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Rokhlin dimension: duality, tracial properties, and crossed products

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Abstract. We study compact group actions with finite Rokhlin dimension, particularly in relation to crossed products. For example, we characterize the duals of such actions, generalizing previous partial results for the Rokhlin property. As an application, we determine the ideal structure of their crossed products. Under the assumption of so-called commuting towers, we show that taking crossed products by such actions preserves a number of relevant classes of C^* -algebras, including: D -absorbing C^* -algebras, where D is a strongly self-absorbing C^* -algebra; stable C^* -algebras; C^* -algebras with finite nuclear dimension (or decomposition rank); C^* -algebras with finite stable rank (or real rank); and C^* -algebras whose K -theory is either trivial, rational, or n -divisible for $n \in \mathbb{N}$. The combination of nuclearity and the universal coefficient theorem (UCT) is also shown to be preserved by these actions. Some of these results are new even in the well-studied case of the Rokhlin property. Additionally, and under some technical assumptions, we show that finite Rokhlin dimension with commuting towers implies the (weak) tracial Rokhlin property. At the core of our arguments is a certain local approximation of the crossed product by a continuous $C(X)$ -algebra with fibers that are stably isomorphic to the underlying algebra. The space X is computed in some cases of interest, and we use its description to construct a \mathbb{Z}_2 -action on a unital AF-algebra and on a unital Kirchberg algebra satisfying the UCT, whose Rokhlin dimensions with and without commuting towers are finite but do not agree.

Key words: C^* -algebra, Rokhlin dimension, crossed products

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1. Introduction

The goal of this paper is to study a variety of structural properties for actions of finite (and more generally compact) groups on C^* -algebras with finite Rokhlin dimension. The concept of Rokhlin dimension for actions of finite groups and actions of the integers was introduced by the second author, Winter and Zacharias in [36] as a generalization of the well-studied Rokhlin property for group actions. (See [29, 38, 53] for finite group actions with the Rokhlin property, and [19, 23, 35] for compact group actions.) The Rokhlin property can be viewed as a regularity condition for the group action, which can be used to show that various structural properties pass from a C^* -algebra to its crossed product; see, for example, [19, 35, 53, 63]. However, particularly in the case of finite group actions, the Rokhlin property is a very restrictive hypothesis to place on the action, as it implies that the unit of the algebra can be written non-trivially as a sum of projections indexed by the group. In the context of Rokhlin dimension, the tower of projections indexed by the group is replaced by several towers consisting of positive contractions, each of which is indexed by the group. In this formulation, Rokhlin dimension zero agrees with the Rokhlin property, while the higher values allow for greater flexibility. In particular, actions with finite Rokhlin dimension may exist, and even be generic, on algebras that do not admit any action with the (tracial) Rokhlin property.

A different and earlier weakening of the Rokhlin property is the tracial Rokhlin property [57], in which the Rokhlin projections are not required to add up to unity, but rather just up to a small error in trace. This property can be shown to hold for many actions of interest which do not have the Rokhlin property. Nevertheless, as it demands the existence of projections, there are many C^* -algebras that do not admit any such action. There exist a number of weakenings of the tracial Rokhlin property, in which projections are replaced by positive elements; see [2, 31, 49]. In general, there is no particular reason for the tracial Rokhlin property (in either its weak version, with positive elements, or strong version, with projections) to imply finite Rokhlin dimension or vice versa, and one of our goals is to find sufficient conditions for such implications to hold.

The main interest in establishing that an action has finite Rokhlin dimension, from the standpoint of structure and classification of nuclear C^* -algebras, stems from the fact that this can be used to show that various structural properties of interest pass to crossed products; see, for example, [20, 32, 36]. We briefly review the main structural

properties of interest, particularly in the context of the Elliott classification program, which has recently essentially been resolved [16, 69], and refer the reader to [17] for a survey of the Elliott program and regularity properties in this context. A separable unital infinite-dimensional C^* -algebra D is said to be *strongly self-absorbing* [71] if $D \cong D \otimes D$, and furthermore the first coordinate embedding $x \mapsto x \otimes 1$ is approximately unitarily equivalent to an isomorphism. (Any strongly self-absorbing C^* -algebra has to be nuclear regardless of which tensor product is *a priori* used in the definition, hence there is no ambiguity.) If D is a strongly self-absorbing C^* -algebra, then a C^* -algebra A is said to be *D -stable* or *D -absorbing* if $A \cong A \otimes D$. The *Jiang–Su algebra*, denoted \mathcal{Z} , is a particularly important strongly self-absorbing C^* -algebra. The explicit construction was first introduced in [39]; however, it can be better characterized as an initial object in the category of strongly self-absorbing C^* -algebras, in the sense that $D \cong D \otimes \mathcal{Z}$ for any strongly self-absorbing C^* -algebra D [73]. The Jiang–Su algebra is KK -equivalent to the complex numbers, and tensoring by it does not change the Elliott invariant of a simple separable nuclear C^* -algebra, assuming its K_0 -group is unperforated, and indeed, \mathcal{Z} -stability turned out to be an essential ingredient in classifiability. Capping extensive work on the Toms–Winter conjecture, for simple separable unital nuclear C^* -algebras, \mathcal{Z} -stability was recently shown to be equivalent to finite *nuclear dimension* [10], another key hypothesis for classification. Nuclear dimension [44, 74] is a generalization of covering dimension for nuclear C^* -algebras. Recall that a C^* -algebra A is nuclear if, for any finite set $F \subseteq A$ and any $\varepsilon > 0$, there exists a finite-dimensional algebra B and a diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ & \searrow \varphi & \nearrow \psi \\ & B & \end{array}$$

such that φ and ψ are completely positive maps and $\|\psi(\varphi(a)) - a\| < \varepsilon$ for all $a \in F$. A completely positive map θ between C^* -algebras is said to be an *order-zero* map if it preserves orthogonality, that is, whenever x, y are positive elements such that $xy = 0$ then $\theta(x)\theta(y) = 0$. Returning to the diagram above, the map $\psi: B \rightarrow A$ is said to be *d -decomposable* if one can decompose B as a direct sum $B \cong \bigoplus_{k=0}^d B_k$ of C^* -subalgebras such that the restriction of ψ to each summand is of order zero. Such approximations can always be found [30], if one does not bound the number of summands, and A is said to have nuclear dimension at most d if one can always find decompositions as above involving at most $d + 1$ summands. Other structural properties we consider include finite real rank and stable rank, other notions of dimension which were introduced much earlier; we refer the reader to Blackadar’s book [6]. We note that, for simple separable nuclear unital and stably finite C^* -algebras, Rørdam [61] showed that \mathcal{Z} -stability implies stable rank one, obtained conditions for real rank zero, and also showed that the Cuntz semigroup is weakly unperforated. The Cuntz semigroup is a semigroup constructed out of positive elements in a way that is analogous to the construction of the Murray–von Neumann semigroup of projections. Weak unperforation of the Cuntz semigroup, for simple separable nuclear unital C^* -algebras, is conjectured to be equivalent to \mathcal{Z} -stability, and this was verified under certain conditions on the trace space [41, 64, 70], and for certain C^* -algebras

with stable rank one [67]. The universal coefficient theorem (UCT) of Rosenberg and Schochet [62] states that, for C^* -algebras A in a suitable ‘bootstrap’ class of C^* -algebras, for any separable C^* -algebra B , the canonical map $KK(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B))$ is surjective, and the kernel is identified with an appropriate Ext group. The UCT plays a central role in Elliott’s program, and it is a major open problem whether any separable nuclear C^* -algebra satisfies the UCT. This is known to be equivalent to the question whether the UCT is preserved under crossed products by the circle or finite cyclic groups [6, 23.15.12].

Recognizing when a given action has finite Rokhlin dimension is not always straightforward (and computing its actual dimension is even more challenging). When the acting group is abelian, one of our main results allows one to detect the Rokhlin dimension of an action $\alpha : G \rightarrow \text{Aut}(A)$ by looking at its dual $\widehat{\alpha} : \widehat{G} \rightarrow \text{Aut}(A \rtimes_{\alpha} G)$. To this end, we define (see Definition 2.10) the *representability dimensions*, $\dim_{\text{rep}}(\beta)$ and $\dim_{\text{rep}}^c(\beta)$, with and without commuting colors, of a discrete group action $\beta : \Gamma \rightarrow \text{Aut}(B)$, and prove the following theorem.

THEOREM A. (See Theorem 2.14) *Let G be a compact abelian group, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. Then*

$$\dim_{\text{Rok}}(\alpha) = \dim_{\text{rep}}(\widehat{\alpha}) \quad \text{and} \quad \dim_{\text{Rok}}^c(\alpha) = \dim_{\text{rep}}^c(\widehat{\alpha}).$$

Theorem A is used to show that actions with finite Rokhlin dimension have full strong Connes spectrum, and hence that ideals in the crossed product are induced by invariant ideals in the algebra; see Proposition 2.16 and Corollary 2.17.

Actions with finite Rokhlin dimension are very closely connected to free actions on spaces. For example, it was shown in [32, Lemma 2.3] and [21, Theorem 4.5] that, for a finite-dimensional metrizable space X , an action of a compact group G on $C_0(X)$ has finite Rokhlin dimension if and only if the induced action of G on X is free. The connections go far beyond the commutative case, at least in the formulation of Rokhlin dimension with commuting towers. Indeed, for every compact group G and every $d \geq 0$, there exists a universal compact free G -space $X_{G,d}$ such that an action $\alpha : G \rightarrow \text{Aut}(A)$ of G on a C^* -algebra A satisfies $\dim_{\text{Rok}}^c(\alpha) \leq d$ if and only if there exists an asymptotically central equivariant homomorphism from $C(X_{G,d})$ into A ; see Theorem 4.7 (for finite groups, this is implicit in [32, Lemma 1.9]). This observation was used in [32, Theorem 4.6] to show that there do not exist actions of non-trivial compact Lie groups on the Jiang–Su algebra \mathcal{Z} or the Cuntz algebra \mathcal{O}_{∞} with finite Rokhlin dimension with commuting towers.

In this work we take these ideas further. By identifying the spaces $X_{G,d}$ as simplicial complexes, we prove the following theorem.

THEOREM B. (See Theorem 3.4) *Let A be an infinite-dimensional, simple, finite, unital C^* -algebra with strict comparison and at most countably many extreme quasitraces. Let G be a finite group and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. If $\dim_{\text{Rok}}^c(\alpha) < \infty$, then α has the weak tracial Rokhlin property.*

We say a few words about the proof of Theorem B. Let $X_{G,d}$ be the universal free G -space associated to G and $d = \dim_{\text{Rok}}^c(\alpha)$. For the sake of argument, suppose that

there is a unital, central, and equivariant inclusion $C(X_{G,d}) \rightarrow A$. The restriction of each (extremal) quasitrace on A to $C(X_{G,d})$ induces a Borel probability measure on $X_{G,d}$. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be the collection of such measures. In order to prove that the given action has the weak tracial Rokhlin property (see Definition 3.2), it suffices to find an open set $U \subseteq X_{G,d}$ such that $gU \cap hU = \emptyset$ for $g, h \in G$ with $g \neq h$, and $\mu_n(X_{G,d} \setminus \bigcup_{g \in G} gU)$ is small for all $n \in \mathbb{N}$. (Given such a set, one chooses a positive contraction supported on U , and considers its G -translates.) The existence of such an open set is proved in Theorem 3.1.

The argument described above breaks down in the absence of traces. However, different methods yield an even stronger result in this case. Indeed, when A is simple, exact, and has strict comparison, it has no non-zero traces if and only if it is purely infinite. In the nuclear case, it must therefore be a Kirchberg algebra. In this context, we use a lemma of Kishimoto from [46] to prove that a number of generally inequivalent notions actually coincide for actions on Kirchberg algebras.

THEOREM C. (See Theorem 3.11) *Let G be a finite group, let A be a Kirchberg algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. Then the following statements are equivalent:*

- (1) α has the (weak or strong) tracial Rokhlin property;
- (2) $\dim_{\text{Rok}}(\alpha) \leq 1$ (or just $\dim_{\text{Rok}}(\alpha) < \infty$);
- (3) α_g is not inner for all $g \in G \setminus \{1\}$.

Suppose that $\alpha : G \rightarrow \text{Aut}(A)$ is an action with $\dim_{\text{Rok}}^c(\alpha) \leq d < \infty$. With $X_{G,d}$ as above, it can be shown that the crossed product $A \rtimes_{\alpha} G$ can be locally approximated by a continuous C^* -bundle with base space $X_{G,d}/G$ and fibers isomorphic to $A \otimes \mathcal{K}(L^2(G))$; see Propositions 4.5 and 4.11. This fact allows us to transfer a number of regularity properties, many of which are relevant from the point of view of classification, from A to $A \rtimes_{\alpha} G$ and A^{α} . Indeed, we show the following theorem.

THEOREM D. (See Theorems 4.17 and 4.24) *Let G be a second countable compact group with $\dim(G) < \infty$, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action with finite Rokhlin dimension (with commuting towers). The following properties pass from A to $A \rtimes_{\alpha} G$ and A^{α} :*

- (1) absorbing a given strongly self-absorbing C^* -algebra [71];
- (2) having finite nuclear dimension (or decomposition rank);
- (3) having finite stable rank (or real rank);
- (4) satisfying the UCT, being nuclear and having (uniquely) divisible K -theory or trivial K -theory;
- (5) being separable, nuclear, and satisfying the UCT;
- (6) if A has ‘no K_1 -obstructions’ (see Definition 4.23) and $\dim_{\text{Rok}}^c(\alpha) \leq 1$, having almost unperforated Cuntz semigroup.

See also [26] for other permanence results.

Recall (see [16, 69]) that separable, simple unital C^* -algebras, with finite nuclear dimension and satisfying the UCT, are classified by their Elliott invariant (which essentially consists of K -theory and traces; see [17] for a survey), and the same is true in this case for $A \otimes \mathcal{K}$. We obtain the following consequence of Theorem D.

COROLLARY E. Let G be a compact group of finite covering dimension, let A be a unital C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. If A is in the classifiable class mentioned above, then so are $A \rtimes_{\alpha} G$ and A^{α} .

The free G -spaces $X_{G,d}$ can be explicitly computed, although their description is not always simple to state. For $G = \mathbb{Z}_2$, the space $X_{\mathbb{Z}_2,d}$ is equivariantly isomorphic to S^d with the antipodal action; see Lemma 4.25. We use this description to produce the following example.

Example F. (See Example 4.29) There exist a unital Kirchberg algebra satisfying the UCT and an action $\alpha : \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ with $\dim_{\text{Rok}}^c(\alpha) = 2$ and $\dim_{\text{Rok}}(\alpha) = 1$.

There is a similar example on a simple, approximately finite-dimensional (AF) C^* -algebra; see Example 4.28. These are the first examples of actions whose Rokhlin dimensions with and without commuting towers are both finite but do not agree. Similar examples for actions of \mathbb{Z} are presently not known. Finally, the explicit description of the spaces $X_{\mathbb{Z}_2,d}$ is used to show that in some cases of interest, the Rokhlin dimension with commuting towers of a given action is either zero or infinite; see Proposition 4.32 and Theorem 4.34.

2. Finite Rokhlin dimension and duality

We begin by introducing some notation and terminology.

Definition 2.1. Let A be a C^* -algebra. Let $\ell^{\infty}(\mathbb{N}, A)$ denote the set of all bounded sequences in A with the supremum norm and pointwise operations. Set

$$c_0(\mathbb{N}, A) = \{(a_n)_{n \in \mathbb{N}} \in \ell^{\infty}(\mathbb{N}, A) : \lim_{n \rightarrow \infty} \|a_n\| = 0\}.$$

Then $c_0(\mathbb{N}, A)$ is an ideal in $\ell^{\infty}(\mathbb{N}, A)$, and we denote the quotient $\ell^{\infty}(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ by A_{∞} . We write $\eta_A : \ell^{\infty}(\mathbb{N}, A) \rightarrow A_{\infty}$ for the quotient map. We identify A with the subalgebra of $\ell^{\infty}(\mathbb{N}, A)$ consisting of the constant sequences, and with a subalgebra of A_{∞} by taking its image under η_A . If D is any subalgebra of A , then $A_{\infty} \cap D'$ denotes the relative commutant of D inside of A_{∞} .

For a subalgebra $D \subseteq A$, write $\text{Ann}(D, A_{\infty})$ for the annihilator of D in A_{∞} , which is an ideal in $A_{\infty} \cap D'$. Following Kirchberg [40], we set

$$F(D, A) = (A_{\infty} \cap D')/\text{Ann}(D, A_{\infty}),$$

and write $\kappa_{D,A} : A_{\infty} \cap D' \rightarrow F(D, A)$ for the quotient map. When $D = A$, we abbreviate $F(A, A)$ to $F(A)$, and $\kappa_{A,A}$ to κ_A . Observe that $F(D, A)$ is unital whenever D is σ -unital.

If $\alpha : G \rightarrow \text{Aut}(A)$ is an action of G on A , and D is an α -invariant subalgebra of A , then there are (not necessarily continuous) actions of G on $\ell^{\infty}(\mathbb{N}, A)$, on A_{∞} , on $A_{\infty} \cap D'$, and on $F(D, A)$, respectively denoted, in a slight abuse of notation, by α^{∞} , α_{∞} , α_{∞} and $F(\alpha)$. Following Kishimoto [47], we set

$$\ell_{\alpha}^{\infty}(\mathbb{N}, A) = \{a \in \ell^{\infty}(\mathbb{N}, A) : g \mapsto \alpha_g^{\infty}(a) \text{ is continuous}\}.$$

We also set $A_{\infty,\alpha} = \eta_A(\ell_{\alpha}^{\infty}(\mathbb{N}, A))$ and $F_{\alpha}(A) = \kappa_{D,A}(A_{\infty,\alpha} \cap D')$. By construction, $A_{\infty,\alpha}$ and $F_{\alpha}(D, A)$ are invariant under α_{∞} and $F(\alpha)$, respectively, so the restrictions of α_{∞} and $F(\alpha)$ to $A_{\infty,\alpha}$ and $F_{\alpha}(D, A)$, which we also denote by α_{∞} and $F(\alpha)$, are continuous. Again, $F_{\alpha}(D, A)$ is unital whenever D is σ -unital.

Remark 2.2. In the context of the definition above, the algebra $A_{\infty, \alpha}$ also agrees with the subalgebra of A_{∞} where the induced action α_{∞} acts continuously, by the main theorem of [7].

Given a compact group G , we denote by $\mathbb{L}t: G \rightarrow \text{Aut}(C(G))$ the action of left translation. If H is a subgroup of G , we also denote by $\mathbb{L}t: H \rightarrow \text{Aut}(C(G))$ the restriction of $\mathbb{L}t$ to H .

We reproduce the definition of Rokhlin dimension for compact group actions as it appears in [21, Definition 3.2]. Recall that a completely positive contractive map $\varphi: C \rightarrow A$ between C^* -algebras is said to have *order zero* if $\varphi(c_1)\varphi(c_2) = 0$ whenever $c_1, c_2 \in C_+$ satisfy $c_1c_2 = 0$. Given a completely positive contractive order-zero map $\varphi: C \rightarrow A$, [75, Theorem 2.3] asserts that there exist a positive contraction $h \in A^{**}$ and a homomorphism $\pi: C_0((0, 1]) \otimes C \rightarrow A$ such that $\varphi(c) = h\pi(\text{id}_{(0,1]} \otimes c)$ for all $c \in C$. In this case, we write $\varphi = h\pi$ for short.

Definition 2.3. Let G be a compact group, let A be a C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. We say that α has *Rokhlin dimension d* , if d is the least integer such that, for every σ -unital α -invariant subalgebra $D \subseteq A$, there exist equivariant completely positive contractive order-zero maps

$$\varphi_0, \dots, \varphi_d: (C(G), \mathbb{L}t) \rightarrow (F_{\alpha}(D, A), F(\alpha))$$

such that $\varphi_0(1) + \dots + \varphi_d(1) = 1$. We denote the Rokhlin dimension of α by $\text{dim}_{\text{Rok}}(\alpha)$. If no integer d as above exists, we say that α has *infinite Rokhlin dimension*, and write $\text{dim}_{\text{Rok}}(\alpha) = \infty$.

If one can always choose the maps $\varphi_0, \dots, \varphi_d$ to have commuting ranges, then we say that α has *Rokhlin dimension d with commuting towers*, and write $\text{dim}_{\text{Rok}}^c(\alpha) = d$.

Remark 2.4. For any compact group action α , we always have $\text{dim}_{\text{Rok}}(\alpha) \leq \text{dim}_{\text{Rok}}^c(\alpha)$. The inequality can be strict (see [21, Example 4.8]), even when both dimensions are finite (see Examples 4.28 and 4.29).

It is straightforward to check that when G is finite and A is separable, Definition 2.3 agrees with [32, Definition 1.14].

We need a series of easy lemmas about (equivariant) order-zero maps. The following is [20, Proposition 2.3]. The part about the dual coaction can be proved by considering the induced homomorphism from the cone as in the proof of [20, Proposition 2.3], and using [59, Theorem 3.5].

LEMMA 2.5. *Let G be a locally compact group, let A and B be C^* -algebras, and let $\alpha: G \rightarrow \text{Aut}(A)$ and $\beta: G \rightarrow \text{Aut}(B)$ be actions. Given an equivariant completely positive contractive order-zero map $\rho: A \rightarrow B$, the expression*

$$\sigma(\xi)(g) = \rho(\xi(g)),$$

for $\xi \in L^1(G, A, \alpha)$ and $g \in G$, determines a completely positive contractive order-zero map $\sigma: A \rtimes_{\alpha} G \rightarrow B \rtimes_{\beta} G$. Moreover, when G is amenable, this map is equivariant with respect to the dual coactions of G on $A \rtimes_{\alpha} G$ and $B \rtimes_{\beta} G$.

In the next lemma, $f(\rho)$ denotes the order-zero map obtained from ρ using continuous functional calculus as in [75, §2]. Explicitly, if $\rho = h\pi$ is a decomposition as in [75, Theorem 2.3] (see the comments above Definition 2.3), then $f(\rho) = f(h)\pi$.

LEMMA 2.6. *Let A and B be C^* -algebras, let G be a locally compact group, and let $\alpha: G \rightarrow \text{Aut}(A)$ and $\beta: G \rightarrow \text{Aut}(B)$ be actions. Let $\rho: A \rightarrow B$ be an equivariant completely positive contractive order-zero map, and let $f \in C_0((0, 1])$ be a positive function. Let $f(\rho): A \rightarrow B$ be the completely positive order-zero map obtained from [75, Corollary 3.2]. Then $f(\rho)$ is equivariant.*

Proof. Denote by $\pi: C_0((0, 1]) \otimes A \rightarrow B$ the homomorphism induced by ρ as in the conclusion of [75, Theorem 2.3]. Give $C_0((0, 1])$ the trivial G -action. Then π is equivariant by [21, Corollary 2.10].

The homomorphism $\pi_f: C_0((0, 1]) \otimes A \rightarrow B$ determined by $f(\rho)$ is determined by $\pi_f(\text{id}_{(0,1]} \otimes a) = \pi(f \otimes a)$ for $a \in A$. It is clear that π_f is also equivariant, and, again by [21, Corollary 2.10], it follows that $f(\rho)$ is equivariant. \square

We will need the notion of an order-zero representation of a group, which we define below. It generalizes the notion of a unitary representation, in the same way that order-zero maps generalize $*$ -homomorphisms.

