\{x\}\delta_XA$,

(2) $A\delta_XB$, $A, B \subset X$ if and only if there exists an element $x \in X$ such that $\{x\}\delta_XA$ and $\{x\}\delta_XB$.

We shall give another functor G_2: TOP $\to$ SEP. Let (X, u_X') be a closure space. Define a relation δ_X as follows: $A\delta_XB$ if and only if $(u_X'A \cap B) \cup (A \cap u_X'B) \neq \emptyset$. Then (X, δ_X) is an s-space and, by putting $G_2'(X, u_X') =$

(X, δ_X), we have a functor $G_2': \mathrm{CLS} \to \mathrm{SEP}$. Define a functor $G_2: \mathrm{TOP} \to \mathrm{SEP}$ by $G_2 = G_2'D$.

PROPOSITION 2.6 *G_2 preserves monomorphisms, epimorphisms, extremal monomorphisms and M_0.*

PROOF. Let $f:(X, u_X) \to (Y, u_Y)$ belong to M_0 in TOP, $(X, \delta_X) = G_2(X, u_X)$, $(Y, \delta_Y) = G_2(Y, u_Y)$ and let $fA\delta_Y fB$, $A, B \subset X$. Then $(u_Y fA \cap fB) \cup (fA \cap u_Y fB) \neq \emptyset$ and hence $(f^{-1}u_Y fA \cap B) \cup (A \cap f^{-1}u_Y fB) \neq \emptyset$. Since f belongs to M_0 in TOP, we have that $(u_X A \cap B) \cup (A \cap u_X B) \neq \emptyset$, that is, $A\delta_X B$ and this implies that f belongs to M_0 in SEP.

For a topological space (X, u_X), let $(X, \delta_X) = G_2(X, u_X)$ and $(X, v_X) = F(X, \delta_X)$. Then the identity mapping $1_X: X \to X$ induces a morphism $(\alpha_2)_X: (X, u_X) \to (X, v_X)$ in TOP and we have a natural transformation $\alpha_2: 1 \Rightarrow FG_2$.

PROPOSITION 2.7 *If a topological space (X, u_X) satisfies the axiom* (R_0) *then $(\alpha_2)_X$ is an isomorphism in* TOP.

This is similar to Proposition 2.4. We can also obtain the fact that G_2 reflects products and extremal monomorphisms. An s-space (X, δ_X) belongs to $G_2(R_0\text{-TOP})$ if and only if the following are satisfied:

(1) if $\{x\}\delta_X\{x \in X \mid \{x\}\delta_X A\}$, $A \subset X$, then $\{x\}\delta_X A$,

(2) $A\delta_X B$, A, $B \subset X$ if and only if there exists a point $a \in A$ with $\{a\}\delta_X B$ or a point $b \in B$ with $\{b\}\delta_X A$.

PROPOSITION 2.8 *There exists a natural transformation $\kappa: G_2 \Rightarrow G_1$ such that each $\kappa_X: G_2(X, u_X) \to G_1(X, u_X)$ is a bimorphism in* SEP.

In fact, κ_X is induced by the identity mapping $1_X: X \to X$.

Let (X, d) be a metric space and (X, u_X) an associated topological space. Define an s-space (X, δ_X) as follows: $A\delta_X B$ if and only if $d(A, B) = 0$. Then we have that $G_1(X, u_X) = (X, \delta_X)$, while $G_2(X, u_X)$ is usually different from $G_1(X, u_X)$.

Now we shall define two kinds of separations of pairs of subsets in TOP. Proposition 2.2 implies that $\mathfrak{g}_i = (\mathrm{TOP}, \mathrm{SEP}, \mathrm{ENS}, T_R, T_S, F, G_i, \alpha_i)$ is a separation system for $i = 1, 2$. Let (Σ, η) be a $\mathfrak{g}_i$-separation and denote $\Sigma(X, u_X)$ by (X', σ_X'). Then we can obtain an operator σ which associates a topological space (X, u_X) to a binary relation σ_X on $P(X)$ as follows: $A\sigma_X B$ if and only if $\eta_X A \sigma_X' \eta_X B$. σ satisfies the conditions (s1), (s2), (s4) mentioned at the beginning of this section and

(s3′) If $u_X A \cap u_X B \neq \emptyset$, then $A\sigma_X B$.

