SHEETS OF REAL ANALYTIC VARIETIES

ANDREW H. WALLACE

Introduction. In a previous paper **(4)** the author worked out some results on the analytic connectivity properties of real algebraic varieties, that is to say, properties associated with the joining of points of the variety by analytic arcs lying on the variety. It is natural to ask whether these properties can be carried over to analytic varieties, since the proofs in the algebraic case depend mainly on local properties. But although this generalization can be carried out to a large extent, there are, nevertheless, difficulties in the analytic case, owing mainly to the fact (cf. **2**, § 11) that a real analytic variety may not be definable by means of a set of global equations. Thus, although the general idea of the treatment given here is the same as in **(4)**, some variation in the details of the method has proved to be necessary, and some of the final results are slightly weaker in form.

As in **(4)** the key result is an approximation theorem for piecewise analytic curves on a variety; this theorem is stated in § 2 and then proved in §§ 3 and 4. In § 5 the approximation theorem is applied to the discussion of the sheets, that is, the maximal analytically connected sets, of a real analytic variety.

As regards further literature on the subject of real analytic varieties, see Whitney **(5** and **6)**; in the former paper an approximation theorem of the type just mentioned is proved for analytic manifolds (that is, varieties without singularities), while in the latter certain decomposition theorems are obtained for a wide class of varieties.

1. Real analytic varieties. In this paper the term real analytic variety will be applied to a set V in a fixed Euclidean space E_n such that V is closed in E_n and each point p of V has a neighbourhood U in the ambient space such that $U \cap V$ is the set of zeros of a finite collection of functions analytic in U. Thus the term is equivalent to "sous-ensemble analytique" as in **(2)**. At each point p of V, V defines a germ of a real analytic variety V_p. Write V_p' for the complexification of V_p, that is to say, the smallest germ of a complex analytic variety containing V_p and contained in the complex n-space obtained by allowing the co-ordinates in E_n to take complex values. The dimension of V_p' (that is to say, the dimension of the highest dimensional component of V_p') will be called the local dimension of V at p, to be written as $\dim_p V$. $\operatorname{Dim}_p V$ has a maximum ($< n$) over all points p of V; this maximum will be called $\dim V$.

Received November 3, 1958.

A regular or simple point of V is a point p at which $\dim_p V = \dim V$ and at which local analytic co-ordinates can be set up in E_n in such a way that V has locally the equations $x_{r+1} = x_{r+2} = \ldots = x_n = 0$. A singular point of V is a point which is not regular; note that this includes any point where the local dimension is less than the maximum for V. The set of all singular points of V will be called the singular locus of V.

It is essential at this point to note that the singular locus of a real analytic variety is not necessarily an analytic variety. For example, consider the analytic variety defined in E_3 by the single equation $x^2y^3 - z^2(y + z) = 0$. The cross-section of this surface by a plane y = constant is a cubic curve with a loop, and as y tends to zero this loop flattens out into the line segment on the x-axis joining the points $x = \pm 2/3\sqrt{3}$. It is then easy to check that the singular locus of this variety consists of this line segment along with the whole y-axis, and this set is certainly not an analytic variety. Of course the surface under consideration could be regarded as a real algebraic variety, in which case the whole of the x-axis would be included in the singular locus, and this locus would be an algebraic sub-variety. The essential difference is that for real analytic varieties the regularity or otherwise of a point is determined by local equations, and not, as in the algebraic case, by global equations (which in general do not exist in the analytic case; cf. **(2)**).

Another feature of the example just given concerns the approximation of analytic arcs (definition at the beginning of § 2) on the surface V with the equation $x^2y^3 - z^2(y + z) = 0$. Take two regular points of V on the x-axis and on opposite sides of the origin, say the points $(\pm 1, 0, 0)$, and call them p and q. The segment pq of the x-axis is an analytic arc joining p and q but there is no other arc joining these points in such a direct manner. In fact if C is any other arc on V joining p and q and if K is its projection on the (x, y)-plane, then the part of K lying in the strip defined by $|x| \leqslant 2/3\sqrt{3}$ is covered three times by part of C. It follows, for example, from this that an approximation of K, however good, cannot be lifted to an approximation of C in V in the manner of **(4)**. The trouble is that in **(4)** the success of the method used depended on the fact that the arcs studied never had more than finitely many points in common with the singular locus of the variety. But, as can be seen from the present example, an arc on a real analytic variety can have a sub-arc in common with the singular locus, even when the end-points are regular.

The example just given indicates that, in order to study properties of real analytic varieties analogous to those studied in **(4)** for algebraic varieties, a weaker form of approximation for curves will have to be used, in which the approximation C' of a given curve C will lie in a preassigned neighbourhood of C, but C' will be mapped on C by a mapping which may be (at least along certain arcs) many-one.