Definition 2.7. Let G be a locally compact group, and let B be a C^* -algebra. We say that a strongly continuous function $u: G \rightarrow B$ is an *order-zero representation* of G on B if the following conditions are satisfied:

- (1) u_g is a normal contraction for all $g \in G$, and u_1 is positive;
- (2) $u_g u_h = u_1 u_{gh}$ for all $g, h \in G$; and
- (3) $u_g^* = u_{g^{-1}}$ for all $g \in G$.

The next result shows how to ‘dilate’ an order-zero representation to a unitary representation, similarly to how one dilates an order-zero map to a homomorphism.

PROPOSITION 2.8. *Let G be a locally compact group, let B be a C^* -algebra, and let $u: G \rightarrow B$ be an order-zero representation. Then there exist a projection $p \in B^{**}$ and a unitary representation $v: G \rightarrow \mathcal{U}(pB^{**}p)$ commuting with u_1 , such that $u_g = u_1 v_g$ for all $g \in G$.*

Proof. We denote $h = u_1$, which is a positive contraction that commutes with u_g for all $g \in G$. For $t \in (0, 1]$, use Borel functional calculus to define a projection $p_t \in B^{**}$ by $p_t = \chi_{(t,1]}(h)$. Then p_t commutes with u_g for all $g \in G$, and the strong operator limit of p_t , as $t \rightarrow 0$, is p_0 . We write p in place of p_0 .

Represent B faithfully on a Hilbert space \mathcal{H} . Denote by T the (unbounded) inverse of ph , which is defined on $p(\mathcal{H})$. Observe that Tp_t is a bounded and positive operator on $p_t(\mathcal{H})$. For $g \in G$, set $V_g = Tu_g$, which is a possibly unbounded operator defined on all of \mathcal{H} . Set $v_g = V_g p$, regarded as an operator on $p(\mathcal{H})$. We claim that v_g is a unitary (and, in particular, bounded). To show this, let $t \in (0, 1)$, and let $\xi, \eta \in \mathcal{H}$. We use that

u_g commutes with Tp_t at the fourth step, to get

$$\begin{aligned} \langle v_g p_t(\xi), v_g p_t(\eta) \rangle &= \langle Tu_g p_t(\xi), v_g p_t(\eta) \rangle \\ &= \langle Tp_t u_g p_t(\xi), v_g p_t(\eta) \rangle \\ &= \langle u_g p_t(\xi), Tv_g p_t(\eta) \rangle \\ &= \langle p_t(\xi), Tu_g^* v_g p_t(\eta) \rangle \\ &= \langle p_t(\xi), v_{g^{-1}} v_g p_t(\eta) \rangle \\ &= \langle p_t(\xi), p_t(\eta) \rangle, \end{aligned}$$

since $v_g v_{g^{-1}} = Tu_g Tu_{g^{-1}} = T^2 u_1^2$ acts as a unit on $p_t(\mathcal{H})$. It follows that the restriction of v_g to $p_t(\mathcal{H})$ is unitary, with inverse $v_{g^{-1}}$. A similar computation shows that $g \mapsto v_g|_{p_t(\mathcal{H})}$ is a unitary representation. Since $p_t \rightarrow p$ strongly as $t \rightarrow 0$, we conclude that v_g is a unitary operator on $p(\mathcal{H})$, and that the resulting map $v : G \rightarrow \mathcal{U}(p(\mathcal{H}))$ is a unitary representation. In other words, $v_g : G \rightarrow \mathcal{U}(pB^{**}p)$ is a unitary representation.

Finally, it is clear that $u_g = hv_g = v_g h$ for all $g \in G$, so the proof is finished. □

In particular, it follows that order-zero representations of a group G are in one-to-one correspondence with (completely positive contractive) order-zero maps from $C^*(G)$. Instead of giving a proof for this, we prove a more general result for covariant representations of crossed products.

We denote by $A \rtimes_\alpha G$ the maximal crossed product of A by α . In the following proposition, when u is a unitary representation, the conclusion is precisely the universal property of the crossed product.

PROPOSITION 2.9. *Let G be a locally compact group, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. Let B be another C^* -algebra, let $\varphi : A \rightarrow M(B)$ be a homomorphism, and let $u : G \rightarrow B$ be an order-zero representation, satisfying $u_g \varphi(a) = \varphi(\alpha_g(a))u_g$ for all $g \in G$ and for all $a \in A$. Define a map*

$$\psi : L^1(G, A, \alpha) \subseteq A \rtimes_\alpha G \rightarrow B$$

by $\psi(\xi) = \int_G \varphi(\xi(g))u_g \, dg$ for all $\xi \in L^1(G, A, \alpha) \subseteq A \rtimes_\alpha G$. Then ψ extends to a completely positive contractive order-zero map $A \rtimes_\alpha G \rightarrow B$.

Proof. Let $p \in B^{**}$ be the projection and let $v : G \rightarrow \mathcal{U}(pB^{**}p)$ be the unitary representation provided by Proposition 2.8. It is clear that u_1 commutes with the image of φ . Represent B faithfully on a Hilbert space \mathcal{H} . An argument similar to the one used in the proof of Proposition 2.8 (using the unbounded operator T and the projections p_t), shows that (v, φ) is a covariant representation of (G, A, α) . Let $\pi : A \rtimes_\alpha G \rightarrow \mathcal{B}(\mathcal{H})$ be its integrated form, which on $L^1(G, A, \alpha)$ is given by $\pi(f) = \int_G \varphi(f(g))v_g \, dg$ for all $f \in L^1(G, A, \alpha)$. Then u_1 commutes with the image of π , and it is clear that $\pi(f)u_1 = \psi(f)$ for all $f \in L^1(G, A, \alpha)$. It thus follows that ψ extends to a completely positive contractive order-zero map, as desired. □

In the rest of this section we characterize actions that are dual to compact groups actions with finite Rokhlin dimension. As it turns out, there is a dimensional notion

dual to Rokhlin dimension (with and without commuting towers), which we call the representability dimension (with and without commuting colors); see the definition below. The relationship between Rokhlin dimension and representability dimension is clarified in Theorem 2.14.

Definition 2.10. Let Γ be a discrete group, let B be a C^* -algebra, and let $\beta: \Gamma \rightarrow \text{Aut}(B)$ be an action. Given $d \in \mathbb{N}$, we say that β is *approximately representable with d colors*, and write $\dim_{\text{rep}}(\beta) \leq d$, if, for all finite subsets $F \subseteq B$ and $K \subseteq \Gamma$, and for every $\varepsilon > 0$, there exist contractions $x_\gamma^{(j)} \in B$, for $\gamma \in \Gamma$ and $j = 0, \dots, d$, with $x_1^{(j)}$ positive for all $j = 0, \dots, d$, and satisfying:

- (1) $\|((x_\gamma^{(j)})^* x_\gamma^{(j)} - x_\gamma^{(j)} (x_\gamma^{(j)})^*)b\| < \varepsilon$ for all $\gamma \in K$ and $b \in F$;
- (2) $\|(x_\gamma^{(j)} x_\delta^{(j)} - x_1^{(j)} x_{\gamma\delta}^{(j)})b\| < \varepsilon$ for all $\gamma \in K, \delta \in \Gamma$, and $b \in F$;
- (3) $\|(\beta_\gamma(x_\delta^{(j)}) - x_{\gamma\delta\gamma^{-1}}^{(j)})b\| < \varepsilon$ for all $\gamma \in K, \delta \in \Gamma$, and $b \in F$;
- (4) $\|(\sum_{j=0}^d (x_1^{(j)})^* x_1^{(j)})b - b\| < \varepsilon$ for all $b \in F$;
- (5) $\|x_\gamma^{(j)}b - \beta_\gamma(b)x_\gamma^{(j)}\| < \varepsilon$ for all $\gamma \in K$, and $b \in F$.

We write $\dim_{\text{rep}}(\beta)$ for the smallest integer d satisfying $\dim_{\text{rep}}(\beta) \leq d$, and call it the *representability dimension* of β .

Similarly, given $d \in \mathbb{N}$, we say that β is *approximately representable with d commuting colors*, and write $\dim_{\text{rep}}^c(\beta) \leq d$, if, for every finite subset $F \subseteq B$ and for every $\varepsilon > 0$, there exist contractions $x_\gamma^{(j)} \in B$, for $\gamma \in \Gamma$ and $j = 0, \dots, d$, satisfying conditions (1) through (5) above, in addition to:

- (6) $\|(x_\gamma^{(j)} x_\delta^{(k)} - x_\delta^{(k)} x_\gamma^{(j)})b\| < \varepsilon$ for all $\gamma, \delta \in \Gamma$, for all $j, k = 0, \dots, d$, and for all $b \in F$.

As before, we write $\dim_{\text{rep}}^c(\beta)$ for the smallest integer d satisfying $\dim_{\text{rep}}^c(\beta) \leq d$, and call it the *representability dimension with commuting colors* of β .

In the definition above, we get an equivalent notion if condition (5) is replaced by

$$(5') \quad \|(x_\gamma^{(j)}b - \beta_\gamma(b)x_\gamma^{(j)})c\| < \varepsilon \text{ for all } \gamma \in K, \text{ and } b, c \in F.$$

To see the equivalence, apply (5') to the finite set $\tilde{F} = \bigcup_{\gamma \in K} \beta_\gamma(F \cup F^*)$, so that

$$\beta_\gamma(b)x_\gamma^{(j)}c \approx_\varepsilon \beta_\gamma(b)\beta_\gamma(c)x_\gamma^{(j)},$$

for all $b, c \in F$, since $\beta_\gamma(b) \in \tilde{F}$.

Remark 2.11. In the context of the above definition, when Γ is finite and B is unital, then $\dim_{\text{rep}}(\beta) = 0$ if and only if β is approximately representable in the sense of [38, Definition 3.6 and Remark 3.7]. More generally, our notion of representability dimension zero generalizes [3, Definition 4.21], except that the said definition is only for separable C^* -algebras and countable groups.

Let $\beta: \Gamma \rightarrow \text{Aut}(B)$ be an action of a discrete group Γ on a C^* -algebra B . For $\gamma \in \Gamma$, we let $v_\gamma \in M(B \rtimes_\beta \Gamma)$ be the canonical unitary implementing β_γ . We define an (inner) action

$$\lambda^\beta: \Gamma \rightarrow \text{Aut}(B \rtimes_\beta \Gamma)$$

by $\lambda_\gamma^\beta = \text{Ad}(v_\gamma)$ for $\gamma \in \Gamma$. Whenever $B \rtimes_\beta \Gamma$ is regarded as a Γ -algebra, it is with respect to λ^β . Finally, for $\gamma \in \Gamma$, we write $u_\gamma \in C^*(\Gamma)$ for the canonical unitary.

In the next two results, the assumption that the acting group be amenable is unnecessary if the algebra on which it acts is unital. However, we will need these results for dual actions of infinite compact abelian groups, which always act on non-unital algebras (see Theorem 2.14).

THEOREM 2.12. *Let Γ be an amenable countable group, let B be a C^* -algebra, let $\beta: \Gamma \rightarrow \text{Aut}(B)$ be an action, and let $d \in \mathbb{N}$. Then the following statements are equivalent.*

- (1) $\dim_{\text{rep}}(\beta) \leq d$.
- (2) *There exist completely positive contractive order-zero maps*

$$\rho_0, \dots, \rho_d: C^*(\Gamma) \rightarrow B_\infty$$

satisfying the following conditions for all $\gamma, \delta \in \Gamma$ and all $j = 0, \dots, d$:

- (2.a) $(\beta_\infty)_\gamma(\rho_j(u_\delta))b = \rho_j(u_{\gamma\delta\gamma^{-1}})b$ for all $b \in B \subseteq B_\infty$;
 - (2.b) $(\sum_{j=0}^d \rho_j(1)^2)b = b$ for all $b \in B \subseteq B_\infty$;
 - (2.c) $\rho_j(u_\gamma)b = (\beta_\infty)_\gamma(b)\rho_j(u_\gamma)$ for all $b \in B \subseteq B_\infty$.
- (3) *There exist completely positive contractive, Γ -equivariant order-zero maps*

$$\psi_0, \dots, \psi_d: B \rtimes_\beta \Gamma \rightarrow B_\infty$$

satisfying the following conditions for all $j = 0, \dots, d$, for all $b \in B$, and for all $x \in B \rtimes_\beta \Gamma$:

- (3.a) $\psi_j(bx) = b\psi_j(x)$;
- (3.b) $\sum_{j=0}^d \psi_j(b) = b$.

There is an analogous statement for representability dimension with commuting colors: in (2) above, the maps ρ_j must have commuting ranges, while in (3) we must have $[\psi_j(x), \psi_k(y)] = 0$ for all $j, k = 0, \dots, d$, whenever $[x, y] = 0$.

Proof. We only prove the theorem for the case of $\dim_{\text{rep}}(\beta) \leq d$; the case of commuting colors is analogous.

That (1) implies (2) follows immediately from Proposition 2.8 and the universal property of $C^*(\Gamma)$. We show that (3) implies (1). Let $\varepsilon > 0$, and let $F \subseteq B$ and $K \subseteq \Gamma$ be finite subsets. Without loss of generality, we assume that F contains only contractions. For $j = 0, \dots, d$, define $\varphi_j = \psi_j^{1/2}$ using functional calculus for order-zero maps (see [75, Corollary 3.2]). If ψ_j has the form $\psi_j(z) = h_j\pi_j(z)$ for all $z \in C^*(\Gamma)$, where h_j is a positive contraction and π_j is a homomorphism as in [75, Theorem 2.3], then φ_j has the form $\varphi_j(z) = h_j^{1/2}\pi_j(z)$ for all $z \in C^*(\Gamma)$. By Lemma 2.6, the map φ_j is also equivariant.

Recall that any approximate unit in B is also an approximate unit in $B \rtimes_\beta \Gamma$. Choose a positive contraction $e \in B$ satisfying the following conditions for all $k = 1, 2$, for all $b \in F$, and for all $\gamma \in K$:

$$\|e^k b - b\| < \varepsilon, \quad \|be^k - b\| < \varepsilon, \quad \text{and} \quad \|\beta_{\gamma^{-1}}(e^k) - e^k\| < \varepsilon.$$

(One can take e to be a suitable element in an approximate unit for B , and average its images under β over a Følner set $L \subseteq \Gamma$ satisfying $|K L \Delta L|/|L| < \varepsilon$. Recall also that if $(a_\lambda)_{\lambda \in \Lambda}$ is an approximate unit, then so is $(a_\lambda^2)_{\lambda \in \Lambda}$.)

For $j = 0, \dots, d$ and $\gamma \in \Gamma$, set $x_\gamma^{(j)} = \varphi_j(e v_\gamma) \in B_\infty$. We check the first three conditions in Definition 2.7, since the other ones are similar.

For $\gamma \in K$, for $b \in B$, and for $j = 0, \dots, d$, we have

$$\begin{aligned} \|b((x_\gamma^{(j)})^* x_\gamma^{(j)} - x_\gamma^{(j)} (x_\gamma^{(j)})^*)\| &= \|b(\varphi_j(ev_\gamma)^* \varphi_j(ev_\gamma) - \varphi_j(ev_\gamma) \varphi_j(ev_\gamma)^*)\| \\ &= \|h_j^{1/2} \varphi_j(bv_\gamma^* e^2 v_\gamma) - h_j^{1/2} \varphi_j(be^2)\| \\ &= \|h_j^{1/2} \varphi_j(b\beta_{\gamma^{-1}}(e^2) - be^2)\| \\ &\leq \|h_j^{1/2}\| \|\varphi_j(b\beta_{\gamma^{-1}}(e^2) - be^2)\| \\ &\leq \|b\| \|\beta_{\gamma^{-1}}(e^2) - e^2\| < \varepsilon, \end{aligned}$$

which proves the first condition.

Recall that $v_1 = 1$. For $\gamma \in K$, for $\delta \in \Gamma$, for $b \in F$, and for $j = 0, \dots, d$, we have

$$\begin{aligned} \|b(x_\gamma^{(j)} x_\delta^{(j)} - x_1^{(j)} x_{\gamma\delta}^{(j)})\| &= \|\varphi_j(bev_\gamma) \varphi_j(ev_\delta) - \varphi_j(bev_1) \varphi_j(ev_{\gamma\delta})\| \\ &= \|\varphi_j(be) \varphi_j(v_\gamma ev_\delta) - \varphi_j(be) \varphi_j(ev_{\gamma\delta})\| \\ &= \|\varphi_j(be)\| \|\varphi_j(v_\gamma ev_\delta - ev_{\gamma\delta})\| \\ &\leq \|\beta_\gamma(e) - e\| < \varepsilon, \end{aligned}$$

which proves the second condition.

For $\gamma \in K$, for $\delta \in \Gamma$, for $b \in B$, and for $j = 0, \dots, d$, we use that φ_j is equivariant at the second step to get

$$\begin{aligned} \|b((\beta_\infty)_\gamma(x_\delta^{(j)} - x_{\gamma\delta\gamma^{-1}}^{(j)}))\| &= \|b((\beta_\infty)_\gamma(\varphi_j(ev_\delta) - \varphi_j(ev_{\gamma\delta\gamma^{-1}})))\| \\ &= \|b(\varphi_j(v_\gamma ev_\delta v_{\gamma^{-1}}) - \varphi_j(ev_{\gamma\delta\gamma^{-1}}))\| \\ &= \|\varphi_j(bv_\gamma ev_\delta v_{\gamma^{-1}}) - \varphi_j(bev_{\gamma\delta\gamma^{-1}})\| \\ &= \|v_\gamma e - ev_\gamma\| = \|\beta_\gamma(e) - e\| < \varepsilon, \end{aligned}$$

which proves the third condition.

We leave the verification of the remaining two conditions to the reader.

We now show that (2) implies (3). Define a linear map $\psi_j: B \rtimes_\beta \Gamma \rightarrow B_\infty$ by

$$\psi_j(bv_\gamma) = b\rho_j^2(u_\gamma)$$

for all $j = 0, \dots, d$, for all $b \in B$, and for all $\gamma \in \Gamma$. (Recall that ρ_j^2 is defined using functional calculus for order-zero maps.) The assignment $\gamma \mapsto \rho_j^2(v_\gamma)$ is an order-zero representation of Γ in the sense of Definition 2.7, and, by condition (2.c), it satisfies the covariance condition $\rho_j^2(v_\gamma)b = \beta_\gamma(b)\rho_j^2(v_\gamma)$ for all $\gamma \in \Gamma$ and $b \in B$ (it suffices to multiply the identity in (2.c) by $\rho_j(1)$ on both sides). Hence, it follows from Proposition 2.9 that ψ_j is a completely positive, contractive order-zero map. It clearly satisfies $\psi_j(bx) = b\psi_j(x)$ for all $b \in B$ and for all $x \in B \rtimes_\beta \Gamma$. Using condition (2.a) at the second step, we deduce that

$$\sum_{j=0}^d \psi_j(b) = b \sum_{j=0}^d \rho_j(1)^2 = b$$

for all $b \in B$.

Finally, we show that ψ_j is equivariant. First, note that condition (2.a) implies that

$$(\beta_\infty)_\gamma(\rho_j^2(u_\delta))b = \rho_j^2(u_{\gamma\delta\gamma^{-1}})b$$

for all $\gamma, \delta \in \Gamma$ and all $b \in B$. Indeed, to obtain the identity above, it suffices to multiply the equality in (2.a) by $\rho_j(1)$ on both sides, since $\rho_j(1)\rho_j(x) = \rho_j^2(x)$ for all $x \in C^*(\Gamma)$. Let $\gamma, \delta \in \Gamma$, and let $b \in B$. Then

$$\begin{aligned} \psi_j(\lambda_\gamma^\beta(bv_\delta)) &= \psi_j(v_\gamma b v_\delta v_{\gamma^{-1}}) \\ &= \psi_j(\beta_\gamma(b)v_{\gamma\delta\gamma^{-1}}) \\ &= \beta_\gamma(b)\rho_j^2(u_{\gamma\delta\gamma^{-1}}) \\ &= \beta_\gamma(b)\beta_\gamma(\rho_j^2(u_\gamma)) \\ &= \beta_\gamma(\psi_j(bv_\delta)), \end{aligned}$$

as desired. This finishes the proof. □

LEMMA 2.13. *Let G be an amenable second countable locally compact group, let A and C be C^* -algebras, with C unital, and let α and γ be actions of G on A and C , respectively. Given an α -invariant, σ -unital subalgebra $D \subseteq A$ and $d \in \mathbb{N}$, the following statements are equivalent.*

- (1) *There exist equivariant completely positive contractive order-zero maps $\varphi_0, \dots, \varphi_d: C \rightarrow F_\alpha(D, A)$ satisfying $\sum_{j=0}^d \varphi_j(1) = 1$.*
- (2) *There exist equivariant completely positive contractive order-zero maps $\theta_0, \dots, \theta_d: C \otimes_{\max} D \rightarrow A_{\infty, \alpha}$ satisfying*
 - $\theta_j(c \otimes a_1 a_2) = a_1 \theta_j(c \otimes a_2)$ for all $j = 0, \dots, d$, all $a_1, a_2 \in D$, and all $c \in C$;
 - and
 - $\sum_{j=0}^d \theta_j(1 \otimes a) = a$ for all $a \in D$.

Proof. That (1) implies (2) is easily seen by tensoring each φ_j with the identity on D and using [19, Lemma 2.3]. We show that (2) implies (1). Using an easy diagonal argument, it is enough to show the following: for every $\varepsilon > 0$, for every self-adjoint finite subset $F \subseteq D$, and for every compact subset $K \subseteq G$, there exist completely positive contractive order-zero maps $\varphi_0, \dots, \varphi_d: C \rightarrow A_{\infty, \alpha}$ satisfying the following conditions for all $j = 0, \dots, d$, for all $c \in C$, and for all $a \in F$:

- (a) $\|\varphi_j(c)a - a\varphi_j(c)\| < \varepsilon\|c\|$;
- (b) $\sup_{g \in K} \|(\alpha_\infty)_g(\varphi_j(c)) - \varphi_j(\gamma_g(c))\| < \varepsilon\|c\|$;
- (c) $\|(\sum_{j=0}^d \varphi_j(1))a - a\| < \varepsilon$.

Let $\varepsilon > 0$, $F \subseteq D$, and $K \subseteq G$ as above be given. Find a positive contraction $e \in D$ satisfying

$$\sup_{g \in K} \|\alpha_g(e) - e\| < \varepsilon, \quad \|ea - ae\| < \varepsilon, \quad \|ea - a\| < \varepsilon, \quad \|ae - a\| < \varepsilon$$

for all $a \in F$. (It is enough to take a suitable element in an approximate unit for D , and average its images under α over a Følner set $L \subseteq G$ satisfying $|KL\Delta L|/|L| < \varepsilon$.) For $j = 0, \dots, d$, define $\varphi_j: C \rightarrow A_{\infty, \alpha}$ by $\varphi_j(c) = \theta_j(c \otimes e)$ for $c \in C$. It is clear that φ_j is

a completely positive contractive order-zero map. Moreover, given $a \in F$ and $c \in C$, we have

$$\begin{aligned} \|\varphi_j(c)a - a\varphi_j(c)\| &= \|\theta_j(c \otimes e)a - a\theta_j(c \otimes e)\| \\ &= \|\theta_j(c \otimes (ea - ae))\| \\ &\leq \|c\| \|ea - ae\| < \varepsilon \|c\|, \end{aligned}$$

which verifies condition (a). To check condition (b), given $c \in C$ and $j = 0, \dots, d$, we have

$$\begin{aligned} \sup_{g \in K} \|(\alpha_\infty)_g(\varphi_j(c)) - \varphi_j(\gamma_g(c))\| &= \sup_{g \in K} \|(\alpha_\infty)_g(\theta_j(c \otimes e)) - \theta_j(\gamma_g(c) \otimes e)\| \\ &= \sup_{g \in K} \|\theta_j(\gamma_g(c) \otimes \alpha_g(e)) - \theta_j(\gamma_g(c) \otimes e)\| \\ &\leq \|\gamma_g(c)\| \sup_{g \in K} \|\alpha_g(e) - e\| < \varepsilon \|c\|, \end{aligned}$$

as desired. To check condition (c), we let $a \in F$ and compute

$$\begin{aligned} \left\| \left(\sum_{j=0}^d \varphi_j(1) \right) a - a \right\| &= \left\| \left(\sum_{j=0}^d \theta_j(1 \otimes e) \right) a - a \right\| \\ &= \left\| \sum_{j=0}^d \theta_j(1 \otimes ea) - a \right\| \\ &= \|ea - a\| < \varepsilon. \end{aligned}$$

This completes the proof. □

The following is the main result of this section. It generalizes previously known characterizations of duals of Rokhlin actions of Izumi when G is finite and A is separable [38, Theorem 3.8]. In particular, since we do not make any cardinality assumptions on G or on A , we obtain new information even in the well-studied case of Rokhlin actions (when $d = 0$).