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(s5) For any continuous mapping $f:(X, u_X)\to(Y, u_Y)$, if $A\sigma_X B$ then $fA\sigma_Y fB$.

It is obvious that there is a one-to-one correspondence between equivalence classes of $\mathfrak{g}_1$-separations (Σ, η) and operators σ satisfying the above five conditions. Hence an operator σ satisfying the above conditions is called a $\mathfrak{g}_1$-*separation.*

Similarly there is a one-to-one correspondence between equivalence classes of $\mathfrak{g}_2$-separations and operators τ satisfying the conditions (s1), (s2), (s4), (s5) and

(s3″) If $(u_X A \cap B)\cup(A \cap u_X B)\neq\emptyset$, then $A\tau_X B$.

Such an operator τ is also called a $\mathfrak{g}_2$-*separation.*

As an application of Theorem 1.3 we have the following.

THEOREM 2.9 *The following statements on a subcategory $\mathscr{A}$ of* R_0-TOP *are equivalent for $i = 1, 2$, respectively.*

(a) *If $(X, u_X)\in Ob$* R_0-TOP *and $(X_\lambda, u_\lambda)\in Ob\,\mathscr{A}$ for each $\lambda\in\Lambda$ and if there is a morphism $f: G_i(X, u_X)\to\Pi_{\lambda\in\Lambda}G_i(X_\lambda, u_\lambda)$ belonging to M_0 in* SEP, *then $(X, u_x)\in Ob\,\mathscr{A}$.*

(b) *There exists a bireflective subcategory $\mathscr{B}$ of* SEP *such that $G_i(\mathscr{A}) = \mathscr{B}\cap G_i(R_0\text{-}TOP)$.*

(c) *There exists a $\mathfrak{g}_i$-separation σ such that $Ob\,\mathscr{A}$ consists of all σ-separated R_0-topological spaces.*

From Proposition 2.6 we have

COROLLARY 2.10 *Let τ be a $\mathfrak{g}_2$-separation. If $f:(X, u_X)\to(Y, u_Y)$ is a morphism in* R_0-TOP *belonging to M_0 and if (Y, u_Y) is τ-separated, then (X, u_X) is τ-separated.*

A $\mathfrak{g}_1$-separation σ can be considered as a $\mathfrak{g}_2$-separation which will be denoted by $\hat{\sigma}$. The following is obvious.

PROPOSITION 2.11 *If a topological space (X, u_X) is $\hat{\sigma}$-separated, it is σ-separated.*

For a $\mathfrak{g}_2$-separation τ and a topological space (X, u_X), define a relation $\check{\tau}_X$ on $P(X)$ as follows: for $A, B\subset X$, $A\check{\tau}_X B$ if and only if $u_X A\ \tau_X u_X B$. Then $\check{\tau}$ is a $\mathfrak{g}_1$-separation. Let (Θ, ζ) and $(\check{\Theta}, \check{\zeta})$ be the pairs of functors and natural transformations associated with τ and $\check{\tau}$, respectively. Then there exists a natural transformation $\mu:\Theta\Rightarrow\check{\Theta}$ such that $\check{\zeta}\kappa = \mu\zeta: G_2\Rightarrow\check{\Theta}$.

PROPOSITION 2.12 *If $f:(X, u_X)\to(Y, u_Y)$ belongs to M_0 in* R_0-TOP *and (Y, u_Y) is τ-separated for a $\mathfrak{g}_2$-separation τ, then (X, u_X) is $\check{\tau}$-separated. Hence τ-separated spaces are hereditarily $\check{\tau}$-separated.*

PROOF. By Corollary 2.10, (X, u_X) is τ-separated. It is obvious that τ-separated spaces are $\check{\tau}$-separated.

We shall consider the following condition on $\mathfrak{g}_2$-separations τ.

(H) If $f:(X, u_X) \to (Y, u_Y)$ is an open embedding in TOP, then $fA\tau_Y fB$, $A, B \subset X$ implies $A\tau_X B$.