2. Statement of the approximation theroem. For convenience some of the definitions of **(4)** will be repeated here. An analytic arc in Euclidean n-space E_n is an arc given by parametric equations $x_i = f_i(t)$, $i = 1, 2, \ldots, n$, where the f_i are real analytic functions of t. The end-points of such an arc are assumed to be non-singular; that is to say, the given equations define a simple linear branch at each end-point. A piecewise analytic curve is a union of finitely many analytic arcs joined end to end, in such a way that at a common end-point P of two of the arcs, say C_1 and C_2, exactly these two arcs meet and no others, and C_1 and C_2 have distinct tangents at P. Each point P of the type described is called a joint of the curve. A curve C' is said to be an ϵ-approximation of a curve C if there is a homeomorphism f of C' on C such that the distance of p from $f(p)$ is less than ϵ for each p. C' and C are said to be analytically equivalent at p if $f(p) = p$, and if, in a sufficiently small neighbourhood U of p, there is defined a mapping T of the form $T(x_i) = x_i + h_i(x)$, where the h_i are real analytic functions of $x_1, x_2, \ldots, x_n$ at p, such that $T(C \cap U) = C' \cap U$. Here the h_i, expanded in power series at p, are assumed to be of order $\geqslant 2$; if they are of order $\geqslant r$, the analytic equivalence is said to be of order $\geqslant r$.

One of the main results of **(4)**, which will be required here is:

LEMMA 2.1. *Let C be a piecewise analytic curve in E_n and let S be a finite set of points on C including all singular points of C (note that the joints of C are not to be counted as singularities of C). Then for any preassigned ϵ and r there is an analytic curve C' which is an ϵ-approximation of C with analytic equivalence of order $\geqslant r$ at each point of S.*

For any set A in E_n the ϵ-neighbourhood of A is the set of points in the union of all spheres of radius ϵ with centres at the points of A. With this terminology the approximation theorem to be proved in this paper can be stated.

THEOREM 1. *Let V be a real analytic variety in E_n and let C be a piecewise analytic curve on V, all the joints of C being regular on V. Then for any preassigned ϵ there is an analytic arc C' on V and contained in the ϵ-neighbourhood of C. In particular the end-points of C' are within ϵ-neighbourhoods of those of C, and if C is closed so is C'.*

The idea of the proof of this theorem is as follows. If $\dim V = r$, C will be projected on a suitable r-dimensional linear subspace E_r of E_n. Writing the projected curve as K, Lemma 2.1 will be applied to give an approximation K' of K. K' is then to be lifted into V. In the case of a real algebraic variety V (or more generally a real analytic variety given by global equations, with singular points defined globally as in algebraic geometry), this lifting is in general carried out by a one-one correspondence, yielding the stronger approximation theorem of **(4)**. Here, however, the lifting has to be done by means of the local equations of V, splitting K' into a sequence of arcs for the purpose, and lifting each one in turn. The lifted arcs are then to be strung together to give the required curve C'. As illustrated in the example of § 1 the sequence of lifted arcs may double back on itself covering parts of K' several times. It has to be checked of course that C' cannot break up into closed loops, or the last condition to be proved in the theorem would not hold.

Most of the proof is taken up with the process of choosing a set of co-ordinates so that the projection just referred to is the orthogonal projection onto the space $x_{r+1} = x_{r+2} = \ldots = x_n = 0$. The method is to take a point p on C, and using local equations for V in a neighbourhood $U(p)$ of p, to make a list of the various conditions unfavourable to the projection, approximation and lifting process described above. It turns out that, if co-ordinates are to be changed by orthogonal transformations, then the choices which are unfavourable, in $U(p)$, correspond to an analytic subvariety in the space of orthogonal transformations. Since the curve C can be contained in a finite number of neighbourhoods of the type $U(p)$ it follows that a choice of co-ordinates can be made which is favourable for the whole of C.

3. Choice of co-ordinates. The procedure sketched at the end of the last section will now be carried out in detail. Take a point p of V as origin. In a neighbourhood $U(p)$ of p, V is defined as the set of zeros of a finite number of power series in $x_1, x_2, \ldots, x_n$ with real coefficients, convergent in $U(p)$. Alternatively, V is the set of real zeros of an ideal I in the ring of power series in $x_1, x_2, \ldots, x_n$ with real coefficients convergent near p. Write V' for the complexification of V at p, that is to say, the smallest complex analytic variety, defined in a complex neighbourhood $U'(p)$ of p (obtained by allowing all the co-ordinates to assume complex values) whose set of real points coincides with $V \cap U(p)$. If I is the ideal of V at p then the ideal of V' is generated by I in the ring of power series with complex coefficients convergent around p.