THEOREM 2.14. *Let A be a C^* -algebra, let G be a second countable compact abelian group, let $\alpha: G \rightarrow \text{Aut}(A)$ be an action, and denote by $\widehat{\alpha}: \widehat{G} \rightarrow \text{Aut}(A \rtimes_\alpha G)$ its dual action. Then*

$$\dim_{\text{Rok}}(\alpha) = \dim_{\text{rep}}(\widehat{\alpha}) \quad \text{and} \quad \dim_{\text{Rok}}^c(\alpha) = \dim_{\text{rep}}^c(\widehat{\alpha}).$$

Proof. We only show the statement for the formulation without commuting towers, since the other one is proved analogously. It is enough to show that, for every $d \in \mathbb{N}$, we have

$$\dim_{\text{Rok}}(\alpha) \leq d \quad \text{if and only if} \quad \dim_{\text{rep}}(\widehat{\alpha}) \leq d.$$

We divide the proof into proving the equivalence of the following statements, where we identify D with the subalgebra of $C(G, D)$ of constant functions.

- (a) $\dim_{\text{Rok}}(\alpha) \leq d$.

- (b) For every α -invariant σ -unital subalgebra $D \subseteq A$, there exist G -equivariant completely positive contractive order-zero maps

$$\theta_0, \dots, \theta_d: C(G, D) \rightarrow A_{\infty, \alpha}$$

satisfying:

- $\theta_j(af) = a\theta_j(f)$ for all $j = 0, \dots, d$, all $a \in D$, and all $f \in C(G, D)$; and
 - $\sum_{j=0}^d \theta_j(a) = a$ for all $a \in D$.
- (c) For every α -invariant σ -unital subalgebra $D \subseteq A$, there exist \widehat{G} -equivariant completely positive contractive order-zero maps

$$\psi_0, \dots, \psi_d: C(G, D) \rtimes_{\text{Lt} \otimes \alpha} G \rightarrow (A \rtimes_{\alpha} G)_{\infty}$$

satisfying:

- $\psi_j(bx) = b\psi_j(x)$ for all $b \in D \rtimes_{\alpha} G$, where we regard $D \rtimes_{\alpha} G$ as a subset of both $A \rtimes_{\alpha} G$ and $C(G, D) \rtimes_{\text{Lt} \otimes \alpha} G$, and for all $x \in C(G, D) \rtimes_{\text{Lt} \otimes \alpha} G$; and
 - $\sum_{j=0}^d \psi_j(b) = b$ for all $b \in D \rtimes_{\alpha} G$.
- (d) $\dim_{\text{rep}}(\widehat{\alpha}) \leq d$.

We show that (a) is equivalent to (b). By definition, $\dim_{\text{Rok}}(\alpha) \leq d$ is equivalent to the existence, for every α -invariant σ -unital subalgebra $D \subseteq A$, of equivariant completely positive contractive maps $\varphi_0, \dots, \varphi_d: C(G) \rightarrow F_{\alpha}(D, A)$ satisfying $\sum_{j=0}^d \varphi_j(1) = 1$. By Lemma 2.13, that is equivalent to the statement in (b).

We prove that (b) implies (c). For $j = 0, \dots, d$, let $\psi_j: C(G, D) \rtimes_{\text{Lt} \otimes \alpha} G \rightarrow (A \rtimes_{\alpha} G)_{\infty}$ be the \widehat{G} -equivariant completely positive contractive order-zero map associated to θ_j as in Lemma 2.5. For $\xi \in L^1(G, C(G, D), \text{Lt} \otimes \alpha)$, this map is given by $\psi_j(\xi)(g) = \theta_j(\xi(g))$ for all $g \in G$. We claim that these maps satisfy the conditions under item (c) above. To verify the first one, it is enough to assume that $b \in D \rtimes_{\alpha} G$ has the form $b = \sum_{k=1}^n f_k a_k$ for some $f_1, \dots, f_n \in C(G)$ and $a_1, \dots, a_n \in D$, since elements of this form are dense in $L^1(G, D, \alpha)$, and hence in $D \rtimes_{\alpha} G$. Similarly, we can assume that $x \in C(G, D) \rtimes_{\text{Lt} \otimes \alpha} G$ has the form $x = \sum_{\ell=1}^m h_{\ell} y_{\ell}$ for some $h_1, \dots, h_m \in C(G)$ and $y_1, \dots, y_m \in C(G, D)$. Given $g \in G$, we use the second identity in (b) at the third step to get

$$\begin{aligned} \psi_j(bx)(g) &= \psi_j\left(\sum_{k=1}^n \sum_{\ell=1}^m f_k a_k h_{\ell} y_{\ell}\right)(g) \\ &= \theta_j\left(\sum_{k=1}^n \sum_{\ell=1}^m f_k(g) a_k h_{\ell}(g) y_{\ell}\right) \\ &= \sum_{k=1}^n \sum_{\ell=1}^m f_k(g) h_{\ell}(g) \theta_j(a_k y_{\ell}) \\ &= \sum_{k=1}^n f_k(g) a_k \sum_{\ell=1}^m h_{\ell}(g) \theta_j(y_{\ell}) \\ &= b(g) \theta_j(x(g)) = (b\psi_j(x))(g), \end{aligned}$$

as desired. Similarly, for $g \in G$, we have

$$\sum_{j=0}^d \psi_j(b)(g) = \sum_{j=0}^d \theta_j(b(g)) = b(g),$$

and hence $\sum_{j=0}^d \psi_j(b) = b$. This proves (c).

We now show that (c) implies (a). We take \widehat{G} -crossed products in the statement of (c) and apply Takai duality together with Lemma 2.5 to obtain G -equivariant completely positive contractive order-zero maps

$$\widetilde{\theta}_0, \dots, \widetilde{\theta}_d : C(G, D \otimes \mathcal{K}(L^2(G))) \rightarrow (A \otimes \mathcal{K}(L^2(G)))_{\infty, \alpha}$$

satisfying the conditions under item (b) above. (The verification of these conditions is analogous to how we verified that (b) implies (c), by working with appropriate dense subalgebras of the crossed product.) By the equivalence between (a) and (b) applied to the G -action $\alpha \otimes \text{Ad}(\lambda)$ on $A \otimes \mathcal{K}(L^2(G))$, we conclude that $\dim_{\text{Rok}}(\alpha \otimes \text{Ad}(\lambda)) \leq d$. Since $\dim_{\text{Rok}}(\alpha \otimes \text{Ad}(\lambda)) = \dim_{\text{Rok}}(\alpha)$ by part (3) of [24, Proposition 6.8], this shows that (c) implies (a).

That (c) is equivalent to (d) is the content of Theorem 2.12, since the \widehat{G} -algebra $(C(G, D) \rtimes_{\text{Lt} \otimes \alpha} G, \text{Lt} \widehat{\otimes} \alpha)$ is equivariantly isomorphic to $(D \rtimes_{\alpha} G \rtimes_{\widehat{\alpha}} \widehat{G}, \lambda^{\widehat{\alpha}})$ by Takai duality. This completes the proof of the theorem. □

We present some applications of Theorem 2.14 to the ideal structure of crossed products by actions with finite Rokhlin dimension; see Corollary 2.17. Recall the following definition from [45].

Definition 2.15. Let G be a second countable compact abelian group, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be a continuous action. Given τ in \widehat{G} , denote by A_{τ} its associated eigenspace, that is,

$$A_{\tau} = \{a \in A : \alpha_g(a) = \tau(g)a \text{ for all } g \in G\}.$$

The *strong Arveson spectrum* of α , denoted $\widetilde{\text{Sp}}(\alpha)$, is the set

$$\widetilde{\text{Sp}}(\alpha) = \{\tau \in \widehat{G} : \overline{A_{\tau}^* A A_{\tau}} = A\}.$$

If $\widetilde{\text{Sp}}(\alpha) = \widehat{G}$, then we say that α is *saturated*.

Denote by $\text{Her}_{\alpha}(A)$ the set of all α -invariant hereditary subalgebras of A . Then the *strong Connes spectrum* of α , denoted $\widetilde{\Gamma}(\alpha)$, is the set

$$\widetilde{\Gamma}(\alpha) = \bigcap_{B \in \text{Her}_{\alpha}(A)} \widetilde{\text{Sp}}(\alpha|_B).$$

If $\widetilde{\Gamma}(\alpha) = \widehat{G}$, then we say that α is *hereditarily saturated*.

PROPOSITION 2.16. *Let G be a compact abelian group, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. If $\dim_{\text{Rok}}(\alpha) < \infty$, then α is hereditarily saturated, that is, $\widetilde{\Gamma}(\alpha) = \widehat{G}$.*

Proof. By [45, Lemma 3.4], we have

$$\tilde{\Gamma}(\alpha) = \{\tau \in \widehat{G} : \widehat{\alpha}_\tau(I) \subseteq I \text{ for all ideals } I \subseteq A \rtimes_\alpha G\}.$$

Let I be an ideal in $A \rtimes_\alpha G$, and let τ be an element in \widehat{G} . Fix b in I and set $d = \dim_{\text{Rok}}(\alpha)$. For every m in \mathbb{N} , use Theorem 2.14 to find contractions $x_\gamma^{(j),m}$, for $\gamma \in \Gamma$ and $j = 0, \dots, d$ such that, in particular,

$$\left\| \widehat{\alpha}_\tau(b) - \sum_{j=0}^d (x_\gamma^{(j),m})^* b x_\gamma^{(j),m} \right\| < \frac{1}{m}.$$

Since I is an ideal, it follows that $\sum_{j=0}^d (x_\gamma^{(j),m})^* b x_\gamma^{(j),m}$ belongs to I . We conclude that $\widehat{\alpha}_\tau(b)$ is the limit in norm of elements of I , so it belongs to I itself. Hence $\widehat{\alpha}_\tau(I) \subseteq I$. Since I and τ are arbitrary, we conclude that $\tilde{\Gamma}(\alpha) = \widehat{G}$. □

A similar result for \mathbb{R} -actions has been proved in [34, Proposition 3.11].

COROLLARY 2.17. *Let G be a second countable compact abelian group, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. If $\dim_{\text{Rok}}(\alpha) < \infty$, then every ideal J in $A \rtimes_\alpha G$ has the form $J = I \rtimes_\alpha G$ for some α -invariant ideal I in A . Moreover, A^G is Morita equivalent to $A \rtimes_\alpha G$. In particular, if A is simple, then so is $A \rtimes_\alpha G$.*

Proof. We have $\tilde{\Gamma}(\alpha) = \widehat{G}$ by Proposition 2.16. The first claim now follows from [56, Theorem 5.14] (see [45] for a proof).

It follows from a result in [28] (reproduced as [56, Theorem 5.10]) that α is hereditarily saturated. Hence it is saturated, and the second claim follows from [54, Proposition 7.1.3]. □

The following application was announced in [20, Example 5.1].

Example 2.18. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let A_θ be the associated irrational rotation algebra, that is, the universal C^* -algebra generated by two unitaries u and v subject to the relation $uv = e^{\pi i \theta} vu$. Define an action $\gamma : \mathbb{T} \rightarrow \text{Aut}(A_\theta)$ by $\gamma_\zeta(u) = \zeta u$ and $\gamma_\zeta(v) = v$ for all $\zeta \in \mathbb{T}$. We claim that $\dim_{\text{Rok}}(\gamma) = \infty$. Indeed, the crossed product $A_\theta \rtimes_\gamma \mathbb{T}$ is easily seen to be isomorphic to $C(S^1) \otimes \mathcal{K}$, so it is not simple. Since A_θ is simple, the result follows from Corollary 2.17.

A similar argument shows that other related actions, such as $\beta : \mathbb{T}^2 \rightarrow \text{Aut}(A_\theta)$ given by $\beta_{(\zeta, \omega)}(u) = \zeta u$ and $\beta_{(\zeta, \omega)}(v) = \omega v$ for $(\zeta, \omega) \in \mathbb{T}^2$, have infinite Rokhlin dimension. This is in stark contrast to the fact, proved in [32], that the restrictions of γ and β to finite subgroups of \mathbb{T} have finite Rokhlin dimension with commuting towers. (See also Proposition 3.9 for a more general result.)

3. Tracial properties

In this section, we show how actions of finite groups with finite Rokhlin dimension with commuting towers enjoy a weak form of the tracial Rokhlin property from [57], where projections are replaced by positive elements, and the remainder is assumed to be small in all tracial states. This notion is called the *weak tracial Rokhlin property*; see Definition 3.2

for the precise definition. We point out that similar notions have been considered by a number of other authors [2, 49, 72], and that all of these notions agree for reasonably well-behaved C^* -algebras (for example, those which have strict comparison; see Definition 3.3 below).

If X is a simplicial complex and $k \in \mathbb{N}$, we denote its k th skeleton by $X^{(k)}$.

THEOREM 3.1. *Let G be a finite group, let X be a finite simplicial complex. Let G act freely on X in such a way that, for all k in \mathbb{N} , no point in one k -cell of X is mapped to another point in the same k -cell. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of finite Borel measures on X . Then there exists an open set $U \subseteq X$ such that:*

- (a) $gU \cap hU = \emptyset$ for all $g, h \in G$ with $g \neq h$;
- (b) $\mu_n(X \setminus \bigcup_{g \in G} gU) = 0$ for all n in \mathbb{N} .

Proof. We abbreviate the orbit space X/G to Y throughout, and observe that the assumptions on the action imply that Y is also a simplicial complex of the same dimension as X , and that $Y^{(k)} = X^{(k)}/G$ for $k = 0, \dots, \dim(X)$. We write $\pi: X \rightarrow Y$ for the quotient map, which is a local homeomorphism since the action is free and the group is finite. This means that, for every $y \in Y$, there exist an open set O_y in Y containing y , and a continuous function $s: O_y \rightarrow X$ satisfying $\pi \circ s = \text{id}_{O_y}$. (The function s is called a *local cross-section*.) Set $d = \dim(X)$. By the assumptions on the action, every d -cell in Y is contained in an open subset where a local cross-section is defined. Similarly, there is a local cross-section defined on all of $Y^{(d)} \setminus Y^{(d-1)}$. Finally, for $n \in \mathbb{N}$, we will denote by ν_n the measure on Y defined as the pushforward of μ_n , that is, $\nu_n = \pi_*(\mu_n)$.

Assume first that $\dim(X) = 0$, so that X is finite. We claim that X is equivariantly homeomorphic to $Y \times G$, where G acts trivially on Y and by translation on G . Since Y is finite, there exists a section $s: Y \rightarrow X$. Define a map $\lambda: X \rightarrow G$ by $x = \lambda(x)s(\pi(x))$. Note that the assignment $x \mapsto \lambda(x)$ is well defined because the action is free. The map $\phi: X \rightarrow Y \times G$ given by $\phi(x) = (\pi(x), \lambda(x))$ for $x \in X$, is easily seen to be an equivariant homeomorphism, with inverse given by $(y, g) \mapsto gs(y)$ for $y \in Y$ and $g \in G$. This proves the claim. With ϕ as before, set $U = \phi^{-1}(Y \times \{e\})$. It is straightforward to check that $\bigcap_{g \in G} gU = \emptyset$ and that $\bigcup_{g \in G} gU = X$.

We may assume, from now on, that $\dim(X) > 0$. We will show that there exists an open set $V \subseteq Y$ satisfying:

- (A) $\nu_n(Y^{(k)} \setminus V) = 0$ for all $n \in \mathbb{N}$ and all $k = 0, \dots, \dim(X)$; and
- (B) there is a local cross-section defined on all of V .

Once this is proved, let U be the image of V under some local cross-section. Then U is an open set in X which clearly satisfies condition (a) in the statement, and, moreover,

$$X \setminus \bigcup_{g \in G} gU = X \setminus \pi^{-1}(V) = \pi^{-1}(Y \setminus V).$$

In particular, $\mu_n(X \setminus \bigcup_{g \in G} gU) = 0$ for all $n \in \mathbb{N}$, as desired.

We obtain this set by constructing its intersection with the lower-dimensional skeletons, and using induction. (In fact, our construction yields uncountably many pairwise distinct open sets satisfying conditions (A) and (B) above.) Recall that Y has only finitely many cells of each dimension. For $y \in Y^{(0)}$, let $E_1^{(y)}, \dots, E_{d_y}^{(y)}$ denote the collection of 1-cells

in Y whose boundaries contain y , and let O_y be an open set in Y which contains y and is contained in the domain of some local cross-section for the quotient map $X \rightarrow Y = X/G$.

Claim. There exist continuous functions $f_{y,j} : [0, 1] \rightarrow E_j^{(y)}$, for $y \in Y^{(0)}$ and $j = 1, \dots, d_y$, satisfying the following conditions.

- (1) The image of $f_{y,j}$ is contained in O_y , for all $y \in Y^{(0)}$ and all $j = 1, \dots, d_y$.
- (2) $f_{y,j}$ is a homeomorphism onto its image, for all $y \in Y^{(0)}$ and all $j = 1, \dots, d_y$.
- (3) $f_{y,j}(0) = y$, for all $y \in Y^{(0)}$ and all $j = 1, \dots, d_y$.
- (4) $v_n(f_{y,j}(1)) = 0$, for all $n \in \mathbb{N}$, all $y \in Y^{(0)}$, and all $j = 1, \dots, d_y$.
- (5) $f_{y,j}([0, 1]) \cap f_{z,k}([0, 1]) = \emptyset$ for all $y, z \in Y^{(0)}$ with $y \neq z$, and for all $j = 1, \dots, d_y$ and $k = 1, \dots, d_z$.

To construct these functions, start with any collection of functions $h_{y,j} : [0, 1] \rightarrow E_j^{(y)}$ satisfying (1), (2), and (3). By restricting them to an initial segment of the form $[0, r]$, for $r \in (0, 1)$, we can ensure that condition (5) also holds. We now explain how to fulfill condition (4). Set

$$T = \{t \in (0, 1) : v_n(h_{y,j}(t)) = 0 \text{ for all } n \in \mathbb{N}, y \in Y^{(0)}, j = 1, \dots, d_y\}.$$

Then the complement of T is countable (and, in particular, T is uncountable). Indeed, if $[0, 1] \setminus T$ were uncountable, there would exist $m \in \mathbb{N}$, $z \in Y^{(0)}$, and $k \in \{1, \dots, d_z\}$ such that $v_m(h_{z,k}(t)) > 0$ for uncountably many $t \in [0, 1]$. Since $h_{z,k}$ is a homeomorphism, it follows that v_m has an uncountable set of atoms, which contradicts the fact that it is a probability measure. Hence the complement T is countable.

The claim then follows by fixing $t_0 \in T$, and letting $f_{y,j}$ be given by $f_{y,j}(t) = h_{y,j}(t/t_0)$ for all $t \in [0, 1]$.

Set

$$V = Y \setminus \left(\bigcup_{y \in Y^{(0)}} \bigcup_{j=1}^{d_y} \{f_{y,j}(1)\} \right),$$

which is an open subset of Y satisfying $v_n(Y^{(1)} \setminus V) = 0$ for all $n \in \mathbb{N}$. By the choice of the functions (specifically, by condition (1)), there exists a local cross-section defined on all of V (obtained by considering cross-sections on the connected components of V). The base step of the induction is complete.

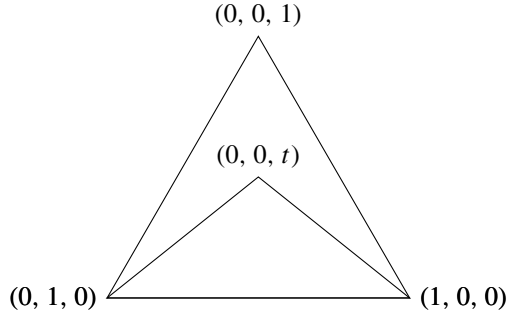
Let $m \leq \dim(X)$, and suppose we have constructed an open W satisfying conditions (A) and (B) above, for $k = 1, \dots, m - 1$. We will construct an open set V satisfying conditions (A) and (B) for $k = 1, \dots, m$. Denote by C_1, \dots, C_ℓ the connected components of W_i . For $k = 1, \dots, \ell$, let $E_{k,1}, \dots, E_{k,d_k}$ denote the collection of all m -cells in Y whose boundaries intersect C_k .

We write Δ_m for the m -dimensional simplex, which we identify as

$$\Delta_m = \left\{ (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m : \sum_{k=1}^m \lambda_k \leq 1 \right\}.$$

We also identify Δ_{m-1} with the subset of Δ_m of vectors whose last coordinate vanishes. For $t \in [0, 1]$, set

$$\Delta_m^{(t)} = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m : (\lambda_1, \dots, \lambda_m/t) \in \Delta_m\}.$$



Note that $\Delta_m^{(1)} = \Delta_m$, and $\Delta_m^{(0)} = \Delta_{m-1}$ if $m \geq 3$. When $m = 2$, the subsimplex $\Delta_2^{(t)}$ can be identified with the subinterval $[0, t]$. For $m = 3$, the subsimplex $\Delta_3^{(t)}$ is shown in the picture above.

For a set Z , we write Z° for its interior.

Claim. There exist continuous functions $f_{k,j}: \Delta_m \rightarrow E_{k,d_k}$, for $k = 1, \dots, \ell$ and $j = 1, \dots, d_k$, satisfying the following conditions:

- (1') The image of $f_{k,j}$ is contained in W_i , for all $k = 1, \dots, \ell$ and all $j = 1, \dots, d_k$.
- (2') $f_{k,j}$ is a homeomorphism onto its image, for all $k = 1, \dots, \ell$ and all $j = 1, \dots, d_k$.
- (3') $f_{k,j}(\Delta_{m-1}^\circ) \subseteq C_k \cap E_{k,d_k}$, for all $k = 1, \dots, \ell$ and all $j = 1, \dots, d_k$.
- (4') $\nu_n(f_{k,j}(\partial\Delta_m \setminus \Delta_{m-1}^\circ)) = 0$, for all $n \in \mathbb{N}$, all $k = 1, \dots, \ell$, and all $j = 1, \dots, d_k$.
- (5') $f_{k,j}(\Delta_m) \cap f_{k',i}(\Delta_m) = \emptyset$ for all $k, k' = 1, \dots, \ell$ with $k \neq k'$, and for all $j = 1, \dots, d_k$ and $i = 1, \dots, d_{k'}$.