PROPOSITION 2.13 *Suppose that a $\mathfrak{g}_2$-separation τ satisfies the condition (H). Then an R_0-topological space (X, u_X) is τ-separated if and only if it is hereditarily $\check{\tau}$-separated.*

PROOF. Let (X, u_X) be hereditarily $\check{\tau}$-separated. For $A, B \subset X$ with $(u_X A \cap B) \cup (A \cap u_X B) = \emptyset$, let $Y = X - u_X(A \cap B)$ and $f:(Y, u_Y) \to (X, u_X)$ be the embedding. Then $u_Y f^{-1}A \cap u_Y f^{-1}B = f^{-1}u_X(A \cap B) = \emptyset$. Since (Y, u_Y) is $\check{\tau}$-separated, $f^{-1}A\check{\tau}f^{-1}B$ and hence $f^{-1}A\bar{\tau}_Y f^{-1}B$. Since f is an open embedding and τ satisfies (H), we have that $A\bar{\tau}_X B$. This implies that (X, u_X) is τ-separated.

REMARK. Examples in §4 show that Proposition 2.13 does not hold without the condition (H) on τ.

For a $\mathfrak{g}_1$-separation σ, $(\hat{\sigma})^\vee$-separatedness coincides with σ-separatedness. For a $\mathfrak{g}_2$-separation τ, however, $(\check{\tau})^\wedge$-separatedness is different from τ-separatedness. In fact it will be shown in §4 that there exist $\mathfrak{g}_2$-separations τ, τ' with τ-separatedness $\neq \tau'$-separatedness and $\check{\tau}$-separatedness = $\check{\tau}'$-separatedness.

3. Separations of pairs consisting of a point and a subset in TOP

In this section we shall consider the separation system $\mathfrak{k} =$ (TOP, CLS, ENS, T_T, T_C, C, D, 1). For a $\mathfrak{k}$-separation (Λ, λ) and a topological space (X, u_X), let $\Lambda(X, u_X) = (X', l'_X)$ and let $l_X A = \lambda_X^{-1} l'_X \lambda_X A$ for $A \subset X$. Then the following are satisfied:

($l1$) $u_X A \subset l_X A$ for $A \subset X$.

($l2$) $l_X(A \cup B) = l_X A \cup l_X B$ for A, $B \subset X$.

($l3$) $l_X \phi = \emptyset$.

($l4$) For any morphism $f:(X, u_X) \to (Y, u_Y)$ in TOP and for any $A \subset X$, $fl_X A \subset l_Y fA$.

There is a one-to-one correspondence between equivalence classes of $\mathfrak{k}$-separations (Λ, λ) and operators l which associate with any topological space (X, u_X) a mapping $l_X : P(X) \to P(X)$ satisfying the above conditions ($l1$) ~ ($l4$). Such an operator l is also called a *$\mathfrak{k}$-separation*.

For $\mathfrak{k}$-separations we can apply Corollary 1.4 and obtain the following.

THEOREM 3.1 *A subcategory $\mathscr{A}$ of* TOP *is bireflective in* TOP *if and only if there exists a* $\mathfrak{k}$*-separation* l *such that Ob* $\mathscr{A}$ *consists of all* l*-separated topological spaces.*

Let TOP_0 be the full subcategory of TOP consisting of all T_0-spaces. It is known that TOP_0 is extremal epi-reflective in TOP. The class M_0 in TOP is used to characterize T_0-spaces.

PROPOSITION 3.2 *A topological space* (X, u_X) *satisfies the separation axiom* T_0 *if and only if any morphism* $f:(X, u_X) \to (Y, u_Y)$ *belonging to* M_0 *is an embedding.*

PROOF. Let (X, u_X) be not a T_0-space. Then there are two distinct points $x, y \in X$ such that every open set containing one of x, y contains them both. By identifying x and y we can obtain a quotient space (Y, u_Y). Then it is shown that the quotient mapping $f:(X, u_X) \to (Y, u_Y)$ belongs to M_0. The converse is obvious.

