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ALGEBRAIC STRUCTURE OF DEGENERATE SYSTEMS

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This thesis is concerned with the problem of degenerate systems, that is, given a theory with nonphysical degrees of freedom, how to find a canonical method for extracting the physical subtheory from it. The work falls into two parts. The first part, the classical theory surveys the existing mathematical work on this problem, and it traces a continuous line of development from the emergence of degeneracies in the variational framework, through the local Dirac-Bergman theory of constraints [*1*], [*2*], [*3*], the problems arising from field theoretical aspects, and culminating in Gotay, Nester and Hinds' global method [*4*] on infinite dimensional manifolds, which allows for curvature. The theory is applied to two examples; electromagnetism, and the Yang-Mills field.

The second part, quantum theory in a C^*-algebra framework comprises the main thrust of the thesis. After preparing the ground with mathematical and philosophical discussions of the quantization process, we argue that degeneracy occurs as supplementary conditions on either the field algebra itself, or on the set of possible expectation values; respectively called *algebraic* and *state* conditions. The structure of

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these conditions is determined by the underlying physics.

The study of C^*-degenerate systems starts with the assumption of a unital C^*-algebra F as the field algebra, within which is specified two sets of unitaries U, V, denoted state and algebraic conditions respectively. State conditions are imposed by specifying the physical states ω by the condition $\omega(U) = 1$. This condition is seen to be equivalent to $\omega(AU) = \omega(A) = \omega(UA)\ \forall\ A \in F$, $U \in U$, and such states exist if and only if $C^*(U - 1) \not\ni 1$. Next we find the unique maximal C^*-algebra D annihilated by all physical states, and its multiplier algebra O in F is taken as the physical observables, this choice being justified by showing $U' \subset O$, where the commutant U' is the traditional observable algebra. The physical algebra is defined as $R := O/D$, and it is constraint free. A connection with the heuristic structures is made by showing O to be the "weak" commutant of D. Next, the state spaces of the various algebras and their connections are investigated, and we show that in the GNS-representation of a physical state ω, that $\pi_\omega(D)\Omega_\omega = 0$, with Ω_ω the cyclic state. This condition is similar in form to the usual heuristic state conditions.

To define the physical transformations, we examine the automorphisms of F which are compatible with the degenerate structure. Automorphisms which preserve D will define automorphisms on R, and those automorphisms on F which define the identity on R are taken as the gauge transformations. We find various reasonable types of gauge transformations to be in this set.

We argue that the algebraic conditions V should be imposed either at the point of definition of F, or that it should be done identically to the state conditions in order to avoid ordering problems.

The general theory finds its application in an example; the Dirac form of electromagnetism, which was prepared in the classical part. It turns out that Dirac electromagnetism is the prototype of any linear boson theory with linear hermitian constraints, and it is nontrivial. This example takes the C^*-algebra of the CCR as defined by Manuceau [5], for the field algebra F.

Next we examine the compatibility of the general structures with various reasonable physical requirements, such as R being simple, and the physical automorphisms on R being inner, and develop internal criteria for these.

Finally, since indefinite inner product (*IIP*) representations arise only in the context of degenerate systems, we consider the relation between these and the algebraic structures above. An important finding is that the triple $\langle D, O, F \rangle$ relates in a simple and direct fashion to a triple $\langle H'', H', H \rangle$ which always occur in *IIP*-representations, where H is the *IIP*-representation space, H' is some positive subspace on which the observables are represented, and H'' is its null-space. This relation is given by the *GNS*-type representations of non positive-definite functionals on F, satisfying additional conditions. The use of *IIP*-representations turns out to be just a device for obtaining a covariant representation of the physical algebra, which may not be obtainable from a covariant physical state on F. This theory is applied as before to a linear boson field with linear hermitian constraints, and Landau electromagnetism is treated in detail as an example of an algebraic field theory with an *IIP*-representation. The latter also serves as an example of a system which involves both state and algebraic conditions.

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