To prove the existence of these functions, one argues similarly as in the inductive step: start with any set $\{h_{k,j}: k = 1, \dots, \ell, j = 1, \dots, d_k\}$ of functions satisfying (1'), (2'), (3'). Since this collection of functions is finite, there exists $r \in (0, 1]$ such that the restrictions of the functions $h_{k,j}$ to $\Delta_m^{(r)}$ have disjoint ranges (so that (5') is satisfied). Arguing as before, and using that ν_n is a probability measure for all $n \in \mathbb{N}$, it follows that the set

$$T = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\ell} \bigcap_{j=1}^{d_k} \{t \in (0, 1): \nu_n(h_{k,j}(\partial\Delta_m^{(t)} \setminus \Delta_{m-1}^\circ)) = 0\}$$

has countable complement in $[0, 1]$. Fix $t_0 \in T$, and let $f_{\ell,j}$ be the function given by $f_{y,j}(\lambda_1, \dots, \lambda_m) = h_{y,j}(\lambda_1, \dots, \lambda_m/t_0)$ for all $(\lambda_1, \dots, \lambda_m) \in \Delta_m$. Then conditions (1') through (5') are satisfied, and the claim is proved.

Set

$$V = Y \setminus \left(\bigcup_{k=1}^{\ell} \bigcup_{j=1}^{d_k} f_{k,j}(\partial\Delta_m \setminus \Delta_{m-1}^\circ) \right),$$

which is an open subset of Y satisfying $\nu_n(Y \setminus V) = 0$ for all $n \in \mathbb{N}$. By the choice of the functions (specifically, by condition (1')), there exists a local cross-section defined on all of V (obtained by considering cross-sections on the connected components of V). This finishes the induction, and proves the theorem. □

We remark that our argument in the proof of Theorem 3.1 depends on having at most countably many finite Borel measures. We do not know whether the conclusion of Theorem 3.1 holds if this assumption is dropped.

The following is a variant of [31, Definition 5.2]. It is closely related to the projectionless free tracial Rokhlin property of Archey [2], and the weak Rokhlin properties of Matui and Sato [49] and Wang [72].

Definition 3.2. Let A be a simple unital C^* -algebra, let G be a finite group, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action.

- (1) We say that α has the *weak tracial Rokhlin property* if, for every finite set $F \subseteq A$, for every $\varepsilon > 0$, and for every positive element $x \in A$ of norm one, there exist positive contractions a_g in A for $g \in G$, such that:
 - (a) $\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ for all $g, h \in G$;
 - (b) $\|a_g a_h\| < \varepsilon$ for all $g, h \in G$ with $g \neq h$;
 - (c) $\|a_g b - b a_g\| < \varepsilon$ for all $g \in G$ and all $b \in F$;
 - (d) with $a = \sum_{g \in G} a_g$, the element $1 - a$ is Cuntz subequivalent to x ;
 - (e) $\|axa\| > 1 - \varepsilon$.
- (2) We say that α has the *tracial Rokhlin property* if the positive contractions in (1) can be chosen to be orthogonal projections.

Condition (e) is automatically satisfied whenever A is finite and infinite-dimensional; see [57, Lemma 1.16].

We now want to elaborate on an observation made in [32]. Fix a finite group G and a non-negative integer $d \in \mathbb{N}$. By [32, Lemma 1.9] (see also Theorem 4.7 below for a more general argument), there exists a compact free G -space Y (depending on both d and G) that is universal for actions with $\dim_{\text{Rok}}^c \leq d$, in the following sense: an action $\alpha : G \rightarrow \text{Aut}(A)$ of G on a unital C^* -algebra A has $\dim_{\text{Rok}}^c(\alpha) \leq d$ if and only if there is a unital equivariant homomorphism $C(Y) \rightarrow A_\infty \cap A'$.

The space Y has a very concrete description, which we proceed to describe. Denote by C the universal commutative unital C^* -algebra generated by positive contractions $f_g^{(j)}$, for $g \in G$ and $j = 0, \dots, d$, satisfying the following relations:

- (1) $f_g^{(j)} f_h^{(j)} = 0$ whenever $h \neq g$, for all $j = 0, \dots, d$;
- (2) $\sum_{g \in G} \sum_{j=0}^d f_g^{(j)} = 1$.

Define an action $\gamma : G \rightarrow \text{Aut}(C)$ on generators by $\gamma_g(f_h^{(j)}) = f_{gh}^{(j)}$ for all $g, h \in G$ and all $j = 0, \dots, d$. Set $Y = \widehat{C}$, the maximal ideal space of C . Then Y is a free G -space, and it is readily checked that an action $\alpha : G \rightarrow \text{Aut}(A)$ of G on a unital C^* -algebra A has $\dim_{\text{Rok}}^c(\alpha) \leq d$ if and only if there is a unital equivariant homomorphism $C(Y) \rightarrow A_\infty \cap A'$. (This homomorphism may not be injective.)

The description of $C(Y)$ as a universal C^* -algebra allows one to identify the space Y as a simplicial complex. We briefly describe this structure: each of the contractions $f_g^{(j)}$ determines a 0-simplex. Moreover, there is a 1-simplex between $f_g^{(j)}$ and $f_h^{(k)}$ whenever $f_g^{(j)} f_h^{(k)} \neq 0$. In general, for $n \in \mathbb{N}$, there is an n -simplex with zero-dimensional boundary $\{f_{g_0}^{(j_0)}, \dots, f_{g_n}^{(j_n)}\}$ whenever $f_{g_0}^{(j_0)} \cdots f_{g_n}^{(j_n)} \neq 0$. In particular, Y is a d -dimensional simplicial complex. (An explicit computation of the G -space Y when $G = \mathbb{Z}_2$ is given

in Lemma 4.25: one gets S^d with the antipodal map.) Moreover, the action of G on Y is easily seen to be compatible with its simplicial structure, and has the property that, for all $k = 0, \dots, d$, no point in one k -cell of Y is mapped to another point in the same k -cell. It follows that actions constructed in this way satisfy the assumptions of Theorem 3.1.

In the following, we work with quasitraces because we do not assume the algebra to be exact. For a quasitrace τ on a C^* -algebra A , we denote by $d_\tau : (A \otimes \mathcal{K})_+ \rightarrow [0, \infty]$ its associated dimension function, which is given by $d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^{1/n})$ for all $a \in (A \otimes \mathcal{K})_+$.

Definition 3.3. We say that a C^* -algebra A has *strict comparison of positive elements by quasitraces*, usually referred to as ‘strict comparison’ for short, if for every $a, b \in (A \otimes \mathcal{K})_+$ satisfying $d_\tau(a) < d_\tau(b)$ for all quasitraces τ on A , then $a \prec\prec b$.

Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group G on a simple, unital C^* -algebra A . If A has strict comparison, then the weak tracial Rokhlin property and the tracial Rokhlin property for α can be reformulated using quasitracial states. Indeed, it is easy to see that under these assumptions, condition (d) in Definition 3.2 can be replaced by the condition that, for every $\varepsilon > 0$, there is a positive contraction a satisfying $d_\tau(1 - a) < \varepsilon$ for all $\tau \in QT(A)$. Equivalently, we may only require $d_\tau(1 - a) < \varepsilon$ for all *extreme* quasitracial states on A .

THEOREM 3.4. *Let A be an infinite-dimensional, simple, finite, unital C^* -algebra with strict comparison and at most countably many extreme quasitraces. Let G be a finite group and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. If $\dim_{\text{Rok}}^c(\alpha) < \infty$, then α has the weak tracial Rokhlin property.*

Proof. Let $\{\tau_n\}_{n \in \mathbb{N}}$ be an enumeration of the set of extreme quasitraces on A , allowing for repetition if this set is finite. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter, and, for $n \in \mathbb{N}$, denote by $\sigma_n : A_\omega \rightarrow \mathbb{C}$ the quasitrace obtained from τ_n . Denote by Y the free G -simplicial complex from the discussion before this theorem; see also [32, Lemma 1.9] and Theorem 4.7. Let $\varphi : C(Y) \rightarrow A_\omega \cap A'$ be a unital equivariant homomorphism. For $n \in \mathbb{N}$, the map $\sigma_n \circ \varphi : C(Y) \rightarrow \mathbb{C}$ is a quasitrace, and since $C(Y)$ is commutative, it is a tracial state. By the Riesz representation theorem, there exists a Borel probability measure μ_n on Y such that $(\sigma_n \circ \varphi)(f) = \int_Y f(y) d\mu_n(y)$ for all $f \in C(Y)$.

To prove that α has the weak tracial Rokhlin property, we will show that there exists a completely positive contractive, G -equivariant, order-zero map

$$\psi : (C(G), \text{Lt}) \rightarrow (A_\omega \cap A', \alpha_\omega)$$

such that $\sigma_n(1 - \psi(1)) = 0$ for all $n \in \mathbb{N}$.

Fix $m \in \mathbb{N}$. Let $U \subseteq Y$ be an open subset as in the conclusion of Theorem 3.1 for the sequence $(\mu_n)_{n \in \mathbb{N}}$. Choose a continuous function $0 \leq f^{(m)} \leq 1$ supported on U satisfying

$$\mu_n(\{x \in U : f^{(m)}(x) \neq 1\}) < \frac{1}{m}$$

for all $n = 1, \dots, m$. For $g \in G$, set $f_g^{(m)} = g \cdot f^{(m)} \in C(Y)$.

Let $(F_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of A with dense union. Use the Choi–Effros lifting theorem for $\varphi: C(Y) \rightarrow A_\omega \cap A'$ to find a unital completely positive linear map $\psi_m: C(Y) \rightarrow A$ such that the following conditions hold:

- (1) $\|\psi_m(f_g^{(m)})\psi_m(f_h^{(m)}) - \psi_m(f_g^{(m)} f_h^{(m)})\| < 1/m$ for all $g, h \in G$;
- (2) $\|\alpha_h(\psi_m(f_g^{(m)})) - \psi_m(f_{hg}^{(m)})\| < 1/m$ for all $g, h \in G$;
- (3) $\|\psi_m(f_g^{(m)})b - b\psi_m(f_g^{(m)})\| < 1/m$ for all $g \in G$ and all $b \in F_m$;
- (4) for every $n = 1, \dots, m$, we have

$$d_{\tau_n}\left(1 - \sum_{g \in G} \psi_m(f_g^{(m)})\right) < \frac{1}{m}.$$

For $g \in G$, set $a_g^{(m)} = \psi_m(f_g^{(m)}) \in A$ for $m \in \mathbb{N}$, and set

$$a_g = \eta_A((a_g^{(m)})_{m \in \mathbb{N}}) \in A_\omega.$$

(See Definition 2.1 for the definition of the canonical map $\eta_A: \ell^\infty(A) \rightarrow A_\omega$.) It is then easy to verify that the assignment $g \mapsto a_g$ extends to a completely positive contractive order-zero map $\psi: C(G) \rightarrow A_\omega \cap A'$ which is G -equivariant and satisfies $\sigma_n(1 - \psi(1)) = 0$ for all $n \in \mathbb{N}$. This finishes the proof. □

Remark 3.5. In the theorem above, simplicity is only needed because it is required in the definition of the weak tracial Rokhlin property. If the algebra is not assumed to be simple or to have strict comparison, the conclusion is as follows: for every $\varepsilon > 0$ and for every finite subset $F \subseteq A$, there exist positive contractions $f_g \in A$, for $g \in G$, satisfying:

- (1) $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;
- (2) $\|f_g f_h\| < \varepsilon$ for all $g, h \in G$ with $g \neq h$;
- (3) $\|f_g a - a f_g\| < \varepsilon$ for all $g \in G$ and for all $a \in F$;
- (4) with $f = \sum_{g \in G} f_g$, we have $d_\tau(1 - f) < \varepsilon$ for all $\tau \in T(A)$.

COROLLARY 3.6. *Let A be a simple, unital, infinite-dimensional C^* -algebra with tracial rank zero and at most countably many extreme tracial states. Let G be a finite group and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action. If $\dim_{\text{Rok}}^c(\alpha) < \infty$, then α has the tracial Rokhlin property.*

Proof. It follows from Theorem 3.4 that α has the weak tracial Rokhlin property. Since A has tracial rank zero and is infinite-dimensional, [58, Theorem 1.9] implies that α has the tracial Rokhlin property. □

Theorem 3.4 and Corollary 3.6 hold in greater generality for actions on Kirchberg algebras. Indeed, in this case, finite Rokhlin dimension (without commuting towers) is in fact equivalent to the (weak) tracial Rokhlin property, and, moreover, equivalent to pointwise outeriness; see Theorem 3.11.

COROLLARY 3.7. *Finite group actions with finite Rokhlin dimension with commuting towers preserve the class of simple, nuclear, unital, separable C^* -algebras with tracial rank zero, as long as the algebra has at most countably many extreme tracial states.*

Proof. The result follows immediately from Corollary 3.6, together with [57, Theorem 2.6]. \square

One advantage of Corollary 3.6 is that finite Rokhlin dimension with commuting towers is sometimes easier to establish than the tracial Rokhlin property. Cyclic group actions on higher-dimensional non-commutative tori provide one instance where this is the case; see Proposition 3.9. We need some preparation first.

Let $d \in \mathbb{N}$ and let $\Theta = (\theta_{jk})_{1 \leq j, k \leq d} \in M_d(\mathbb{R})$ be a skew symmetric matrix. The higher-dimensional non-commutative torus A_Θ is the universal unital C^* -algebra generated by unitaries u_j , for $1 \leq j \leq d$, satisfying the commutation relations $u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j$ for $1 \leq j, k \leq d$. Define a rotation map $h_\Theta: \mathbb{Z}^d \rightarrow \mathbb{T}^d$ by

$$h_\Theta(m) = (e^{2\pi i \sum_{j=1}^d \theta_{1j} m_j}, \dots, e^{2\pi i \sum_{j=1}^d \theta_{dj} m_j})$$

for $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$. It is clear that h_Θ is a group homomorphism.

Recall that a matrix $\Theta \in M_d(\mathbb{R})$ as above is said to be *non-degenerate* if whenever $x \in \mathbb{Z}^d$ satisfies $e^{2\pi i \langle x, \theta y \rangle} = 1$ for all $y \in \mathbb{Z}^d$, then $x = 0$. (This condition is equivalent to the rows of Θ forming a rationally linearly independent set in \mathbb{R}^d .) The case $d = 1$ of the following lemma is well known. Recall that, given $n \in \mathbb{Z}^d$, there exists a (continuous) group homomorphism $\gamma_n: \mathbb{T}^d \rightarrow \mathbb{C}$, given by $\gamma_n(z_1, \dots, z_d) = z_1^{n_1} \cdots z_d^{n_d}$ for all $z = (z_1, \dots, z_d) \in \mathbb{T}^d$, and that every continuous homomorphism $\mathbb{T}^d \rightarrow \mathbb{C}$ has this form (this is a restatement of Pontryagin duality for the group \mathbb{T}^d). Moreover, γ_n is non-trivial if and only if $n \neq 0$.

The following lemma is folklore, but since we were unable to find a reference, we include a proof for the convenience of the reader.

LEMMA 3.8. *Adopt the notation of the discussion above. Then Θ is non-degenerate if and only if h_Θ has dense range.*

Proof. Set $G = \overline{h_\Theta(\mathbb{Z}^d)}$, which is a closed subgroup of \mathbb{T}^d . Observe that G is proper if and only if there is $n \in \mathbb{Z}^d \setminus \{0\}$ such that $\gamma_n(G) = \{1\}$. Continuity of γ_n implies that this is in turn equivalent to $\gamma_n(h_\Theta(m)) = 1$ for all $m \in \mathbb{Z}^d$. By definition, we have

$$\begin{aligned} \gamma_n(h_\Theta(m)) &= \gamma_n(e^{2\pi i \sum_{j=1}^d \theta_{1j} m_j}, \dots, e^{2\pi i \sum_{j=1}^d \theta_{dj} m_j}) \\ &= e^{2\pi i [n_1 \sum_{j=1}^d \theta_{1j} m_j + \dots + n_d \sum_{j=1}^d \theta_{dj} m_j]} \\ &= e^{2\pi i [m_1 \sum_{k=1}^d n_k \theta_{k1} + \dots + m_d \sum_{k=1}^d n_k \theta_{kd}]} \end{aligned}$$

For $j = 1, \dots, d$, let $e_j \in \mathbb{Z}^d$ be the canonical j th basis vector. Since h_Θ is a group homomorphism, one has $\gamma_n(h_\Theta(m)) = 1$ for all $m \in \mathbb{Z}^d$ if and only if $\gamma_n(h_\Theta(e_j)) = 1$ for all $j = 1, \dots, d$. Using the computation above, one sees that this is the case if and only if $\sum_{k=1}^d n_k \theta_{kj}$ is an integer for all $j = 1, \dots, d$.

Summing up, we argued that h_Θ has dense range if and only if there does not exist a non-zero $n \in \mathbb{Z}^d$ such that $n\Theta$ belongs to \mathbb{Z}^d , which is precisely the definition of non-degeneracy for Θ . This finishes the proof. \square

Next, we use ideas from [47] to show that certain quasifree actions of finite cyclic groups on A_Θ have Rokhlin dimension one with commuting towers. Our result contains

[32, Example 1.12] as a particular case. For $d \geq 2$, we write the coordinates of a vector $r \in \mathbb{Z}^d$ as $(r^{(1)}, \dots, r^{(d)})$.

PROPOSITION 3.9. *Let $\Theta \in M_d(\mathbb{R})$ be a non-degenerate skew symmetric matrix, let $m_1, \dots, m_d \in \mathbb{N}$ be mutually coprime positive integers, and set $m = m_1 \cdots m_d$. Suppose that $m > 1$. Let α be the automorphism of A_Θ determined by $\alpha(u_j) = e^{2\pi i/m_j} u_j$ for $j = 1, \dots, d$. Denote also by α the action of \mathbb{Z}_m on A_Θ that it determines. Then $\dim_{\text{Rok}}^c(\alpha) = 1$.*

In particular, α has the tracial Rokhlin property.

Proof. We claim that there exists a sequence $(r_n)_{n \in \mathbb{N}}$ in \mathbb{Z}^d such that:

- (1) $\lim_{n \rightarrow \infty} \text{dist}(\Theta \cdot r_n, \mathbb{Z}^d) = 0$; and
- (2) $r_n^{(j)} = 1 \pmod{m_j}$ for all $j = 1, \dots, d$ and for all $n \in \mathbb{N}$.

As a first step, we show that $h_\Theta(m\mathbb{Z}^d + (1, \dots, 1))$ is dense in \mathbb{T}^d . Since $h_\Theta(m\mathbb{Z}^d)$ equals $h_{m\Theta}(\mathbb{Z}^d)$ and $m\Theta$ is also non-degenerate, Lemma 3.8 implies that this set is dense in \mathbb{T}^d . Finally, $h_\Theta(m\mathbb{Z}^d + (1, \dots, 1)) = h_\Theta(m\mathbb{Z}^d)h_\Theta(1, \dots, 1)$ is just a translate of $h_\Theta(m\mathbb{Z}^d)$, so it is also dense.

Let $(r_n)_{n \in \mathbb{N}}$ be a sequence in $m\mathbb{Z}^d + (1, \dots, 1)$ such that $\lim_{n \rightarrow \infty} h_\Theta(r_n) = (1, \dots, 1)$. It is immediate that $(r_n)_{n \in \mathbb{N}}$ satisfies conditions (1) and (2) above, and the claim is proved.

Let $(r_n)_{n \in \mathbb{N}}$ be a sequence as in the claim above. Set

$$v_n = u_1^{r_n^{(1)}} \cdots u_d^{r_n^{(d)}},$$

which is a unitary in A_Θ . For $1 \leq j \leq d$, it is easy to check that

$$v_n u_j = e^{2\pi i \sum_{k=1}^d \theta_{jk} m_k} u_j v_n.$$

Since $\sum_{k=1}^d \theta_{jk} m_k = (\Theta \cdot r_n)^{(j)}$, condition (1) above implies that $(v_n)_{n \in \mathbb{N}}$ is a central sequence of unitaries in A_Θ .

Denote by $\gamma \in \text{Aut}(C(\mathbb{T}))$ the automorphism induced by rotation by the angle $e^{2\pi i(1/m_1 + \dots + 1/m_d)}$. It is clear that $\gamma^m = \text{id}$, so it determines an action of \mathbb{Z}_m on $C(\mathbb{T})$, which we also denote by γ . Using that the integers m_1, \dots, m_d are coprime, it is easy to verify that γ is free.

In the following computation, we use condition (2) at the second step to get

$$\begin{aligned} \alpha(v_n) &= (e^{2\pi i r_n^{(1)}/m_1} u_1^{r_n^{(1)}}) \cdots (e^{2\pi i r_n^{(d)}/m_d} u_d^{r_n^{(d)}}) \\ &= (e^{2\pi i/m_1} u_1^{r_n^{(1)}}) \cdots (e^{2\pi i/m_d} u_d^{r_n^{(d)}}) \\ &= e^{2\pi i(1/m_1 + \dots + 1/m_d)} v_n. \end{aligned}$$

We conclude that $(v_n)_{n \in \mathbb{N}}$ determines a \mathbb{Z}_m -equivariant homomorphism

$$(C(\mathbb{T}), \gamma) \rightarrow (A_\infty \cap A', \alpha_\infty).$$

Since γ is free, [32, Lemma 1.9] (see also Theorem 4.7) implies that $\dim_{\text{Rok}}^c(\alpha) \leq 1$.

To show that $\dim_{\text{Rok}}^c(\alpha) = 1$, we must argue that α does not have the Rokhlin property. In fact, we show that no non-trivial finite group can act on A_Θ with the Rokhlin property.

To this end, suppose that G is a finite group and $\beta: G \rightarrow \text{Aut}(A_\Theta)$ is an action with the Rokhlin property. Find projections e_g in A_Θ , for $g \in G$, satisfying

$$\|\beta_g(e_h) - e_{gh}\| < 1 \quad \text{and} \quad \sum_{g \in G} e_g = 1$$

for all $g, h \in G$. In particular, $\beta_g(e_h)$ is Murray–von Neumann equivalent to e_{gh} . Observe that, by [65, Lemma 3.1], A_Θ has a unique tracial state τ , which must therefore be invariant under β . Thus $\tau(e_g) = \tau(e_h)$ for all $g, h \in G$. We denote by $\tau_*: K_0(A_\Theta) \rightarrow \mathbb{R}$ the group homomorphism induced by τ . We deduce that $\tau([1])$ is divisible by the cardinality of G in $\tau(K_0(A_\Theta))$. However, the range of τ on projections has been computed by Elliott in [15], and his computation yields that the class of the unit of A_Θ is not divisible in $K_0(A_\Theta)$. This forces the cardinality of G to be one, and the assertion follows.

For the last claim, observe that A_Θ is a simple AT-algebra by the main result of [55], and hence it has tracial rank zero. Since A_Θ has a unique trace, the result is a consequence of Corollary 3.6. □

It is not in general true that a finite group action with the tracial Rokhlin property has finite Rokhlin dimension with commuting towers, since there are K -theoretic obstructions to admitting such an action. For example, it is easy to show that the order-two automorphism $\bigotimes_{n \in \mathbb{N}} \text{Ad}(\text{diag}(1, 1, -1))$ of the uniformly hyperfinite (UHF) algebra of type 3^∞ determines a \mathbb{Z}_2 -action $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(M_{3^\infty})$ with the tracial Rokhlin property; see [57]. It is clear that α does not have the Rokhlin property since the unit of M_{3^∞} is not 2-divisible in K -theory. It then follows from [21, Theorem 4.19] that α does not have finite Rokhlin dimension with commuting towers. More generally, it is a consequence of part (2) in [32, Corollary 4.8] that there are *no* \mathbb{Z}_2 -actions on M_{3^∞} with finite Rokhlin dimension with commuting towers.

There are also more subtle obstructions, related to the equivariant K -theory of an action (as opposed to the K -theory of the algebra), as we show in the next example. We refer the reader to [54, §2] for the definitions of equivariant K -theory and the augmentation ideal $I_G \subseteq R(G)$ of a compact group G . (A quicker introduction containing the notions that are needed here is given in [18, §III.3].)