THEOREM 3.3 *A subcategory $\mathscr{A}$ of* TOP_0 *is epireflective in* TOP *if and only if there exists a* $\mathfrak{k}$*-separation* l *such that Ob* $\mathscr{A}$ *consists of all* l*-separated* T_0*-spaces.*

PROOF. Suppose that $\mathscr{A}$ is epireflective in TOP and denote the reflector by $R: TOP \to \mathscr{A}$ and the reflection of (X, u_X) by $r_X:(X, u_X) \to R(X, u_X)$. Define an operator l by $l_X A = r_X^{-1} u_{RX} r_X A$ for $A \subset X$. Then we have a $\mathfrak{k}$-separation l. It is obvious that any object in $\mathscr{A}$ is l-separated. Let (X, u_X) be an l-separated T_0-space. Then $u_X A = r_X^{-1} u_{RX} r_X A$ holds for any $A \subset X$ and this implies that r_X belongs to M_0. By Proposition 3.2 we have that r_X is an isomorphism and (X, u_X) belongs to $\mathscr{A}$. The converse follows from Theorem 3.1.

Sharpe, Beattie and Marsden (1966) gave a uniform definition of point separation axioms and Wyler gave a characterization of separated spaces.

PROPOSITION 3.4 (Wyler) *A subcategory $\mathscr{A}$ of* TOP *is extremal epi-reflective in* TOP *if and only if there exists a point separation axiom* ρ *such that Ob* $\mathscr{A}$ *consists of all* ρ*-separated spaces.*

A point separation axiom ρ will be called *trivial* if any topological space is ρ-separated.

COROLLARY 3.5 *Suppose that a point separation axiom* ρ *is non-trivial. Then the full subcategory* $\mathscr{A}_\rho$ *of* TOP *consisting of all* ρ*-separated spaces is an intersection of* TOP_0 *and a full subcategory* $\mathscr{A}_l$ *of* TOP *consisting of all* l*-separated spaces for some* $\mathfrak{k}$*-separation* l*.*

PROOF. From the non-triviality of ρ and Proposition 3.4, we have that $\mathscr{A}_\rho \subset \mathrm{TOP}_0$. Hence we can apply Theorem 3.3 and obtain the result.

4. Examples

Let (X, u_X) be a topological space and define relations σ_X^1, σ_X^2, τ_X^1 and τ_X^2 as follows:

$A\sigma_X^1 B$ if and only if any open subsets U, $V \subset X$ with $U \supset u_X A$, $V \supset u_X B$ have a non-empty intersection,

$A\sigma_X^2 B$ if and only if there is no continuous mapping $f:(X, u_X) \to [0, 1]$ with $f(A) = 0$ and $f(B) = 1$,

$A\tau_X^1 B$ if and only if any open subsets U, $V \subset X$ with $U \supset A$, $V \supset B$ have a non-empty intersection,

$A\tau_X^2 B$ if and only if there is no continuous mapping $f:(X, u_X) \to [0, 1]$ with $f(A) \subset [0, \frac{1}{2})$ and $f(B) \subset (\frac{1}{2}, 1]$.

Then σ^1, σ^2 are $\mathfrak{g}_1$-separations and τ^1, τ^2 are $\mathfrak{g}_2$-separations. A topological space (X, u_X) is σ^1-, σ^2-, or τ^1-separated if and only if it is a T_4-, T_4- or T_5-space, respectively. τ^2- and $(\sigma^2)^\wedge$-separated spaces are considered by Terada (1975), too. He uses them for characterizing z-embedded spaces. $(\tau^1)^\vee$- and $(\tau^2)^\vee$- separatedness coincide with the axiom T_4, while it can be shown that there is a τ^1-separated space which is not τ^2-separated.

For the unit interval $[0, 1]$ with the usual topology u_I, let $(I_i, \delta_{I_i}) = G_i([0, 1], u_I)$ and $\mathscr{I}_i$ be the bireflective hull of (I_i, δ_{I_i}) in SEP for $i = 1, 2$.