Example 3.10. In [5], Blackadar constructed an example of a \mathbb{Z}_2 -action on the UHF-algebra A of type 2^∞ , whose crossed product is not AF. Denote this action by $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$. Phillips showed in [58, Proposition 3.4] that α has the tracial Rokhlin property. Since $A \rtimes_\alpha \mathbb{Z}_2$ is not AF, the action α does not have the Rokhlin property, because this would otherwise contradict [57, Theorem 2.2]. We claim that α has infinite Rokhlin dimension with commuting towers.

In order to show that $\text{dim}_{\text{Rok}}^c(\alpha) = \infty$, we will show that α does not have discrete K -theory. Once we show this, the result will then follow from [32, Corollary 4.2]. In order to arrive at a contradiction, assume that there exists $n \in \mathbb{N}$ such that $I_{\mathbb{Z}_2}^n \cdot K_*^{\mathbb{Z}_2}(A, \alpha) = 0$. Denote by $\widehat{\alpha}: \mathbb{Z}_2 \rightarrow \text{Aut}(A \rtimes_\alpha \mathbb{Z}_2)$ the dual action of α , and, by a slight abuse of notation, denote also by $\widehat{\alpha} \in \text{Aut}(A \rtimes_\alpha \mathbb{Z}_2)$ the generating order-two automorphism. It is immediate to check that the condition $I_{\mathbb{Z}_2}^n \cdot K_*^{\mathbb{Z}_2}(A, \alpha) = 0$ is equivalent to

$$(\text{id}_{K_*(A \rtimes_\alpha G)} - K_*(\widehat{\alpha}))^n = 0.$$

The proof of [58, Proposition 3.5] shows that there exists $x \in K_1(A \rtimes_\alpha \mathbb{Z}_2)$ with $x \neq 0$ such that $K_1(\widehat{\alpha})(x) = -x$. It is shown in [5, Proposition 5.4.1] that $A \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to the tensor product of the Bunce–Deddens algebra of type 2^∞ with A , so in particular $K_1(A \rtimes_\alpha \mathbb{Z}_2)$ does not have any 2-torsion. We conclude that

$$(\text{id}_{K_*(A \rtimes_\alpha \mathbb{Z}_2)} - K_*(\widehat{\alpha}))^m(x) = 2^m x \neq 0$$

for all $m \in \mathbb{N}$, contradicting the fact that α has discrete K -theory. This contradiction, together with [32, Corollary 4.2], shows that α does not have finite Rokhlin dimension with commuting towers.

Next, we show that Theorem 3.4 and Corollary 3.6 hold in greater generality for finite group actions on Kirchberg algebras. We emphasize that there are no UCT assumptions in the next result.

THEOREM 3.11. *Let G be a finite group, let A be a unital Kirchberg algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action. Then the following statements are equivalent:*

- (1) α has the tracial Rokhlin property;
- (2) α has the weak tracial Rokhlin property;
- (3) $\dim_{\text{Rok}}(\alpha) \leq 1$;
- (4) α is pointwise outer (that is, α_g is not inner for all $g \in G \setminus \{1\}$).

Proof. It is clear that (1) implies (2). That (2) implies (4) is well known; see, for example, [2, Lemma VI.8] or [31, Proposition 5.3]. (These implications hold without restrictions on the algebra A .) In turn, the equivalence between (3) and (4) was proved in [21, Theorem 4.20]. It remains to show that (3) implies (1).

Assume that α is pointwise outer. By [27, Theorem 5.1], there are a pointwise outer action $\gamma : G \rightarrow \text{Aut}(\mathcal{O}_\infty)$ and an equivariant isomorphism

$$(A, \alpha) \cong \left(A \otimes \bigotimes_{n=1}^\infty \mathcal{O}_\infty, \alpha \otimes \bigotimes_{n=1}^\infty \gamma \right).$$

(In fact, any outer quasifree action γ will do.) We identify these two G -algebras in the sequel. Also, for $k \in \mathbb{N}$, we regard $A \otimes \bigotimes_{n=1}^k \mathcal{O}_\infty$ canonically as a subalgebra of $A \otimes \bigotimes_{n=1}^\infty \mathcal{O}_\infty$.

Let $\varepsilon > 0$, let $F \subseteq A \otimes \bigotimes_{n=1}^\infty \mathcal{O}_\infty$ be a finite set, and let $x \in A \otimes \bigotimes_{n=1}^\infty \mathcal{O}_\infty$ be a non-zero positive contraction. Find $m \in \mathbb{N}$, a finite set $\widetilde{F} \subseteq A \otimes \bigotimes_{n=1}^m \mathcal{O}_\infty$, and a positive contraction $y \in A \otimes \bigotimes_{n=1}^m \mathcal{O}_\infty$ such that:

- for every $a \in F$ there exists $b \in \widetilde{F}$ with $\|a - b\| < \varepsilon$; and
- $\|x - y\| < \varepsilon$.

Since γ_g is outer for all $g \in G \setminus \{1\}$, [46, Lemma 1.9] implies that there exists a positive contraction $f \in \mathcal{O}_\infty$ with $\|f\gamma_g(f)\| < \varepsilon$ for all $g \in G \setminus \{1\}$. Since \mathcal{O}_∞ has real rank zero, we may assume that f has finite spectrum, and that $1 \in \text{sp}(f)$. Let $q \in \mathcal{O}_\infty$ be the spectral projection corresponding to $1 \in \text{sp}(f)$. Then $qf q = q$ and thus

$$\|q\gamma_g(q)\| = \|f q f \gamma_g(f) \gamma_g(q) \gamma_g(f)\| < \varepsilon$$

for all $g \in G \setminus \{1\}$. By slightly perturbing q , we may assume without loss of generality that $q\gamma_g(q) = 0$. It follows that $\|\gamma_h(q)\gamma_g(q)\| < \varepsilon$ for all $g, h \in G$ with $g \neq h$.

Given $g \in G$, set

$$e_g = 1_A \otimes \underbrace{1_{\mathcal{O}_\infty} \otimes \cdots \otimes 1_{\mathcal{O}_\infty}}_{m \text{ times}} \otimes \gamma_g(q) \in A \otimes \bigotimes_{n=1}^{m+1} \mathcal{O}_\infty \subseteq A \otimes \bigotimes_{n=1}^\infty \mathcal{O}_\infty.$$

We claim that the projections e_g , for $g \in G$, satisfy the conditions in part (2) of Definition 3.2 for ε , F and x . It is clear that $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$, so the first condition is satisfied. Clearly $\|e_g e_h\| < \varepsilon$ for all $g, h \in G$, so condition (b) follows. Given $a \in F$, choose $b \in \tilde{F}$ satisfying $\|a - b\| < \varepsilon$. For $g \in G$, we have

$$\|e_g a - a e_g\| \leq \|a - b\| + \|e_g b - b e_g\| < \varepsilon,$$

because e_g commutes with b , so condition (c) is also satisfied. Condition (d) is automatic since any two non-zero positive elements in a purely infinite simple C^* -algebra are Cuntz equivalent. Finally, to check condition (e), recall that $y \in A \otimes \bigotimes_{n=1}^m \mathcal{O}_\infty$ satisfies $\|x - y\| < \varepsilon$. Set $e = \sum_{g \in G} e_g$. Then e is a non-zero projection in \mathcal{O}_∞ , and $e y e = e y$ can be represented as

$$\sum_{g \in G} y \otimes q_g \in A \otimes \bigotimes_{n=1}^m \mathcal{O}_\infty \otimes \mathcal{O}_\infty \subseteq A.$$

We deduce that

$$\|e x e\| \geq \|e y e\| - \|x - y\| > \|e y\| - \varepsilon = \|e\| \|y\| - \varepsilon = 1 - \varepsilon,$$

as desired. This shows that α has the tracial Rokhlin property, and finishes the proof. □

4. Structure of the crossed product

In this section we study crossed products by actions of compact groups with finite Rokhlin dimension with commuting towers. The main result of this section, Theorem 4.17, shows that a number of structural properties are inherited under formation of crossed products by such actions. We also point out, in Remark 4.18, that the formulation of Rokhlin dimension without commuting towers is not enough to obtain most of these conclusions. Strict comparison is also inherited, as long as the action has Rokhlin dimension with commuting towers at most one, and the C^* -algebra has ‘no K_1 -obstructions’; see Theorem 4.24 for the precise formulation. Our results are used to construct an action α of \mathbb{Z}_2 on a UCT Kirchberg algebra with $\dim_{\text{Rok}}^c(\alpha) = 2$ and $\dim_{\text{Rok}}(\alpha) = 1$; see Example 4.29. To the best of our knowledge, this is the first example of a group action on a simple C^* -algebra with Rokhlin dimension other than 0, 1, or ∞ .

At the core of these results is the fact that if $\alpha: G \rightarrow \text{Aut}(A)$ has finite Rokhlin dimension with commuting towers, then $A \rtimes_\alpha G$ can be locally approximated by a certain continuous $C(X)$ -algebra with fibers isomorphic to $A \otimes \mathcal{K}(L^2(G))$; see Proposition 4.11. Moreover, the space X can be chosen to satisfy $\dim(X) < \infty$ whenever $\dim(G) < \infty$; see Theorem 4.7.

The following proposition is well known, but we include a proof for the convenience of the reader.

PROPOSITION 4.1. *Let A be a C^* -algebra, let G be a compact group, and let $\alpha : G \rightarrow \text{Aut}(A)$ be a continuous action. Then there is an equivariant isomorphism*

$$\theta : (C(G, A), \text{Lt} \otimes \alpha) \rightarrow (C(G, A), \text{Lt} \otimes \text{id}_A)$$

given by $\theta(f)(g) = \alpha_{g^{-1}}(f(g))$ for all $f \in C(G, A)$ and all $g \in G$. In particular, there is a natural identification

$$C(G, A) \rtimes_{\text{Lt} \otimes \alpha} G \cong A \otimes \mathcal{K}(L^2(G)).$$

Proof. It is clear that θ is an isomorphism. To check that it is equivariant, let $f \in C(G, A)$ and $g, h \in G$ be given. Then

$$\begin{aligned} \theta((\text{Lt}_g \otimes \alpha_g)(f))(h) &= \alpha_{h^{-1}}((\text{Lt}_g \otimes \alpha_g)(f)(h)) \\ &= \alpha_{h^{-1}}(\alpha_g(f(g^{-1}h))) \\ &= \theta(\xi)(g^{-1}h) \\ &= (\text{Lt}_g \otimes \text{id}_A)(f)(h). \end{aligned}$$

The second claim is immediate since there are natural isomorphisms

$$(C(G) \otimes A) \rtimes_{\text{Lt} \otimes \alpha} G \cong (C(G) \rtimes_{\text{Lt}} G) \otimes A \cong \mathcal{K}(L^2(G)) \otimes A. \quad \square$$

With the notation from the above proposition, it follows that the canonical equivariant inclusion $A \rightarrow C(G) \otimes A$ induces an injective homomorphism

$$\iota : A \rtimes_{\alpha} G \rightarrow A \otimes \mathcal{K}(L^2(G)).$$

Denote by $\lambda : G \rightarrow \mathcal{U}(L^2(G))$ the left regular representation, and identify $A \rtimes_{\alpha} G$ with its image under ι . It is then a consequence of non-commutative duality for crossed products that

$$A \rtimes_{\alpha} G = (A \otimes \mathcal{K}(L^2(G)))^{\alpha \otimes \text{Ad}(\lambda)}.$$

The following definitions are standard, and have been independently defined by Kasparov and Phillips. For a C^* -algebra A , we denote its center by $Z(A)$.

Definition 4.2. Let X be a locally compact Hausdorff space.

- (1) A $C_0(X)$ -algebra is a pair (A, μ) consisting of a C^* -algebra A and a homomorphism $\mu : C_0(X) \rightarrow Z(M(A))$ satisfying $\mu(C_0(X))A = A$. We usually suppress μ from the notation, and simply say that A is a $C_0(X)$ -algebra.

If A is a $C_0(X)$ -algebra and $U \subseteq X$ is open, then $\mu(C_0(U))A$ is an ideal in A . Given $x \in X$, the fiber over x is the quotient

$$A(x) = A / C_0(X \setminus \{x\})A.$$

(In principle, the fiber $A(x)$ depends on μ , but we do not incorporate it into the notation.) For $a \in A$, we write $a(x)$ for its image in $A(x)$.

- (2) We say that A is a continuous $C_0(X)$ -algebra if it is a $C_0(X)$ -algebra and, for every $a \in A$, the map $X \rightarrow \mathbb{R}$ given by $x \mapsto \|a(x)\|$ is continuous.

- (3) If A is a continuous $C_0(X)$ -algebra, we say that it is *locally trivial* if, for every $x \in X$, there exist an open set $U \subseteq X$ containing x and an isomorphism $\mu(C_0(U))A \cong C_0(U, A(x))$. (In particular, $A(x) \cong A(y)$ for every $y \in U$.)

Let $G \curvearrowright X$ be a continuous action of a group G on a space X . We write $\pi : X \rightarrow X/G$ for the canonical quotient map. Given $x \in X$, we denote by $G \cdot x$ the set $\{g \cdot x : g \in G\}$. The *stabilizer* of $x \in X$ is the closed subgroup $G_x = \{g \in G : g \cdot x = x\}$, and there is a canonical homeomorphism $G \cdot x \cong G/G_x$.

PROPOSITION 4.3. *Let G be a compact group, let X be a locally compact Hausdorff space, and let G act continuously on X . Then $C_0(X)$ is naturally a continuous $C_0(X/G)$ -algebra.*

Proof. We denote by $\pi : X \rightarrow X/G$ the quotient map. Observe that, since G is compact, $C_0(X/G)$ can be identified with the fixed point algebra of $C_0(X)$ via the map $\pi^* : C_0(X/G) \rightarrow C_0(X)$. Hence $C_0(X)$ is a $C_0(X/G)$ -algebra, and we only need to show continuity of the bundle. Let $U \subseteq X/G$ be an open subset, and set $V = \pi^{-1}(U)$. Using the notation from Definition 4.2, there is a natural identification of the ideal $C_0(X)C_0(U)$ with $C_0(V)$. In particular, for $x \in X$, the fiber of $C_0(X)$ over $\pi(x)$ is naturally identified with $C(G \cdot x)$. Thus, in order to check continuity, we must show that, for all $f \in C_0(X)$, the assignment $\pi(x) \mapsto \sup_{g \in G} |f(g \cdot x)|$ is a continuous map $X/G \rightarrow \mathbb{R}$.

Let $f \in C_0(X)$, let $\varepsilon > 0$ and let $x \in X$. Use continuity of f to find, for every $g \in G$, an open neighborhood W_g of $g \cdot x$ such that $|f(g \cdot x) - f(y)| < \varepsilon$ for all $y \in W_g$. By compactness of G , there exists an open neighborhood W of x such that $g \cdot W \subseteq W_g$ for all $g \in G$. Set $U = \pi(W)$, which is an open neighborhood of $\pi(x)$ in X/G . Let $y \in U$. Then

$$\sup_{h \in G} |f(h \cdot y)| - \varepsilon \leq \sup_{g \in G} |f(g \cdot x)| \leq \sup_{h \in G} |f(h \cdot y)| + \varepsilon.$$

We conclude that $\sup_{\pi(y) \in U} |\sup_{h \in G} |f(h \cdot y)| - \sup_{g \in G} |f(g \cdot x)|| < \varepsilon$, and hence $C_0(X)$ is a continuous $C_0(X/G)$ -algebra. □

Remark 4.4. Note that if X is a compact Hausdorff space and Y is some quotient of X , then $C(X)$ is not necessarily a continuous $C(Y)$ -algebra.

We will need the following fact about compact group actions on algebras of the form $C_0(X, A)$. We will only need it when X is compact and the action is free, but the proof in the general case does not take more work.

PROPOSITION 4.5. *Let G be a compact group, let X be a locally compact space, let A be a C^* -algebra, let $G \curvearrowright X$ be a continuous action, and let $\alpha : G \rightarrow \text{Aut}(A)$ be another action. Endow $C_0(X, A)$ with the diagonal G -action γ . Then $C_0(X, A) \rtimes_{\gamma} G$ is a continuous $C_0(X/G)$ -algebra. Moreover, with $\pi : X \rightarrow X/G$ denoting the quotient map, the fiber over $\pi(x) \in X/G$ is canonically isomorphic to*

$$(A \rtimes_{\alpha} G_x) \otimes \mathcal{K}(L^2(G/G_x)).$$

Proof. Observe that X/G is Hausdorff because X is Hausdorff and G is compact. Also, for $x, y \in X$, the algebras $(A \rtimes_{\alpha} G_x) \otimes \mathcal{K}(L^2(G/G_x))$ and $(A \rtimes_{\alpha} G_y) \otimes \mathcal{K}(L^2(G/G_y))$ are isomorphic whenever $\pi(x) = \pi(y)$.

Regard $C_0(X)$ as a continuous $C_0(X/G)$ -algebra as in Proposition 4.3. By [42, Theorem B], the tensor product $C_0(X, A)$ is a continuous $C_0(X/G)$ -algebra, and the fiber of $C_0(X, A)$ over $\pi(x) \in X/G$ can be identified with $C(G \cdot x, A) \cong C(G/G_x, A)$. Moreover, the action γ is a fiberwise action, in the sense of [43, p. 194], and the induced action on the fiber $C(G/G_x, A)$ over $\pi(x)$ is $\mathbb{L}t \otimes \alpha$. The discussion there shows that $C_0(X, A) \rtimes_\gamma G$ is a $C_0(X/G)$ -algebra with fibers isomorphic to $C(G/G_x, A) \rtimes_{\mathbb{L}t \otimes \alpha} G$. By Green’s imprimitivity theorem, the fiber over $\pi(x)$ is thus isomorphic to $(A \rtimes_\alpha G_x) \otimes \mathcal{K}(L^2(G/G_x))$. Finally, continuity of this $C(X/G)$ -algebra follows from [43, Theorem 4.1], since G is amenable. \square

We now specialize to the case of free actions, and show that when local cross-sections for the action exist, the C^* -bundle from the theorem above is locally trivial. Recall that, for an action $\alpha : G \rightarrow \text{Aut}(A)$ of a compact group, $\iota : A \rtimes_\alpha G \rightarrow A \otimes \mathcal{K}(L^2(G))$ denotes the canonical inclusion.

COROLLARY 4.6. *Let G be a compact group, let X be a compact Hausdorff space, let A be a C^* -algebra, let $G \curvearrowright X$ be a continuous free action, and let $\alpha : G \rightarrow \text{Aut}(A)$ be another action. Endow $C(X, A)$ with the diagonal G -action γ . Then $C(X, A) \rtimes_\gamma G$ is a continuous $C(X/G)$ -algebra with fibers canonically isomorphic to $A \otimes \mathcal{K}(L^2(G))$.*

Let $\theta : A \rtimes_\alpha G \rightarrow C(X, A) \rtimes_\gamma G$ be the map induced by the canonical inclusion $A \rightarrow C(X, A)$. For $z \in X/G$, denote by

$$\pi_z : C(X, A) \rtimes_\gamma G \rightarrow (C(X, A) \rtimes_\gamma G)_z \cong A \otimes \mathcal{K}(L^2(G))$$

the canonical quotient map onto the fiber over z . Then the following diagram is commutative:

$$\begin{array}{ccc} A \rtimes_\alpha G & \xrightarrow{\theta} & C(X, A) \rtimes_\gamma G \\ & \searrow \iota & \downarrow \pi_z \\ & & A \otimes \mathcal{K}(L^2(G)). \end{array}$$

Finally, if there exist local cross-sections for the canonical quotient map $\pi : X \rightarrow X/G$ (for example, if G is a Lie group; see [51, Theorem 8]), then $C(X, A) \rtimes_\gamma G$ is a locally trivial bundle over X/G .

Proof. The first assertion is a consequence of Proposition 4.5, since $G_x = \{1\}$ for all $x \in X$ because the action is free. (Also, in this case, the computation of the fibers can be performed using Proposition 4.1.)

In order to show that the diagram in the statement is commutative, observe first that the map ι is induced by the map $\tilde{\iota} : C(G, A) \rightarrow C(G \times X, A)$ given by $\tilde{\iota}(f)(g, x) = f(g)$ for $f \in C(G, A)$, for $g \in G$, and for $x \in X$. It is then easy to see that, for $z \in X/G$, the continuous function $\pi_z(\iota(f)) : G \rightarrow C(G, A)$ is given by

$$\pi_z(\iota(f))(g)(h) = f(g)$$

for $g, h \in G$. It follows that the restrictions of $\pi_z \circ \iota$ and θ to $C(G, A)$ agree. By density in the crossed product, we deduce that $\pi_z \circ \iota = \theta$, as desired.

We prove the last statement about local triviality. Let $x \in X$. Using the existence of local cross-sections, find an open set $U \subseteq X/G$ containing $G \cdot x$, and a local section $s : U \rightarrow X$. In particular, the open subset

$$\{g \cdot s(U) : g \in G\} \subseteq X$$

is equivariantly homeomorphic to $U \times G$ with the trivial action on U and translation on G . The crossed product of this invariant ideal, which is the ideal of the bundle associated to the open set U , is isomorphic to $C_0(U \times G, A) \rtimes_{\text{id} \otimes \text{Lt} \otimes \alpha} G$, which is itself isomorphic to $C_0(U) \otimes A \otimes \mathcal{K}(L^2(G))$ by Proposition 4.1. The claim follows. \square

The following theorem is a slight variant of [21, Theorem 4.4], extended to non-Lie groups. (The case of unital A and finite G is implicit [32, Lemma 1.9].) The main difference between the theorem below and [21, Theorem 4.4] is that the free G -space is independent of the algebra and the action, and the cost is that the map φ is not necessarily injective. We denote by $\dim(G)$ the covering dimension of the group G .

THEOREM 4.7. *Let G be a compact group. For every $d \in \mathbb{N}$, there exists a free G -space X with local cross-sections with the following property. Given an action $\alpha : G \rightarrow \text{Aut}(A)$ of G on a C^* -algebra A , we have $\dim_{\text{Rok}}^c(\alpha) \leq d$ if and only if, for every σ -unital α -invariant subalgebra $D \subseteq A$, there exists an equivariant unital homomorphism*

$$\varphi : C(X) \rightarrow F_\alpha(D, A).$$

Moreover, we have the following relations between the covering dimension of X and the Rokhlin dimension of α :

$$\begin{aligned} \dim(X) &\leq (\dim_{\text{Rok}}^c(\alpha) + 1)(\dim(G) + 1) - 1, \\ \dim_{\text{Rok}}^c(\alpha) &\leq \dim(X) - \dim(G). \end{aligned}$$

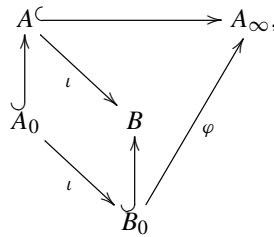
In particular, we can always take the space X above to be finite-dimensional when G is finite-dimensional.

Proof. For the ‘only if’ implication, set $d = \dim_{\text{Rok}}^c(\alpha) < \infty$. An inspection of the proof of [21, Lemma 4.3] shows that the space X can be chosen to be the d -join $G * \dots * G$ with diagonal action (see definition in [50]). This action is free, and has local cross-sections regardless of whether G is Lie or not; see [50, §3]. For the converse implication, an inspection of the proof of [21, Theorem 4.4] reveals that existence of local cross-sections is all that is needed to obtain $\dim_{\text{Rok}}^c(\alpha) < \infty$. \square

We use the above result to show that compact group actions with finite Rokhlin dimension with commuting towers preserve a number of properties upon forming crossed products; see Theorem 4.17. The way it is used is through Definition 4.12. We need an auxiliary notion first.

Definition 4.8. Let A and B be C^* -algebras, and let $\iota : A \rightarrow B$ be an injective homomorphism. We say that ι is a *positively existential embedding* if, for all separable

subalgebras $A_0 \subseteq A$ and $B_0 \subseteq B$ with $\iota(A_0) \subseteq B_0$, there exists a homomorphism $\varphi: B_0 \rightarrow A_\infty$ making the diagram



commute, where all the inclusions are the canonical ones.

When A and B are separable, this notion agrees with that of a *sequential splitting*; see [3]. In general, however, being a positively existential embedding is a weaker condition; see [26, § 4.3]. This terminology agrees with the usual one in model theory for C^* -algebras; see, for example, [26, Definition 2.11] and the comments after it, although we do not use any results from model theory for C^* -algebras in this paper.

Remark 4.9. It is clear that if $\iota: A \rightarrow B$ is a positively existential embedding, then so are all its amplifications $\iota \otimes \text{id}_{M_n}: M_n(A) \rightarrow M_n(B)$, for $n \in \mathbb{N}$, as well as its unitization $\tilde{\iota}: \tilde{A} \rightarrow \tilde{B}$.

Remark 4.10. We need two easy observations about separable subalgebras.

- Let X be a compact space and let A be a C^* -algebra. Then any separable subalgebra of $C(X, A)$ is contained in an algebra of the form $C(X, D)$ for some separable subalgebra $D \subseteq A$.
- Let E be a C^* -algebra, let G be a second countable locally compact group, and let $\beta: G \rightarrow \text{Aut}(E)$ be an action. If $B_0 \subseteq E \rtimes_\beta G$ is a separable subalgebra, then there exists a β -invariant separable subalgebra $E_0 \subseteq E$ such that $B_0 \subseteq E_0 \rtimes_\beta G$. For example, find a countably family $\{f_n: n \in \mathbb{N}\}$ of elements in $C_c(G, E, \beta)$ whose closure in $E \rtimes_\beta G$ contains B_0 . Since the image of each f_n is a compact subset of E and G is second countable, the β -invariant subalgebra E_0 of E generated by $\{f_n(G): n \in \mathbb{N}\}$ is separable, and clearly $B_0 \subseteq E_0 \rtimes_\beta G$.

The combination of Theorem 4.7 with the following proposition constitutes our main technical tool in the study of crossed products by actions with finite Rokhlin dimension with commuting towers.

PROPOSITION 4.11. *Let A be a C^* -algebra, let G be a second countable compact group, let $\alpha: G \rightarrow \text{Aut}(A)$ be an action, and let X be a compact free G -space. Suppose that, for every σ -unital α -invariant subalgebra $D \subseteq A$, there exists a unital equivariant homomorphism $\psi: C(X) \rightarrow F_\alpha(D, A)$. Denote by $\iota: A \rtimes_\alpha G \rightarrow C(X, A) \rtimes G$ the canonical embedding induced by the equivariant inclusion $A \hookrightarrow C(X, A)$, where $C(X, A)$ carries the diagonal G -action. Then ι is a positively existential embedding.*

Proof. Let $B_0 \subseteq C(X, A) \rtimes G$ and $A_0 \subseteq A \rtimes G$ be separable subalgebras satisfying $\iota(A_0) \subseteq B_0$. Use Remark 4.10 to find an α -invariant separable subalgebra $D \subseteq A$

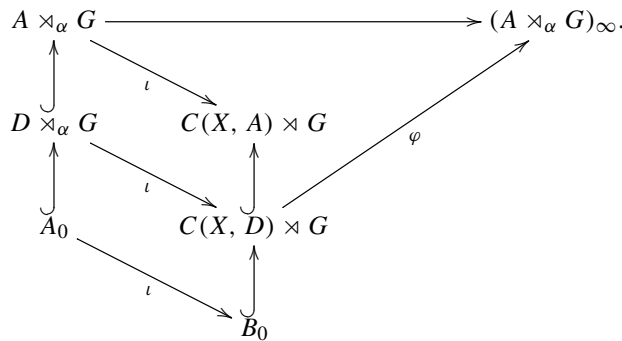
satisfying $B_0 \subseteq C(X, D) \rtimes G$ and $A_0 \subseteq D \rtimes G$. Let $\psi: C(X) \rightarrow F_\alpha(D, A)$ be a unital equivariant homomorphism as in the statement. Upon tensoring with id_D and using [19, Lemma 2.3], we obtain an equivariant homomorphism

$$\theta: C(X) \otimes D \rightarrow A_{\infty, \alpha},$$

which is the identity map on D . We obtain a unital homomorphism

$$\varphi: C(X, D) \rtimes G \rightarrow A_{\infty, \alpha} \rtimes_{\alpha_\infty} G \hookrightarrow (A \rtimes_\alpha G)_\infty$$

which makes the following diagram commute:



This finishes the proof. □

Definition 4.12. Let \mathcal{C} be a class of C^* -algebras. Consider the following permanence properties.

- (P1) *Passage to section algebras.* If Y is a finite-dimensional compact metrizable space and A is a continuous $C(Y)$ -algebra with $A_y \in \mathcal{C}$ for all $y \in Y$, then $A \in \mathcal{C}$.
- (P2) *Passage to positively existential subalgebras.* If A is a C^* -algebra such that there exist another C^* -algebra $B \in \mathcal{C}$, and a positively existential embedding $A \hookrightarrow B$, then $A \in \mathcal{C}$.

Recall [71] that a unital, infinite-dimensional, separable C^* -algebra \mathcal{D} is said to be *strongly self-absorbing* if there exists an isomorphism $\mathcal{D} \cong \mathcal{D} \otimes_{\min} \mathcal{D}$ that is approximately unitarily equivalent to the first factor embedding $d \mapsto d \otimes 1_{\mathcal{D}}$. A C^* -algebra A is said to be \mathcal{D} -stable if $A \cong A \otimes \mathcal{D}$.

We begin by recording the fact that a number of classes of C^* -algebras are closed under passing to section algebras with finite-dimensional bases (property (P1)). Most of these are well known: Theorem 4.13 just collects the relevant references, and provides a proof where one is needed. The only original results are (7), (8), and (9). Recall that a C^* -algebra A is said to be *stable* if $A \cong A \otimes \mathcal{K}$.

THEOREM 4.13. *Let Y be a compact metrizable space of finite covering dimension, and let A be a $C(Y)$ -algebra.*

- (1) *Let \mathcal{D} be a strongly self-absorbing C^* -algebra. If A_y is separable and \mathcal{D} -absorbing for all $y \in Y$, then the same is true for A .*
- (2) *If A_y is stable and separable for all $y \in Y$, then so is A .*

(3) We have

$$\dim_{\text{nuc}}(A) + 1 \leq (\dim(Y) + 1) \left(\sup_{y \in Y} \dim_{\text{nuc}}(A_y) + 1 \right).$$

(4) We have

$$\text{dr}(A) + 1 \leq (\dim(Y) + 1) \left(\sup_{y \in Y} \text{dr}(A_y) + 1 \right).$$

(5) We have

$$\text{sr}(A) \leq \dim(Y) + \sup_{y \in Y} \text{sr}(A_y).$$

(6) We have

$$\text{RR}(A) \leq \dim(Y) + \sup_{y \in Y} \text{RR}(A_y).$$

(7) Let $n \in \mathbb{N}$. If A_y is nuclear, satisfies the UCT, and $K_*(A_y)$ is uniquely n -divisible for all $y \in Y$, then the same is true for A .

(8) If A_y is nuclear, satisfies the UCT, and $K_*(A_y)$ is rational for all $y \in Y$, then the same is true for A .

(9) If A_y is nuclear, satisfies the UCT, and $K_*(A_y) = \{0\}$ for all $y \in Y$, then the same is true for A .

(10) If A_y is nuclear, separable and satisfies the UCT for all $y \in Y$, then the same is true for A .

Proof. (1) and (2) are proved in [33, Theorem 4.6].

For (4), this is [9, Lemma 3.1], while the proof of the said lemma is easily adapted to show (3). For (5), this is a consequence of [66, Theorem 1.3] and the fact that $\text{sr}(C(Y, A)) \leq \dim(Y) + \text{sr}(A)$ for every compact space Y and every C^* -algebra A , by [52, Theorem 1.13]. For (6), the same argument in the proof of [66, Theorem 1.3] applies to real rank, showing that

$$\text{RR}(A) \leq \sup_{y \in Y} \text{RR}(C(Y, A_y)).$$

(In the proof, one has to use the fact that if I is an ideal in a C^* -algebra B , then $\text{RR}(B/I) \leq \text{RR}(B)$ as a replacement of inequality (F2) from [66]. This is [14, Theorem 1.4].) Since, by [52, Corollary 1.12], $\text{RR}(C(Y, A_y)) \leq \dim(Y) + \text{RR}(A_y)$ for all $y \in Y$, the result follows.

We prove (9). By passing to separable sub- $C(Y)$ -algebras as in the proof of (2), we may assume that each A_y is separable. Then A is separable and nuclear. We will prove that $K_*(A) = \{0\}$ by showing that A is KK_Y -equivalent to $A \otimes \mathcal{O}_2$. We regard $A \otimes \mathcal{O}_2$ as a $C(Y)$ -algebra in the obvious way, so that the fiber over $y \in Y$ is $A_y \otimes \mathcal{O}_2$. Since A is exact, this is a *continuous* $C(Y)$ -algebra, by [42, Theorem B]. Let $\rho: A \rightarrow A \otimes \mathcal{O}_2$ denote the first tensor factor embedding. Then ρ is a $C(Y)$ -homomorphism whose fiber map $\rho_y: A_y \rightarrow A_y \otimes \mathcal{O}_2$ is also the first tensor factor embedding. Since A_y has trivial K -theory and satisfies the UCT, we deduce that ρ_y induces a KK -equivalence $A_y \sim_{KK} A_y \otimes \mathcal{O}_2$. Since Y is finite-dimensional and A is separable and nuclear, it follows from [12, Theorem 1.1] that $\xi = KK(\rho) \in KK_Y(A, A \otimes \mathcal{O}_2)$ is invertible. The result follows.

For (7), one uses an identical argument to that for (9), replacing \mathcal{O}_2 by M_{n^∞} and noting that unique n -divisibility of an abelian group is equivalent to absorbing $\mathbb{Z}[1/n]$ tensorially. Also, (8) follows by applying (7) for all $n \in \mathbb{N}$.

Finally, for (10), this is [12, Theorem 1.4]. □

The next result lists a number of classes that are closed under positive existential embeddings (property (P2)). Some of these observations were made, in the separable case, in [19], and later independently in [3], and are also implicit in various earlier papers. For the classes in (6), (7), and (8), the result is new even in the separable case.

PROPOSITION 4.14. *Let A and B be C^* -algebras, and let $\iota: A \hookrightarrow B$ be a positive existential embedding.*

- (1) *Let \mathcal{D} be a strongly self-absorbing C^* -algebra. If B is separable and \mathcal{D} -absorbing, then the same holds for A .*
- (2) *If B is separable and stable, then so is A .*
- (3) *We have*

$$\dim_{\text{nuc}}(A) \leq \dim_{\text{nuc}}(B).$$

- (4) *We have*

$$\text{dr}(A) \leq \text{dr}(B).$$

- (5) *We have*

$$\text{sr}(A) \leq \text{sr}(B).$$

- (6) *We have*

$$\text{RR}(A) \leq \text{RR}(B).$$

- (7) *The map $K_*(\iota): K_*(A) \rightarrow K_*(B)$ is injective. In particular, if either $K_0(B)$ or $K_1(B)$ has any of the following properties, then so does $K_0(A)$ or $K_1(A)$: being trivial; being free; being torsion-free.*
- (8) *Let $n \in \mathbb{N}$. If either $K_0(B)$ or $K_1(B)$ is uniquely n -divisible, then so is $K_0(A)$ or $K_1(A)$. In particular, if either $K_0(B)$ or $K_1(B)$ is rational, then so is $K_0(A)$ or $K_1(A)$.*

Proof. The proof for (1) is contained in the proof of [19, Theorem 4.3] (see also Remark 2.12 there), and we briefly sketch it. Let $\iota: A \rightarrow B$ be a positively existential embedding, and let $\varphi: B \rightarrow A_\infty$ be a left inverse. Find a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of contractive linear maps $\varphi_n: B \rightarrow A$ which are asymptotically $*$ -multiplicative, and such that $\lim_{n \rightarrow \infty} \varphi_n(\iota(a)) = a$ for all $a \in A$.

Let $\varepsilon > 0$, and let $F \subseteq A$ and $E \subseteq \mathcal{D}$ be finite subsets. Use [19, Theorem 4.2] to choose a completely positive map $\psi: \mathcal{D} \rightarrow B$ satisfying:

- (1) $\|\iota(a)\psi(d) - \psi(d)\iota(a)\| < \varepsilon$ for all $a \in F$ and for all $d \in E$;
- (2) $\|\psi(de)\iota(a) - \psi(d)\psi(e)\iota(a)\| < \varepsilon$ for every $d, e \in E$ and every $a \in F$;
- (3) $\|\psi(1)\iota(a) - \iota(a)\| < \varepsilon$ for all $a \in F$.

Consider the completely positive contractive map $\theta = \varphi \circ \psi: \mathcal{D} \rightarrow A_\infty$. Using the Choi–Effros lifting theorem, we can find a completely positive contractive lift $\pi: \mathcal{D} \rightarrow A$ satisfying:

- (1) $\|a\pi(d) - \pi(d)a\| < \varepsilon$ for all $a \in F$ and for all $d \in E$;
- (2) $\|\pi(de)a - \pi(d)\psi(e)a\| < \varepsilon$ for every $d, e \in E$ and every $a \in F$;
- (3) $\|\pi(1)a - a\| < \varepsilon$ for all $a \in F$.

Another application of [19, Theorem 4.2] implies that A is \mathcal{D} -stable, as desired.

We prove (2). By the equivalence between (a) and (c) in [37, Theorem 2.1], and the equivalence between (b) and (c) in [37, Proposition 2.2], a separable C^* -algebra C is stable if and only if, for every positive contraction $c \in C$ and every $\varepsilon > 0$, there exists a contraction $x \in C$ with $\|cxx^*\| < \varepsilon$ and $\|c - xx^*\| < \varepsilon$. Now let $a \in A$ be a positive contraction and let $\varepsilon > 0$. Since B is stable, there exists a contraction $x \in B$ with $\|\iota(a)xx^*\| < \varepsilon$ and $\|\iota(a) - xx^*\| < \varepsilon$. Find a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of contractive linear maps $\varphi_n : B \rightarrow A$ which are asymptotically $*$ -multiplicative, and such that $\lim_{n \rightarrow \infty} \varphi_n(\iota(a)) = a$ for all $a \in A$. Then there exists $n_0 \in \mathbb{N}$ such that $\|a\varphi_n(x)\varphi_n(x)^*\| < \varepsilon$ and $\|a - \varphi_n(x)\varphi_n(x)^*\| < \varepsilon$ for all $n \geq n_0$, so A is stable.

Parts (3) and (4) are almost identical, so we only prove (3). (The definition of nuclear dimension is given on p. 410 of the introduction.) Set $n = \dim_{\text{nuc}}(B)$. Let $F \subseteq A$ be a finite set consisting of contractions, and let $\varepsilon > 0$. Find finite-dimensional C^* -algebras C_0, \dots, C_n , a completely positive contractive map $\varphi : B \rightarrow C = \bigoplus_{j=0}^n C_j$, and completely positive contractive order-zero maps $\psi_j : C_j \rightarrow B$, for $j = 0, \dots, n$, satisfying $\|\sum_{j=0}^n \psi_j(\varphi(\iota(a))) - \iota(a)\| < \varepsilon$ for all $a \in F$. Set

$$\varepsilon_0 = \varepsilon - \max_{a \in F} \left\| \sum_{j=0}^n \psi_j(\varphi(\iota(a))) - \iota(a) \right\| > 0.$$

Using that the cones over the finite-dimensional C^* -algebras C_0, \dots, C_n are projective and the correspondence between order-zero maps from C_j and homomorphisms from its cone, find $\delta > 0$ such that, for all $j = 0, \dots, n$, whenever $\kappa : C_j \rightarrow A$ is a completely positive contractive map which is δ -order zero (meaning that $\|\kappa(c)\kappa(d)\| < \delta$ for all positive orthogonal contractions $c, d \in C_j$), then there exists a completely positive contractive order-zero map $\rho : C_j \rightarrow A$ satisfying $\|\rho - \kappa\| < \varepsilon_0/2(n + 1)$.

Let $A_0 \subseteq A$ be the separable subalgebra generated by F and let $B_0 \subseteq B$ be the separable subalgebra generated by F together with the images of ψ_0, \dots, ψ_n . Denote by $C_j^{(1)}$ the unit ball of C_j , for $j = 0, \dots, n$. Since $\iota : A \rightarrow B$ is positively existential, and arguing as in part (1) above, we can find a completely positive contractive map $\theta : B_0 \rightarrow A$ which is δ -multiplicative on $\bigcup_{j=0}^n \psi_j(C_j^{(1)})$ and satisfies $\|\theta(\iota(a)) - a\| < \varepsilon_0/2$. The maps $\theta \circ \psi_0, \dots, \theta \circ \psi_n$ are thus δ -order zero in the sense above, and hence there exist completely positive contractive order-zero maps $\rho_j : C_j \rightarrow A$, for $j = 0, \dots, n$, satisfying $\|\rho_j - \theta \circ \psi_j\| < \varepsilon_0/2(n + 1)$. For $a \in F$, we have

$$\begin{aligned} \left\| \sum_{j=0}^n \rho_j(\varphi(\iota(a))) - a \right\| &\leq \varepsilon_0/2 + \left\| \sum_{j=0}^n \theta(\psi_j(\varphi(\iota(a)))) - a \right\| \\ &\leq \varepsilon_0/2 + \varepsilon_0/2 + \left\| \sum_{j=0}^n \theta(\psi_j(\varphi(\iota(a)))) - \theta(a) \right\| \\ &\leq \varepsilon_0 + \left\| \sum_{j=0}^n \psi_j(\varphi(\iota(a))) - a \right\| < \varepsilon, \end{aligned}$$

thus showing that $\dim_{\text{nuc}}(A) \leq n$, as desired.

The proofs of (5) and (6) are almost identical, and for the sake of brevity we only prove the result for the real rank. Let $n \in \mathbb{N}$ and suppose that $\text{RR}(B) \leq n$. By passing to unitizations, we may assume that A, B , and ι are unital. Given self-adjoint elements a_0, \dots, a_n in A and $\varepsilon > 0$, find self-adjoint elements $b_0, \dots, b_n \in B$ such that $\sum_{j=1}^n b_j^2$ is invertible in B and $\|\sum_{j=1}^n (\iota(a_j) - b_j)^2\| < \varepsilon$. Denote by $A_0 \subseteq A$ the (separable) subalgebra generated by $\{a_0, \dots, a_n\}$, and denote by $B_0 \subseteq B$ the (separable) subalgebra generated by $\iota(A_0) \cup \{b_0, \dots, b_n\}$. Let $\varphi: B_0 \rightarrow A_\infty$ be a homomorphism as in Definition 4.8. For $j = 0, \dots, n$, set $x_j = \varphi(b_j)$. Then x_j is self-adjoint, satisfies $\|\iota(a_j) - x_j\| < \varepsilon$, and $\sum_{j=1}^n x_j^2$ is invertible in A_∞ . By choosing representing sequences for the x_j and picking suitable elements in these sequences, we obtain self-adjoint elements y_0, \dots, y_n in A with $\sum_{j=0}^n y_j^2$ invertible and satisfying $\|\sum_{j=1}^n (a_j - y_j)^2\| < \varepsilon$. We conclude that $\text{RR}(A) \leq n$, and hence $\text{RR}(A) \leq \text{RR}(B)$.

We prove (7). Observe that $\iota \otimes \text{id}_{M_n}: M_n(A) \rightarrow M_n(B)$ is also positive existential, and so is its unitization. Thus, in order to show that $K_0(\iota)$ is injective, it is enough to show that if p and q are projections in A such that $\iota(p)$ and $\iota(q)$ are Murray–von Neumann equivalent in B , then p and q are Murray–von Neumann equivalent in A .

Let $p, q \in A$ satisfy $\iota(p) \sim_{\text{M-vN}} \iota(q)$ in B , and choose a partial isometry $v \in B$ satisfying $v^*v = \iota(p)$ and $vv^* = \iota(q)$. Set $A_0 = C^*(p, q)$ and $B_0 = C^*(\iota(p), \iota(q), v)$. Find a homomorphism $\varphi: B_0 \rightarrow A_\infty$ as in Definition 4.8, and choose linear maps $\varphi_n: B_0 \rightarrow A$ representing φ . These maps are approximately multiplicative and approximately $*$ -preserving, and are approximately the identity on $\iota(A)$. Thus, for n large enough, we have:

- $\|\varphi_n(v)^* \varphi_n(v) - p\| < \frac{1}{2}$;
- $\|\varphi_n(v) \varphi_n(v)^* - q\| < \frac{1}{2}$;
- $\|q \varphi_n(v) p - \varphi_n(v)\| < \frac{1}{2}$.

For $s = q \varphi_n(v) p$, we have $\|s^*s - p\| < 1$ and $\|s s^* - q\| < 1$. By [48, Lemma 2.5.3], we conclude that p and q are Murray–von Neumann equivalent in A . This shows that $K_0(\iota)$ is injective. The argument for $K_1(\iota)$ is analogous, or can be deduced from the statement for K_0 by taking suspensions.

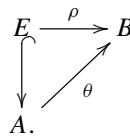
We show (8). Without loss of generality, we assume that $K_0(B)$ is uniquely n -divisible, and will show that so is $K_0(A)$. For this, it suffices to show that $K_0(A)$ is n -divisible, since uniqueness follows from injectivity of $K_0(\iota)$ (by part (7)). By taking unitizations, we may assume that A, B , and ι are unital. As before, upon taking matrix amplifications, it suffices to show that if $p \in A$ is a projection, and there exists a projection $q \in B$ such that $\text{diag}(q, \dots, q)$ is Murray–von Neumann equivalent to $\text{diag}(\iota(p), 0, \dots, 0)$ in $M_n(B)$, then there exists a projection $\tilde{q} \in M_n(A)$ which is Murray–von Neumann equivalent to $\text{diag}(p, 0, \dots, 0)$ in $M_n(A)$ and such that $\iota(\tilde{q})$ is Murray–von Neumann equivalent to $\text{diag}(q, \dots, q)$ in $M_n(B)$. We fix a partial isometry $s \in M_n(B)$ implementing the equivalence $\text{diag}(q, \dots, q) \sim_{\text{M-vN}} \text{diag}(\iota(p), 0, \dots, 0)$.