THEOREM 4.1 *Let* $\mathscr{M}_1$ *and* $\mathscr{M}_2$ *be the full subcategories of* R_0-TOP *consisting of all* σ^2- *and* τ^2-*separated spaces respectively. Then*

$$G_i(\mathscr{M}_i) = \mathscr{I}_i \cap G_i(\mathrm{R}_0\text{-TOP}), \qquad i = 1, 2.$$

PROOF. We shall show that $\mathscr{I}_1$ is the bireflective hull in SEP of $G_1(\mathscr{M}_1)$. Suppose that (X, u_X) is σ^2-separated (= T_4-space). Let Λ be a set consisting of all pairs (A, B) of closed subsets A, $B \subset X$ with $A \cap B = \varnothing$. For $(A, B) \in \Lambda$, there is a continuous mapping $f_{(A,B)}:(X, u_X) \to ([0, 1], u_I)$ in TOP with $f_{(A,B)}(A) = 0$ and $f_{(A,B)}(B) = 1$ and this induces a morphism $f_{(A,B)}: G_1(X, u_X) \to (I_{(A,B)}, \delta_{(A,B)})$ in SEP, where $(I_{(A,B)}, \delta_{(A,B)}) = G_1([0, 1], u_I)$. $(f_{(A,B)})_{(A,B)\in\Lambda}$ defines a morphism $f: G_1(X, u_X) \to \Pi_{(A,B)\in\Lambda}(I_{(A,B)}, \delta_{(A,B)})$ such that $p_{(A,B)}f = f_{(A,B)}$. Then we can show that f belongs to M_0 in SEP and hence $G_1(X, u_X)$ belongs to $\mathscr{I}_1$.

Next, we give examples for $\mathfrak{k}$-separations. Let (X, u_X) be a topological space and define operators l_X^i, $i = 0, 1, 2, 3$ as follows:

$l_X^0 A = \{x \in X \mid u_X\{x\} \cap u_X A \neq \varnothing\}$,

$l^1_X A = \{x \in X \mid$ there is a point $y \in u_X A$ such that any open subsets U, V with $U \ni x,\ V \ni y$ have a non-empty intersection$\}$,

$l^2_X A = \{x \in X \mid$ any open subsets U, V with $U \ni x,\ V \supset A$ have a non-empty intersection$\}$,

$l^3_X A = \{x \in X \mid$ there is no continuous mapping $f:(X, u_X) \to ([0,1], u_t)$ with $f(x) = 0$ and $f(A) = 1\}$.

Then each l^i is a $\mathfrak{k}$-separation. A bireflective subcategory of TOP consisting of all l^i-separated spaces will be denoted by R_i-TOP. It is noted that l^0-separatedness coincides with the axiom (R_0). Let $\Lambda^i : \text{TOP} \to \text{CLS}$ be a functor associated with l^i. Then there are examples in Sharpe, Beattie and Marsden (1966) and Thomas (1968) which show that $C\Lambda^i : \text{TOP} \to \text{TOP}$ does not coincide with the reflector $R^i : \text{TOP} \to R_i\text{-TOP}$ for each $i = 1, 2$, while $\Lambda^3 : \text{TOP} \to \text{TOP}$ coincides with the reflector R^3.

PROPOSITION 4.2. *R_i-TOP $\cap$ TOP$_0$ is an epireflective subcategory whose reflector is given by the composition $T^0 R^i$ for each $i = 0, 1, 2, 3$, where $T^0 : \text{TOP} \to \text{TOP}_0$ is the reflector, and*

$$R_0\text{-TOP} \cap \text{TOP}_0 = \text{TOP}_1 \qquad (T_1\text{-spaces}),$$

$$R_1\text{-TOP} \cap \text{TOP}_0 = \text{TOP}_2 \qquad (T_2\text{-spaces}),$$

$$R_2\text{-TOP} \cap \text{TOP}_0 = \text{REG} \qquad (\text{regular spaces}),$$

$$R_3\text{-TOP} \cap \text{TOP}_0 = \text{CR} \qquad (\text{completely regular spaces}).$$

REMARK. Davis (1961) defines 'axioms of regularity' R_0, R_1 and R_2. His axiom R_i coincides with l^i-separatedness for $i = 0, 2$, while R_1 is rather a point separation axiom and hence differs from l^1-separatedness.

A $\mathfrak{k}$-separation $\tilde{l}$ which gives null-dimensionality is defined as follows:

$\tilde{l}_X A = \{x \in X \mid$ any open and closed subspace U containing x has a non-empty intersection with $A\}$.