Let A_0 be the subalgebra of $M_n(A)$ generated by $\text{diag}(p, 0, \dots, 0)$ and the unit, and let B_0 be the subalgebra of $M_n(B)$ generated by $\text{diag}(q, \dots, q)$ and s . Observe that $\iota \otimes \text{id}_{M_n}: M_n(A) \rightarrow M_n(B)$ is also positive existential, and let $\varphi: B_0 \rightarrow (M_n(A))_\infty$ be a homomorphism as in Definition 4.8. Then $\varphi(s)$ implements a Murray–von Neumann

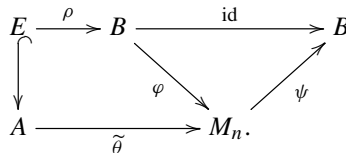
equivalence between $\text{diag}(p, 0, \dots, 0)$ and $\varphi(q, \dots, q)$. If $(q_k)_{k \in \mathbb{N}} \in \ell^\infty(M_n(A))$ is a lift of $\varphi(q, \dots, q)$ consisting of projections, and $(s_k)_{k \in \mathbb{N}} \in \ell^\infty(M_n(A))$ is a lift of $\varphi(s)$ consisting of partial isometries, an argument similar to the one used in part (7) shows that, for k large enough, $\tilde{q} = q_k$ is Murray–von Neumann equivalent, in $M_n(A)$, to $\text{diag}(p, 0, \dots, 0)$, as witnessed by s_k . \square

Our next goal is to show that nuclear C^* -algebras satisfying the UCT are closed under existential embeddings and thus satisfy (P2). This was observed in the simple case, [19, Theorem 3.13], and was then generalized in [3, Theorem 2.12]. We give an alternative proof below, using the following well-known consequence of the Arveson extension theorem. Since we were not able to find this formulation in the literature, we sketch a proof here.

THEOREM 4.15. (Arveson) *Let A and B be C^* -algebras, with A unital and B nuclear; let $E \subseteq A$ be an operator subsystem, and let $\rho: E \rightarrow B$ be a completely positive contractive map. Let $F \subseteq E$ be a finite set and let $\varepsilon > 0$. Then there exists a completely positive contractive map $\theta: A \rightarrow B$ making the following diagram commute on F up to ε :*



Proof. Using nuclearity of B , find a sufficiently large $n \in \mathbb{N}$, and completely positive contractive maps $\varphi: B \rightarrow M_n$ and $\psi: M_n \rightarrow B$ such that $\|\psi(\varphi(b)) - b\| < \varepsilon$ for all $b \in \rho(F)$. By Arveson’s extension theorem (see, for example, [8, Theorem 1.6.1]), there exists a completely positive contractive map $\tilde{\theta}: A \rightarrow M_n$ making the following diagram commute:



The proof is completed by taking $\theta = \psi \circ \tilde{\theta}$. \square

In the following, we denote by $\mathcal{O}_\infty^{\text{st}}$ the unique unital Kirchberg algebra satisfying the UCT with scaled K -theory given by $(\mathbb{Z}, 0, \{0\})$.

THEOREM 4.16. *The class of separable, nuclear C^* -algebras satisfying the UCT is closed under positively existential embeddings (property (P2)). That is, if B is separable, nuclear, and satisfies the UCT, and $A \hookrightarrow B$ is a positively existential embedding, then A is separable, nuclear, and satisfies the UCT.*

Proof. Let A be a C^* -algebra, let B be a nuclear C^* -algebra satisfying the UCT, and let $\iota: A \rightarrow B$ be a positively existential embedding. It is clear that A is separable. Nuclearity of A can be deduced as follows. Take $A_0 = A$ and $B_0 = B$ in Definition 4.8 and apply Choi–Effros to $\varphi: B \rightarrow A_\infty$ to obtain a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of completely

positive contractive linear maps $\varphi_n : B \rightarrow A$ satisfying $\lim_{n \rightarrow \infty} \varphi_n(a) = a$ for all $a \in A$. Let $\varepsilon > 0$ and let $F \subseteq A$ be a finite subset. Since B is nuclear, there exist a finite-dimensional C^* -algebra E and completely positive contractive maps $\xi : B \rightarrow E$ and $\eta : E \rightarrow B$ satisfying $\|(\eta \circ \xi)(\iota(a)) - \iota(a)\| < \varepsilon$ for all $a \in F$. Thus, for large enough n , we have $\|(\varphi_n \circ \eta \circ \xi \circ \iota)(a) - a\| < \varepsilon$ for all $a \in F$, so A is nuclear as well.

For the statement about the UCT, we make some standard reductions first, so as to fit within the framework of [12, Theorem 5.2]. We identify A with its image under ι . By taking unitizations, we may assume that A is a unital subalgebra of B . Since, for a C^* -algebra C , the tensor product $C \otimes \mathcal{O}_\infty^{\text{st}}$ is KK -equivalent to C , it follows that C satisfies the UCT if and only if $C \otimes \mathcal{O}_\infty^{\text{st}}$ does. Thus, by tensoring A and B with $\mathcal{O}_\infty^{\text{st}}$, we may assume that A and B have unital embeddings of \mathcal{O}_2 . By the proof of [12, Theorem 2.5] applied to the one-point space X , there exists a homomorphism $\varphi : B \rightarrow B$ satisfying $\varphi(A) \subseteq A$, such that the stationary inductive limits

$$B^\sharp = \varinjlim(B, \varphi) \quad \text{and} \quad A^\sharp = \varinjlim(A, \varphi|_A)$$

are unital Kirchberg algebras which are KK -equivalent to B and A , respectively. Thus, B^\sharp satisfies the UCT because B does, and it is enough to show that A^\sharp satisfies the UCT.

We claim that the induced inclusion $A^\sharp \hookrightarrow B^\sharp$ is a positively existential embedding. To show this, we will construct a homomorphism $\tilde{\psi} : B^\sharp \rightarrow (A^\sharp)_\infty$ which restricts to the canonical embedding of A^\sharp . Since A^\sharp and B^\sharp are separable, we may choose increasing families $(F_n)_{n \in \mathbb{N}}$ and $(G_n)_{n \in \mathbb{N}}$ of finite subsets of A^\sharp and B^\sharp with dense union. For $n \in \mathbb{N}$, we write A_n for A when we regard it as the n th stage of the stationary inductive limit defining A^\sharp , and similarly with B . Without loss of generality, we may assume that $F_n \subseteq A_n$ and $G_n \subseteq B_n$ for all $n \in \mathbb{N}$. It is enough to construct a sequence $\theta_n : B^\sharp \rightarrow A^\sharp$ of unital completely positive maps satisfying $\|\theta_n(b_1 b_2) - \theta_n(b_1)\theta_n(b_2)\| < 1/n$ for all $b_1, b_2 \in G_n$ and $\|\theta_n(a) - a\| < 1/n$ for all $a \in F_n$.

Fix $n \in \mathbb{N}$. Using that $A \hookrightarrow B$ is a positively existential embedding, find a homomorphism $\psi : B \rightarrow A_\infty$ as in Definition 4.12. Using nuclearity of A and the Choi–Effros lifting theorem, find a unital completely positive map $\rho : B \rightarrow A$ which satisfies:

- $\|\rho(b_1 b_2) - \rho(b_1)\rho(b_2)\| < 1/n$ for all $b_1, b_2 \in G_n$; and
- $\|\rho(a) - a\| < 1/n$ for all $a \in F_n$.

Regard ρ as a unital completely positive map $B_n \rightarrow A_n \hookrightarrow A^\sharp$. Since A^\sharp is nuclear, Arveson’s Extension Theorem (in the form of Theorem 4.15) implies that there is a unital completely positive map $\theta_n : B^\sharp \rightarrow A^\sharp$ which agrees with ρ on G_n up to $1/n$. The induced map

$$\theta = (\theta_n)_{n \in \mathbb{N}} : B^\sharp \rightarrow (A^\sharp)_\infty$$

is clearly a homomorphism which restricts to the standard inclusion of A^\sharp into its sequence algebra. This proves the claim.

The rest of the proof is identical to that of [19, Theorem 3.13]. Denote by \mathcal{C} the class of weakly semiprojective unital Kirchberg algebras satisfying the UCT. Use [19, Lemma 3.12] to write B^\sharp as a direct limit of algebras in \mathcal{C} . Then A^\sharp is a generalized local \mathcal{C} -algebra [19, Definition 3.1]. By Proposition 3.9 (which is a straightforward consequence of [68, Theorem 3.9]), A^\sharp is a unital approximate \mathcal{C} -algebra, that is, a direct limit of C^* -algebras in \mathcal{C} . It follows that A^\sharp satisfies the UCT, and the proof is finished. \square

The following theorem combines the results obtained in this section, proving Theorem D in the introduction.

THEOREM 4.17. *Let G be a compact group of finite covering dimension, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. If A belongs to any of the following classes, then so do $A \rtimes_{\alpha} G$ and A^{α} .*

- (1) *For \mathcal{D} being a strongly self-absorbing C^* -algebra, if A is separable and \mathcal{D} -absorbing, then so are $A \rtimes_{\alpha} G$ and A^{α} .*
- (2) *If A is separable and stable, then so are $A \rtimes_{\alpha} G$ and A^{α} .*
- (3) *We have*

$$\dim_{\text{nuc}}(A^{\alpha}) = \dim_{\text{nuc}}(A \rtimes_{\alpha} G) \leq (\dim_{\text{Rok}}^c(\alpha) + 1)(\dim_{\text{nuc}}(A) + 1) - 1.$$

- (4) *We have*

$$\text{dr}(A^{\alpha}) = \text{dr}(A \rtimes_{\alpha} G) \leq (\dim_{\text{Rok}}^c(\alpha) + 1)(\text{dr}(A) + 1) - 1.$$

- (5) *We have*

$$\text{sr}(A \rtimes_{\alpha} G) \leq \text{sr}(A^{\alpha}) \leq (\dim_{\text{Rok}}^c(\alpha) + 1)(\dim(G) + 1) + \text{sr}(A) - 1.$$

- (6) *We have*

$$\text{RR}(A \rtimes_{\alpha} G) \leq \text{RR}(A^{\alpha}) \leq (\dim_{\text{Rok}}^c(\alpha) + 1)(\dim(G) + 1) + \text{RR}(A) - 1.$$

- (7) *Let $n \in \mathbb{N}$. If A is nuclear, satisfies the UCT, and has uniquely n -divisible K -theory, then the same is true for $A \rtimes_{\alpha} G$ and A^{α} .*
- (8) *If A is nuclear, satisfies the UCT, and has rational K -theory, then the same is true for $A \rtimes_{\alpha} G$ and A^{α} .*
- (9) *If A is nuclear, satisfies the UCT, and has trivial K -theory, then the same is true for $A \rtimes_{\alpha} G$ and A^{α} .*
- (10) *If A is separable, nuclear, and satisfies the UCT, then the same is true for $A \rtimes_{\alpha} G$ and A^{α} .*

Proof. We argue for the fixed point algebra A^{α} first. By Proposition 4.11, it is enough to check that the classes in the statement satisfy conditions (P1) and (P2) of Definition 4.12. For the former, this is the content of Theorem 4.13, and for the latter, it is the content of Proposition 4.14 and Theorem 4.16.

Recall that the crossed product $A \rtimes_{\alpha} G$ is Morita equivalent to $A^{\alpha} \otimes \mathcal{K}(L^2(G))$. Now, the properties in (1)–(2) and (7)–(10) are invariant under stable isomorphism, and the nuclear dimension and decomposition rank of a C^* -algebra and its stabilization are identical, so the result for the crossed product follows. It remains to observe that, for any C^* -algebra B , one has $\text{sr}(B \otimes \mathcal{K}) \leq \text{sr}(B)$ and $\text{RR}(B \otimes \mathcal{K}) \leq \text{RR}(B)$, so the proof is finished. □

Remark 4.18. We make some comments about preservation of the classes in Theorem 4.17 by actions with finite Rokhlin dimension (not necessarily with commuting towers).

- (a) For a strongly self-absorbing C^* -algebra \mathcal{D} , denote by $\mathcal{A}_{\mathcal{D}}$ the class of \mathcal{D} -absorbing separable C^* -algebras. Then $\mathcal{A}_{\mathcal{O}_2}$ and $\mathcal{A}_{M_p^\infty}$, for a prime $p \geq 2$, are not preserved by actions with finite Rokhlin dimension. Indeed, [21, Example 4.8] shows that \mathcal{O}_2 -absorption is not preserved (this action was originally constructed in [38]). It also shows that absorption of UHF-algebras other than the canonical anticommutation relations (CAR) algebra M_{2^∞} is not preserved. In order to rule out preservation of absorption of M_{2^∞} , one may adapt Izumi’s construction to produce an approximately representable action of \mathbb{Z}_3 on \mathcal{O}_2 whose equivariant K -theory is not uniquely 2-divisible. (This is done, for example, in [25].) Whether the classes $\mathcal{A}_{\mathcal{O}_\infty}$ or $\mathcal{A}_{\mathcal{Z}}$ are preserved is still unknown.
- (b) The examples mentioned above also show that the classes in (7), (8), and (9) are not preserved.
- (c) Whether the class of nuclear algebras satisfying the UCT is preserved by actions with finite Rokhlin dimension is in fact equivalent to the UCT problem for separable, nuclear C^* -algebras. Indeed, Barlak has observed that the UCT problem for separable, nuclear C^* -algebras can be reduced to the question whether the crossed products of certain pointwise outer actions of finite groups on \mathcal{O}_2 satisfy the UCT; see [4, Theorem 4.17] for the precise statement. By [21, Theorem 4.19], any such action has Rokhlin dimension at most one, which proves our claim.

In reference to part (1) of Theorem 4.17, we show next that stability (that is, \mathcal{K} -absorption) of $A \rtimes_\alpha G$ is automatic whenever G is infinite, and regardless of A . A similar result for \mathbb{R} -actions has been obtained in [34, Theorem D].

COROLLARY 4.19. *Let G be a second-countable compact group, let A be a separable C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. If G is infinite, then $A \rtimes_\alpha G$ is stable.*

Proof. When G is infinite, $L^2(G)$ is the infinite-dimensional separable Hilbert space. Let X be the finite-dimensional compact free G -space from Theorem 4.7. By Corollary 4.6, the $C(X/G)$ -algebra $C(X, A) \rtimes_\gamma G$ has stable fibers. Since X/G is finite-dimensional, [33, Proposition 3.4] implies that $C(X, A) \rtimes_\gamma G$ is stable. Since the canonical inclusion

$$A \rtimes_\alpha G \hookrightarrow C(X, A) \rtimes_\gamma G$$

is positively existential by Proposition 4.11, it follows from part (2) in Proposition 4.14 that $A \rtimes_\alpha G$ is stable. □

An immediate consequence is the following result.

COROLLARY 4.20. *Let G be an infinite compact group with finite dimension, let A be a C^* -algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. Then $\text{sr}(A \rtimes_\alpha G) \leq 2$ and $\text{RR}(A \rtimes_\alpha G) \leq 1$, regardless of A .*

Adopt the notation of Proposition 4.11. If one knew that $C(X, A) \rtimes G$ is Morita equivalent to $(C(X) \rtimes G) \otimes A$, then it would follow from Situation 2 in [60] that $C(X, A) \rtimes G$ is itself Morita equivalent to $C(X/G) \otimes A$, since the action is free and the group is compact. This is, however, not the case in general, as the next example shows.

Example 4.21. Let the non-trivial element of $G = \mathbb{Z}_2 = \{-1, 1\}$ act on $X = S^1$ via multiplication by -1 , and let it act on $A = \mathcal{O}_\infty^{\text{st}}$ by an order-two automorphism whose induced map on $K_0(\mathcal{O}_\infty^{\text{st}}) \cong \mathbb{Z}$ is multiplication by -1 . Denote by $\gamma: \mathbb{Z}_2 \rightarrow \text{Aut}(C(S^1))$ and $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(\mathcal{O}_\infty^{\text{st}})$ the corresponding actions. Set $B = C(S^1, \mathcal{O}_\infty^{\text{st}})$ and denote by $\beta: \mathbb{Z}_2 \rightarrow \text{Aut}(B)$ the diagonal action $\beta = \gamma \otimes \alpha$, where, as usual, we identify $C(S^1, \mathcal{O}_\infty^{\text{st}})$ with $C(S^1) \otimes \mathcal{O}_\infty^{\text{st}}$.

Let $z \in C(S^1)$ be the canonical generating unitary, and set $v = z \otimes 1 \in C(S^1) \otimes \mathcal{O}_\infty^{\text{st}} \cong B$. Regard v as a unitary in $B \rtimes_\beta \mathbb{Z}_2$ under the canonical unital inclusion $B \rightarrow B \rtimes_\beta \mathbb{Z}_2$.

Claim. $\widehat{\beta}_{-1} \in \text{Aut}(B \rtimes_\beta \mathbb{Z}_2)$ is inner, and it is implemented by v . (In particular, $\widehat{\beta}_{-1}$ acts trivially on K -theory.) Note first that $\beta_{-1}(v) = -v$. Denote by $u \in B \rtimes_\beta \mathbb{Z}_2$ the canonical unitary implementing β . To check that $\widehat{\beta}_{-1} = \text{Ad}(v)$, it suffices to show that these automorphisms agree on the generating set $F = \{u\} \cup B$. Since v commutes with the elements of B (and $\widehat{\beta}_{-1}$ fixes these elements), it is enough to show that $\widehat{\beta}_{-1}(u) = v^*uv$. Since $\widehat{\beta}_{-1}(u) = -u$, this follows from the fact that $\beta_{-1}(v) = uvu^* = -v$, and the claim follows.

Observe that β_{-1} induces the automorphism of multiplication by -1 on both $K_0(B)$ and $K_1(B)$ (essentially because this is the case for α_{-1}). We compute the K -theory of $B \rtimes_\beta \mathbb{Z}_2$ using its Pimsner–Voiculescu six-term exact sequence. The K -groups of $B \rtimes_\beta \mathbb{Z}$ can be obtained from the following sequence:

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-K_0(\beta_{-1})} & K_0(B) & \longrightarrow & K_0(B \rtimes_\beta \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(B \rtimes_\beta \mathbb{Z}) & \longleftarrow & K_1(B) & \xleftarrow{1-K_1(\beta_{-1})} & K_1(B). \end{array}$$

Since $1 - K_j(\beta_{-1})$ is multiplication by 2 and $K_j(B) \cong \mathbb{Z}$ for $j = 0, 1$, it follows that the vertical maps are zero and that $K_0(B \rtimes_\beta \mathbb{Z}) \cong K_1(B \rtimes_\beta \mathbb{Z}) \cong \mathbb{Z}_2$. To compute the K -groups of $B \rtimes_\beta \mathbb{Z}_2$, we use the following sequence:

$$\begin{array}{ccccc} K_0(B \rtimes_\beta \mathbb{Z}_2) & \xrightarrow{1-K_0(\widehat{\beta})} & K_0(B \rtimes_\beta \mathbb{Z}_2) & \longrightarrow & K_1(B \rtimes_\beta \mathbb{Z}_2) \\ \uparrow & & & & \downarrow \\ K_0(B \rtimes_\beta \mathbb{Z}) & \longleftarrow & K_1(B \rtimes_\beta \mathbb{Z}_2) & \xleftarrow{1-K_0(\widehat{\beta})} & K_1(B \rtimes_\beta \mathbb{Z}_2), \end{array}$$

where the vertical maps are the canonical ones induced by the quotient map $B \rtimes_\beta \mathbb{Z} \rightarrow B \rtimes_\beta \mathbb{Z}_2$. The map $K_j(B \rtimes_\beta \mathbb{Z}_2) \rightarrow K_j(B \rtimes_\beta \mathbb{Z}_2)$, for $j = 0, 1$, is zero because $\widehat{\beta}$ acts by inner automorphisms. Since the K -groups of $B \rtimes_\beta \mathbb{Z}$ are both \mathbb{Z}_2 , it follows that either

$$K_0(B \rtimes_\beta \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \text{and} \quad K_1(B \rtimes_\beta \mathbb{Z}_2) \cong \{0\}$$

or

$$K_0(B \rtimes_\beta \mathbb{Z}_2) \cong \{0\} \quad \text{and} \quad K_1(B \rtimes_\beta \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

In either case, it follows that $C(S^1, \mathcal{O}_\infty^{\text{st}}) \rtimes_\beta \mathbb{Z}_2$ and $C(S^1/\mathbb{Z}_2) \otimes \mathcal{O}_\infty^{\text{st}}$ are not Morita equivalent, since $K_0(C(S^1/\mathbb{Z}_2) \otimes \mathcal{O}_\infty^{\text{st}}) \cong K_1(C(S^1/\mathbb{Z}_2) \otimes \mathcal{O}_\infty^{\text{st}}) \cong \mathbb{Z}$ by the Künneth formula.

Recall that the Toms–Winter conjecture predicts that finiteness of the nuclear dimension, \mathcal{Z} -absorption and strict comparison are equivalent for all simple, separable, infinite-dimensional, nuclear unital C^* -algebras. We make some connections between Rokhlin dimension and preservation of the properties in the Toms–Winter conjecture. Theorem 4.17 shows that \mathcal{Z} -absorption, finiteness of the nuclear dimension and finiteness of the decomposition rank are preserved under formation of crossed products (and fixed point algebras) by actions with finite Rokhlin dimension with commuting towers. A similar result for strict comparison is false in general, although it holds with further assumptions, as we show in Theorem 4.24. We need an easy lemma first.

LEMMA 4.22. *Let A and B be C^* -algebras, and let $\iota: A \rightarrow B$ be a positively existential embedding. Then $\text{Cu}(\iota): \text{Cu}(A) \rightarrow \text{Cu}(B)$ is an order embedding. In other words, if $s, t \in \text{Cu}(A)$ satisfy $\text{Cu}(\iota)(s) \leq \text{Cu}(\iota)(t)$ in $\text{Cu}(B)$, then $s \leq t$ in $\text{Cu}(A)$.*

Proof. Since $\iota \otimes \text{id}_{\mathcal{K}}: A \otimes \mathcal{K} \rightarrow B \otimes \mathcal{K}$ is an existential embedding, we may assume that A and B are stable. Let $s, t \in \text{Cu}(A)$ satisfy $\text{Cu}(\iota)(s) \leq \text{Cu}(\iota)(t)$ in $\text{Cu}(B)$, and choose positive elements $a, b \in A$ with $[a] = s$ and $[b] = t$. Since $\iota(a)$ is Cuntz subequivalent to $\iota(b)$, there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in B satisfying $\lim_{n \rightarrow \infty} c_n b c_n^* = a$. Let A_0 and B_0 be, respectively, the separable subalgebras of A and B generated by $\{a, b\}$ and $\{\iota(a), \iota(b), c_n: n \in \mathbb{N}\}$. Let $\varphi: B_0 \rightarrow A_\infty$ be a homomorphism as in Definition 4.8. By choosing suitable elements in representative sequences of the $\varphi(c_n)$, we can find a sequence $(d_n)_{n \in \mathbb{N}}$ in A satisfying $\lim_{n \rightarrow \infty} d_n b d_n^* = a$. Hence $a \preceq b$ in A and thus $s \leq t$. \square

We need a convenient definition from [1].

Definition 4.23. We say that a C^* -algebra A has no K_1 -obstructions if it has stable rank one and $K_1(I) = 0$ for all ideals I in A .

The following is our result on preservation of strict comparison.

THEOREM 4.24. *Let G be a compact group, let A be a C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) \leq 1$. Suppose further that A has stable rank one and that $K_1(I) = 0$ for all ideals I in A . Then the canonical inclusions*

$$A^\alpha \hookrightarrow A \quad \text{and} \quad A \rtimes_\alpha G \hookrightarrow A \otimes \mathcal{K}(L^2(G))$$

induce order embeddings at the level of the Cuntz semigroup. In particular, strict comparison passes from A to A^α and to $A \rtimes_\alpha G$.

Proof. By tensoring A with \mathcal{K} and α with $\text{id}_{\mathcal{K}}$, we may assume that A, A^α , and $A \rtimes_\alpha G$ are stable. We prove the statement for the second inclusion, since the proof for the first one is identical. Denote the canonical map $A \rtimes_\alpha G \hookrightarrow A \otimes \mathcal{K}(L^2(G))$ by θ . Let a and b be positive elements in $A \rtimes_\alpha G$, and assume that $\theta(a) \preceq \theta(b)$ in $A \otimes \mathcal{K}(L^2(G))$. We want to show that $a \preceq b$ in $A \rtimes_\alpha G$.