Let NEAR be the category of all near spaces defined by Herrlich (1974a) and let $\mathfrak{g} = (R_0\text{-TOP}, \text{NEAR}, \text{ENS}, T_R, T_N, F, G, 1)$, where $T_N : \text{NEAR} \to \text{ENS}$ be the forgetful functor, G the embedding functor and F the coreflector. For an R_0-topological space (X, u_X) let $\xi'_X = \{\mathscr{A} \subset P(X) \mid \bigcap \{u_X A \mid A \in \mathscr{B}\} \neq \varnothing$ for any finite subset $\mathscr{B} \subset \mathscr{A}\}$. Then (X, ξ'_X) belongs to NEAR and we have a functor $\Sigma : R_0\text{-TOP} \to \text{NEAR}$ by taking $\Sigma(X, u_X) = (X, \xi'_X)$. An identity mapping $1_X : X \to X$ induces a morphism $\eta_X : G(X, u_X) \to \Sigma(X, u_X)$. Thus we have a $\mathfrak{g}$-separation (Σ, η). An R_0-topological space is (Σ, η)-separated if and only if it is compact. A near space belonging to the subcategory denoted by $\mathscr{B}_{(\Sigma, \eta)}$ in Theorem 1.3 is a contigual space defined in Herrlich (1974a).

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Finally we give another example which concerns collectionwise normality. For this purpose we shall define quasi-near spaces. Let X be a set. If a subset ξ_X of $P(P(X))$ satisfies the following conditions, (X, ξ_X) is called a *quasi-near space*.

(N1) *For* $\mathscr{A} = \{A_\mu \mid \mu \in M\}$, $\mathscr{B} = \{B_\mu \mid \mu \in M\} \subset P(X)$, if $\xi_X \mathscr{A}$ and $A_\mu \subset B_\mu$ for each $\mu \in M$, then $\xi_X \mathscr{B}$.

(N2) If $A \bar{\xi}_X \mathscr{C}$ and $B \bar{\xi}_X \mathscr{C}$, A, $B \subset X$, $\mathscr{C} \subset P(X)$, then $A \cup B \bar{\xi}_X \mathscr{C}$.

(N3) If $\mathscr{A} \subset \mathscr{B} \subset P(X)$ and $\xi_X \mathscr{A}$, then $\xi_X \mathscr{B}$.

(N4) $\{x\} \xi_X \{x\}$ for any $x \in X$.

(N5) $\phi \bar{\xi}_X X$.

Let (X, ξ_X) and (Y, ξ_Y) be quasi-near spaces. A mapping $f: X \to Y$ is called a continuous mapping with respect to ξ_X and ξ_Y provided that if $\xi_X \mathscr{A}$ then $\xi_Y f \mathscr{A}$ for any $\mathscr{A} \subset P(X)$. All quasi-near spaces and all continuous mappings between them form a category Q-NEAR. This category has similar properties to those of SEP.

For a quasi-near space (X, ξ_X), define u'_X by $u'_X A = \{x \in X \mid \{x\} \xi_X A\}$ for $A \subset X$. Then we have a closure space (X, u'_X) and a functor F': Q-NEAR $\to$ CLS with $F'(X, \xi_X) = (X, u'_X)$. Define a functor F: Q-NEAR $\to$ TOP by $F = CF'$. For a topological space (X, u_X), define ξ_X and ξ'_X as follows: for $\mathscr{A} \subset P(X)$, $\xi_X \mathscr{A}$ if and only if $\mathscr{A}$ is a discrete family; for $\mathscr{A} = \{A_\mu \mid \mu \in M\} \subset P(X)$, $\xi'_X \mathscr{A}$ if and only if there exists a discrete family $\hat{\mathscr{A}} = \{\hat{A}_\mu \mid \mu \in M\}$ such that $\hat{A}_\mu$ is open and $\hat{A}_\mu \supset A_\mu$ for each $\mu \in M$. Then we have quasi-near spaces (X, ξ_X) and (X, ξ'_X), functors G, Σ: TOP $\to$ Q-NEAR with $G(X, u_X) = (X, \xi_X)$ and $\Sigma(X, u_X) = (X, \xi'_X)$ and a natural transformation $\eta: G \Rightarrow \Sigma$ such that η_X is induced from the identity mapping 1_X. Thus in the category TOP we can define a separation (Σ, η) such that (Σ, η)-separatedness coincides with collectionwise normality. It can also be shown that a T_1-space is collectionwise normal if and only if it is embedded in a product of Banach spaces in Q-NEAR.

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