Let X be the compact Hausdorff free G -space from Theorem 4.7, and observe that $\dim(X/G) \leq 1$. Denote by

$$\iota: A \rtimes_\alpha G \rightarrow C(X, A) \rtimes_\gamma G$$

the inclusion induced by the equivariant factor embedding $A \rightarrow C(X, A)$. For $z \in X/G$, denote by

$$\pi_z : C(X, A) \rtimes_\gamma G \rightarrow A \otimes \mathcal{K}(L^2(G))$$

the quotient map onto the fiber over z . Then $\pi_z \circ \iota = \theta$ by Corollary 4.6. It follows that $\pi_z(\iota(a)) \lesssim \pi_z(\iota(b))$ for all $z \in X/G$. By [1, Theorem 2.6], it follows that $\iota(a) \lesssim \iota(b)$ in $C(X, A) \rtimes_\gamma G$. Since ι is an existential embedding by Proposition 4.11, it follows from Lemma 4.22 that $a \lesssim b$ in $A \rtimes_\alpha G$, as desired.

That strict comparison passes from A to A^α and $A \rtimes_\alpha G$ is an immediate consequence of the canonical inclusions being order embeddings. □

Obtaining more results may depend on computing X in some concrete cases. We examine the smallest non-trivial case, namely, that of actions of \mathbb{Z}_2 with Rokhlin dimension at most one. In the following lemma, we denote by $C^*(\mathcal{G}, \mathcal{R})$ the universal C^* -algebra generated by a set of generators \mathcal{G} , subject to a set of relations \mathcal{R} .

LEMMA 4.25. *Let $d \in \mathbb{N}$. Consider the set of generators*

$$\mathcal{G} = \{1, f_0^{(0)}, f_1^{(0)}, \dots, f_0^{(d)}, f_1^{(d)}\}$$

and the set of relations

$$\mathcal{R} = \left\{ \begin{array}{l} 0 \leq f_g^{(j)} \leq 1, \quad [f_g^{(j)}, f_k^{(\ell)}] = 0, \\ f_g^{(j)} f_h^{(j)} = 0, \quad \sum_{g \in \mathbb{Z}_2} \sum_{j=0}^d f_g^{(j)} = 1 \end{array} : g, h, k \in \mathbb{Z}_2, g \neq h, j, \ell = 0, \dots, d \right\}.$$

Define an action $\alpha : \mathbb{Z}_2 \rightarrow C^(\mathcal{G}, \mathcal{R})$ by $\alpha_g(f_h^{(j)}) = f_{gh}^{(j)}$ for $g, h \in \mathbb{Z}_2$ and $j = 0, \dots, d$. Then the \mathbb{Z}_2 - C^* -algebra $(C^*(\mathcal{G}, \mathcal{R}), \alpha)$ is canonically equivariantly isomorphic to $C(S^d)$ with the antipodal action.*

Proof. We give a sketch of the proof. Each $f_g^{(j)}$ determines a 0-simplex. Moreover, there is a 1-simplex between $f_g^{(j)}$ and $f_h^{(k)}$ whenever $f_g^{(j)} f_h^{(k)} \neq 0$. More generally, for $n \in \mathbb{N}$, there is an n -simplex connecting $f_{g_0}^{(j_0)}, \dots, f_{g_n}^{(j_n)}$ whenever $f_{g_0}^{(j_0)} \dots f_{g_n}^{(j_n)} \neq 0$. In particular, X is a d -dimensional simplicial complex. This description also allows one to see that X is homeomorphic to S^d , and that α corresponds to the antipodal action. We omit the details. □

Remark 4.26. For cyclic groups of higher order, the space X from Lemma 4.25 is a d -dimensional simplicial complex, but its explicit structure is harder to describe. In particular, it may fail to be a manifold.

We use Lemma 4.25 to provide an explicit computation of the free G -space X in the conclusion of Theorem 4.7.

COROLLARY 4.27. *Let A be a unital C^* -algebra and let $\alpha : \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ be an action. Then $\dim_{\text{Rok}}^c(\alpha) \leq 1$ if and only if there exists a unital equivariant homomorphism*

$$\varphi : (C(S^1), \mathbb{Z}_2) \rightarrow (A_\infty \cap A', \alpha_\infty).$$

Moreover, if $\dim_{\text{Rok}}^c(\alpha) = 1$, then every such map φ is automatically injective.

Proof. Since $\mathbb{L}t: \mathbb{Z}_2 \rightarrow \text{Aut}(C(S^1))$ has Rokhlin dimension one, the ‘if’ implication is clear. The converse is contained in [32, Lemma 1.9], once it is combined with Lemma 4.25; see also the comments before Theorem 3.4.

For the second statement, assume that there is a unital equivariant homomorphism $\varphi: C(S^1) \rightarrow A_\infty \cap A'$ which is not injective. We denote also by $\alpha \in \text{Aut}(A)$ the order-two automorphism generating the given \mathbb{Z}_2 -action. We also write $\gamma \in \text{Homeo}(X)$ for the antipodal homeomorphism, which equals $\mathbb{L}t_{-1}$. Let $Y \subseteq S^1$ be a closed subset satisfying $\gamma(Y) \cap Y = \emptyset$, and such that, with $X = Y \cup \gamma(Y)$, the map φ induces an injective unital equivariant homomorphism $\bar{\varphi}: C(X) \rightarrow A_\infty \cap A'$. Then $\varphi(\chi_Y)$ and $\varphi(\chi_{\gamma(Y)})$ are orthogonal projections in $A_\infty \cap A'$ which witness the fact that α has the Rokhlin property, that is, $\dim_{\text{Rok}}^c(\alpha) = 0$. □

Corollary 4.27 can be used to give lower bounds other than unity for certain actions of cyclic groups. In the following example, the said corollary is used to show that there exists an action α of \mathbb{Z}_2 on a simple AF-algebra with unique trace such that $\dim_{\text{Rok}}^c(\alpha)$ is exactly two. To the best of our knowledge, this is the first example of an action on a simple C^* -algebra with \dim_{Rok}^c different from 0, 1, or ∞ . (For the sake of comparison, we mention here that there is no known example of a simple C^* -algebra with nuclear dimension or decomposition rank other than 0, 1, or ∞ .) We do not know of similar examples for the Rokhlin dimension *without* commuting towers.

Recall that the K_0 -group of an AF-algebra is torsion-free, and that its K_1 -group is trivial.

Example 4.28. We review the construction of a particular case of [58, Example 4.1]. Let $\beta \in \text{Aut}(C(S^2))$ be the automorphism of order two induced by the homeomorphism $x \mapsto -x$ on S^2 . Set

$$A_n = C(S^2) \otimes M_3 \otimes M_5 \otimes \cdots \otimes M_{2n+1}.$$

Let $\{x_n: n \in \mathbb{N}\}$ be a dense subset of S^2 , and, for $n \in \mathbb{N}$, define a homomorphism $\psi_n: C(S^2) \rightarrow M_{2n+1}(C(S^2))$ by

$$\psi_n(f) = \text{diag}(f, f(x_n), f(-x_n), \dots, f(x_n), f(-x_n))$$

for $f \in C(S^2)$. Define maps $\varphi_n: A_{n-1} \rightarrow A_n$ by

$$\varphi_n = \psi_n \otimes \text{id}_{M_3} \otimes \cdots \otimes \text{id}_{M_{2n-1}}.$$

Define an order-two automorphism α of $A = \varinjlim A_n$ as follows. Let

$$w_n = \text{diag} \left(1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \in M_{2n+1},$$

and set

$$\alpha_n = \beta \otimes \text{Ad}(w_1 \otimes w_2 \otimes \cdots \otimes w_n).$$

Then α_n has order two, and there is a direct limit automorphism $\alpha = \varinjlim \alpha_n$, which also has order two. In this example, we will show that α has Rokhlin dimension with commuting towers exactly equal to two.

The C^* -algebra A is unital, simple, separable, nuclear, satisfies the UCT, and has tracial rank zero. Moreover, $K_1(A) = \{0\}$ (since $K_1(A_n) = \{0\}$ for all $n \in \mathbb{N}$), and $K_0(A)$ is

a dimension group. By Lin’s classification of C^* -algebras of tracial rank zero, it follows that A is an AF-algebra. It was shown in part (6) of [58, Proposition 4.2] that $K_0(A \rtimes_{\alpha} \mathbb{Z}_2)$ has torsion isomorphic to \mathbb{Z}_2 , so in particular $A \rtimes_{\alpha} \mathbb{Z}_2$ is not an AF-algebra.

First, note that β has Rokhlin dimension with commuting towers at most two, by [32, Lemma 1.9]. It then follows from part (1) of [21, Theorem 3.8] that $\dim_{\text{Rok}}^c(\alpha_n) \leq 2$, so $\dim_{\text{Rok}}^c(\alpha) \leq 2$ by part (3) of [21, Theorem 3.8]. Since $A \rtimes_{\alpha} \mathbb{Z}_2$ is not an AF-algebra, it follows from [57, Theorem 2.2] that $\dim_{\text{Rok}}^c(\alpha) > 0$.

Suppose that $\dim_{\text{Rok}}^c(\alpha) = 1$. By Corollary 4.27, there exists a unital equivariant embedding $C(S^1) \rightarrow A_{\infty} \cap A'$. Give $C(S^1, A)$ the diagonal action of \mathbb{Z}_2 . By Corollary 4.6, the crossed product $C(S^1, A) \rtimes \mathbb{Z}_2$ is a locally trivial $C(S^1/\mathbb{Z}_2)$ -algebra with fibers $M_2(A)$.

We claim that $K_0(C(S^1, A) \rtimes \mathbb{Z}_2)$ is torsion-free. Choose closed connected sets Y_1 and Y_2 in S^1 such that:

- $C(S^1, A) \rtimes \mathbb{Z}_2$ is trivial over both Y_1 and Y_2 ;
- $Y_1 \cup Y_2 = S^1$;
- $Y_1 \cap Y_2$ is homotopic to $\{-1, 1\}$.

By [13, Proposition 10.1.13] (see also [11, Lemma 2.4]), we can write the crossed product $C(S^1, A) \rtimes \mathbb{Z}_2$ as the pullback

$$\begin{array}{ccc} C(S^1, A) \rtimes \mathbb{Z}_2 & \longrightarrow & (C(S^1, A) \rtimes \mathbb{Z}_2)_{Y_1} \\ \downarrow & & \downarrow \\ (C(S^1, A) \rtimes \mathbb{Z}_2)_{Y_2} & \longrightarrow & (C(S^1, A) \rtimes \mathbb{Z}_2)_{Y_1 \cap Y_2}, \end{array}$$

where all the maps are the canonical quotient (restriction) maps. Observe that $(C(S^1, A) \rtimes \mathbb{Z}_2)_{Y_j}$ is homotopic to $M_2(A)$ for $j = 1, 2$, and that $(C(S^1, A) \rtimes \mathbb{Z}_2)_{Y_1 \cap Y_2}$ is homotopic to $M_2(A) \oplus M_2(A)$. Using homotopy invariance of K -theory, the Mayer–Vietoris exact sequence on K -theory for this pullback (see [6, Theorem 21.5.1]) yields

$$\begin{array}{ccccc} K_0(C(S^1, A) \rtimes \mathbb{Z}_2) & \longrightarrow & K_0(M_2(A)) \oplus K_0(M_2(A)) & \longrightarrow & K_0(M_2(A) \oplus M_2(A)) \\ \uparrow & & & & \downarrow \\ K_1(M_2(A) \oplus M_2(A)) & \longleftarrow & K_1(M_2(A)) \oplus K_1(M_2(A)) & \longleftarrow & K_1(C(S^1, A) \rtimes \mathbb{Z}_2). \end{array}$$

Since A is an AF-algebra, we have $K_1(M_2(A)) = \{0\}$. It follows that the first horizontal map $K_0(C(S^1, A) \rtimes \mathbb{Z}_2) \rightarrow K_0(M_2(A)) \oplus K_0(M_2(A))$ is injective. Since $K_0(M_2(A))$ is torsion-free, we conclude that $K_0(C(S^1, A) \rtimes \mathbb{Z}_2)$ is torsion-free, and the claim is proved.

Recall from Proposition 4.11 that there is a commutative diagram

$$\begin{array}{ccc} A \rtimes_{\alpha} \mathbb{Z}_2 & \xrightarrow{\quad\quad\quad} & (A \rtimes_{\alpha} \mathbb{Z}_2)_{\infty} \\ & \searrow \quad \quad \swarrow & \\ & C(S^1, A) \rtimes \mathbb{Z}_2 & \end{array}$$

Since $K_0(C(S^1, A) \rtimes \mathbb{Z}_2)$ is torsion-free by the claim above, part (7) in Proposition 4.14 implies that $K_0(A \rtimes_{\alpha} \mathbb{Z}_2)$ is also torsion-free. However, this contradicts part (6)

of [58, Proposition 4.2], where it is shown that $K_0(A \rtimes_{\alpha} \mathbb{Z}_2)$ has torsion isomorphic to \mathbb{Z}_2 . This contradiction implies that $\dim_{\text{Rok}}^c(\alpha) = 2$, as desired.

The example above can be modified to produce a \mathbb{Z}_2 -action γ on a unital Kirchberg algebra that satisfies the UCT satisfying $\dim_{\text{Rok}}^c(\gamma) = 2$ and $\dim_{\text{Rok}}(\gamma) = 1$. This is the first example of an action whose Rokhlin dimensions with and without commuting towers are both finite but do not agree. (Examples where \dim_{Rok}^c is infinite but \dim_{Rok} is finite were already known; see [21, Example 4.8], and compare [32, Theorem 4.6] with [23, Theorem 4.20].)

Example 4.29. Let $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ be the action from Example 4.28. Set $B = A \otimes \mathcal{O}_{\infty}$ and let $\gamma: \mathbb{Z}_2 \rightarrow \text{Aut}(B)$ be given by $\gamma_g = \alpha_g \otimes \text{id}_{\mathcal{O}_{\infty}}$ for $g \in \mathbb{Z}_2$. Since α is (pointwise) outer, so is γ . Thus, Theorem 3.11 implies that $\dim_{\text{Rok}}(\gamma) \leq 1$. Since $K_0(B \rtimes_{\gamma} \mathbb{Z}_2) = K_0(A \rtimes_{\alpha} \mathbb{Z}_2)$ has torsion and $K_0(B) = K_0(A)$ is torsion-free, there cannot be any embedding $K_0(B \rtimes_{\gamma} \mathbb{Z}_2) \hookrightarrow K_0(B)$. It then follows from [23, Theorem B] that γ does not have the Rokhlin property. We conclude that $\dim_{\text{Rok}}(\gamma) = 1$.

We claim that $\dim_{\text{Rok}}^c(\gamma) = 2$. Since $\dim_{\text{Rok}}^c(\alpha) = 2$, we deduce from part (1) of [21, Theorem 3.8] that $\dim_{\text{Rok}}^c(\gamma) \leq 2$. It suffices to check that $\dim_{\text{Rok}}^c(\gamma) \neq 1$. Since there are KK -equivalences $A \sim_{KK} B$ and $A \rtimes_{\alpha} \mathbb{Z}_2 \sim_{KK} B \rtimes_{\gamma} \mathbb{Z}_2$, the argument used in Example 4.28 also applies to γ and yields $\dim_{\text{Rok}}^c(\gamma) \neq 1$, as desired.

We now turn to automatic reduction results for Rokhlin dimension. One such result has already appeared as [21, Theorem 4.19], which asserts that, for locally representable AF-actions of finite groups on AF-algebras, finite Rokhlin dimension with commuting towers implies the Rokhlin property. In other words, $\dim_{\text{Rok}}^c(\alpha) < \infty$ implies $\dim_{\text{Rok}}(\alpha) = 0$. Below we present two more instances of this phenomenon: Proposition 4.32 and Theorem 4.34.

We will need a preparatory result, of independent interest. Recall that $F(A)$ denotes the quotient $A_{\infty} \cap A' / \text{Ann}(A, A_{\infty})$, and that we write $\kappa_A: A_{\infty} \cap A' \rightarrow F(A)$ for the quotient map.

PROPOSITION 4.30. *Let G be a second countable compact group, let A be a separable C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. Let \mathcal{D} be a strongly self-absorbing C^* -algebra, and suppose that A is \mathcal{D} -absorbing. Then there exists a unital embedding*

$$\mathcal{D} \rightarrow \kappa_A(A_{\infty}^{\alpha} \cap A').$$

Proof. It is readily checked, by averaging over G , that $\kappa_A(A_{\infty}^{\alpha} \cap A')$ coincides with the fixed point algebra $F(A)^{F(\alpha)}$.

Let X be the free G -space provided by Theorem 4.7, and let

$$\varphi: C(X) \rightarrow F_{\alpha}(A)$$

be the equivariant unital embedding provided by Theorem 4.7. Since G is second countable, $C(X)$ is separable. Using \mathcal{D} -absorption of A , we may therefore choose a unital embedding $\theta_1: \mathcal{D} \rightarrow F_{\alpha}(A)$ whose image commutes with $\varphi(C(X))$. Since φ is equivariant, it follows that $\alpha_g(\theta_1(\mathcal{D}))$ commutes with $\varphi(C(X))$ for all $g \in G$. Define B_1

to be the unital C^* -algebra generated by $\bigcup_{g \in G} \alpha_g(\theta_1(\mathcal{D}))$. Then B_1 is separable, is α_∞ -invariant, commutes with $\varphi(C(X))$, and contains a unital copy of \mathcal{D} . Set $E_1 = B_1$. By separability of E_1 , we can find a unital embedding $\theta_2: \mathcal{D} \rightarrow F_\alpha(A)$ whose image commutes with $E_1 \cup \varphi(C(X))$. Define B_2 to be the unital C^* -algebra generated by $\bigcup_{g \in G} \alpha_g(\theta_2(\mathcal{D}))$. Then B_2 is separable, α_∞ -invariant, commutes with $E_1 \cup \varphi(C(X))$, and contains a unital copy of \mathcal{D} . Set $E_2 = C^*(E_1 \cup B_2)$. Proceed inductively to construct a separable, α_∞ -invariant unital C^* -algebra B_n , which commutes with $E_{n-1} \cup \varphi(C(X))$ and contains a unital copy of \mathcal{D} . Set $E_n = C^*(E_{n-1} \cup B_n)$. Then the inductive limit $E = \varinjlim E_n$ is a separable, α_∞ -invariant unital subalgebra of $F_\alpha(A)$, which commutes with $\varphi(C(X))$ and absorbs \mathcal{D} .

Denote by γ the diagonal action on $C(X, E)$. Then $\varphi \otimes \text{id}_E: C(X, E) \rightarrow F_\alpha(A)$ is an equivariant embedding, which therefore maps $C(X, E)^\gamma$ into $F_\alpha(A)^{F(\alpha)}$. Consider the algebra

$$C = \{f \in C(X, E) : F(\alpha)_g(f(x)) = f(g^{-1} \cdot x) \text{ for all } g \in G, x \in X\} \subseteq C(X, E)^\gamma.$$

Then C is a $C(X/G)$ -algebra with fibers isomorphic to E . Since E absorbs \mathcal{D} and X/G is finite-dimensional, it follows from [33, Theorem 4.6] that C absorbs \mathcal{D} . Thus, there exists a unital embedding $\mathcal{D} \rightarrow C$. The desired map is obtained as the following composition:

$$\mathcal{D} \longrightarrow C \longrightarrow C(X, E)^\gamma \xrightarrow{\varphi \otimes \text{id}_E} F_\alpha(A)^{F(\alpha)}. \quad \square$$

A standard argument, as used in [35], now shows the following corollary.

COROLLARY 4.31. *Let G be a second countable compact group, let A be a separable C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. Let \mathcal{D} be a strongly self-absorbing C^* -algebra, and suppose that A is \mathcal{D} -absorbing. Then α is conjugate to $\alpha \otimes \text{id}_{\mathcal{D}}$.*

The result above is really stronger than the fact that \mathcal{D} -absorption is preserved by taking crossed products by actions with finite Rokhlin dimension with commuting towers.

Here is our first dimension reduction result.

PROPOSITION 4.32. *Let G be a finite group and let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_2)$ be an action. If α has finite Rokhlin dimension with commuting towers, then it has the Rokhlin property.*

Proof. By the equivalence between parts (1) and (3) of [38, Theorem 4.2], it is enough to construct a unital map

$$\mathcal{O}_2 \rightarrow (\mathcal{O}_2^\alpha)_\infty \cap \mathcal{O}_2'.$$

This is an immediate consequence of Proposition 4.30. □

The conclusion of Proposition 4.32 is false if one only assumes $\dim_{\text{Rok}}(\alpha) < \infty$. See, for example, [21, Example 4.8].

For the next reduction result of the Rokhlin dimension, we will need to recall a definition.

Definition 4.33. [38, Definition 3.6] Let G be a finite abelian group, let B be a unital C^* -algebra, and let $\beta: G \rightarrow \text{Aut}(B)$ be an action. We say that β is *strongly approximately inner*, if there exist unitaries $v_g \in (B^\beta)_\infty$, for $g \in G$, satisfying $\beta_g(b) = v_g b v_g^*$ for all $b \in B$ and all $g \in G$.

In the theorem below, we do not know whether we obtain a similar conclusion if we only assume that $\dim_{\text{Rok}}^c(\alpha) < \infty$, or if we replace \mathbb{Z}_2 with a general finite (abelian) group.

We point out that a similar phenomenon was observed in [22].

THEOREM 4.34. *Let A be a separable, unital C^* -algebra, and let $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ be an action. Assume that $\dim_{\text{Rok}}^c(\alpha) \leq 1$ and that A absorbs the UHF-algebra M_{2^∞} tensorially. Then α has the Rokhlin property.*

Proof. By [38, Lemma 3.8], it is enough to show that the dual action $\widehat{\alpha}: \mathbb{Z}_2 \rightarrow \text{Aut}(A \rtimes_\alpha \mathbb{Z}_2)$ is approximately representable. By a slight abuse of notation, we also denote by α and $\widehat{\alpha}$ the order-two automorphisms that generate the given actions on A and $A \rtimes_\alpha \mathbb{Z}_2$.

We claim that $\widehat{\alpha}$ is strongly approximately inner. Use Corollary 4.27 to find a unital equivariant homomorphism

$$\varphi: C(S^1) \rightarrow A_\infty \cap A',$$

and denote by $w \in A_\infty \cap A'$ the image of the canonical unitary in $C(S^1)$. Let $u \in A \rtimes_\alpha \mathbb{Z}_2$ denote the unitary implementing α . Then $uwu^* = \alpha_\infty(w) = -w$ in $(A \rtimes_\alpha \mathbb{Z}_2)_\infty$. In particular,

$$w^*uw = -u = \widehat{\alpha}(u).$$

Moreover, since w commutes with the copy of A in A_∞ , it follows that $w^*aw = a = \widehat{\alpha}(a)$ for all $a \in A$. Since A and u generate $A \rtimes_\alpha \mathbb{Z}_2$, we deduce that $\text{Ad}(w^*)$ coincides with $\widehat{\alpha}$, and thus $\widehat{\alpha}$ is strongly approximately inner. This proves the claim.

By [38, Lemma 3.10], and since A absorbs M_{2^∞} , it is enough to show that there is a unital map

$$M_2 \rightarrow ((A \rtimes_\alpha \mathbb{Z}_2)^{\widehat{\alpha}})_\infty \cap (A \rtimes_\alpha \mathbb{Z}_2)'$$

This is again a consequence of Proposition 4.30, so the proof is finished. □

It is not enough in Theorem 4.34 to assume that $\dim_{\text{Rok}}(\alpha) \leq 1$. See, for example, the comments in part (a) of Remark 4.18.

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