Take an irreducible component V_0 of $V \cap U(p)$ with $\dim V_0 = \dim V$, provided such a component exists. In the applications to follow this choice will always be possible; but to cover the contrary case, if $\dim_p V \neq \dim V$, no restriction will be imposed on the choice of co-ordinates around p. The co-ordinates are now to be changed in such a way that the prime ideal of V_0 at p becomes a regular ideal (**1**, p. 208). This means that, in the new co-ordinates $y_1, y_2, \ldots, y_n$, the ideal is to contain no power series independent of y_{r+1}, $y_{r+2}, \ldots, y_n$ but for each $h > r$ it must contain a power series regular with respect to y_n, that is to say, having an exact power of y_n among its terms of lowest order. The method of regularizing an ideal is explained in (**1**, p. 208, Theorem 4). It involves a sequence of linear changes of co-ordinates, which can in fact be taken to be orthogonal. And at each stage the condition that a change of co-ordinates should not be suitable is that the elements of the corresponding matrix should satisfy certain algebraic equations. However, the procedure followed in (**1**) is not quite suitable for the present purpose. For there the discussion is carried out in terms of formal series, after which a check has to be made as to the region of convergence. This region may well depend on the particular choice of co-ordinates made, whereas here it is necessary to work in a sequence of steps, making sure that at each there is a region of convergence independent of the co-ordinates chosen. The following is a variant of the method of **(1)** designed to meet this requirement.

LEMMA 3.1. *Let C be a compact set of the real analytic variety V. Then co-ordinates can be chosen, making an orthogonal linear transformation from those originally given, such that, in a neighbourhood of each point p of C the co-ordinates of points on V satisfy a polynomial equation in x_n with coefficients analytic in $x_1, x_2, \ldots, x_{n-1}$ at p.*

Proof. The set C used here will eventually be an analytic curve on V, but for the moment compactness is the only property wanted. Let $p \in C$, and for convenience shift the origin of the given co-ordinates $x_1, x_2, \ldots, x_n$ in E_n to p. In a neighbourhood $U(p)$ of p the points of V are the zeros of an ideal I in the ring of power series in the x_i with real coefficients convergent around p. Take a series f in I and assume $U(p)$ is such that f is convergent in $U(p)$. Let A be a generic orthogonal $n \times n$ matrix and define the co-ordinates y_i by $y_i = \sum_{j=1}^{n} a_{ij} x_j$. Clearly there is an algebraic subvariety $W(p)$ of the set O_n of orthogonal $n \times n$ matrices such that any specialization of A not in $W(p)$ will give a set of co-ordinates $y_1, \ldots, y_n$ such that f is regular with respect to y_n. Now, by the Weierstrass preparation theorem, f can be multiplied by a power series in the y_i, not vanishing at p, to give a polynomial g in y_n whose highest coefficient is 1 and whose other coefficients all are power series in $y_1, y_2, \ldots, y_{n-1}$ vanishing at p. Now the proof of the Weierstrass preparation theorem **(1)** shows that all the series involved in the theorem are convergent in a smaller neighbourhood $U'(p)$ than $U(p)$, obtained in fact by reducing the bounds of the various co-ordinates by a factor which depends on the upper bound of f in $U(p)$. Bearing in mind that the linear change of co-ordinates being made here is orthogonal, thus leaving spheres invariant, it follows that if $U(p)$ is taken as a spherical neighbourhood, $U'(p)$ can be taken as a smaller sphere whose radius is a fraction of that of $U(p)$. The fraction depends on the upper bound of f in $U(p)$, but does not depend on the particular choice of the orthogonal matrix A. C, being compact, can be covered by a finite number of neighbourhoods of the type $U'(p)$, and so the union of the corresponding $W(p)$ makes up an algebraic subvariety of the set O_n. If the matrix A changing the co-ordinates from the x_i to the y_j is chosen not in this subvariety it follows at once that the conditions of the lemma are satisfied by the new co-ordinates.

Note that, as the conclusion of the lemma has been left, it is not necessarily true that the polynomial in y corresponding to some point p of C has its coefficients vanishing at p, unless p happens to be one of the finite set of points corresponding to the finite set of $U'(p)$ covering C. However some factor of this polynomial will satisfy this condition if the origin is shifted so that $y_n = 0$ at p. On the other hand, even without taking any further steps beyond the proof described above, the polynomial in y_n corresponding to any point of C will always have the coefficient of the highest power of y_n equal to 1.

The choice of co-ordinates made in the last lemma is to fix y_n once and for all, subsequent changes affecting only the co-ordinates, $y_1, y_2, \ldots, y_{n-1}$. With the choice of co-ordinates just described, let p be any point of C, and repeat the argument of Lemma 3.1, replacing the ideal I of that lemma by the intersection of I with the ring of real analytic functions at p independent of y_n. This argument shows that, for points of V in a neighbourhood of each point of C, y_{n-1} satisfies a polynomial equation with coefficients analytic in $y_1, y_2, \ldots, y_{n-2}$ at p, in particular the coefficient of the highest power of y_{n-1} being 1. Proceeding in this way step by step, the following result is obtained:

LEMMA 3.2. *With the assumptions of the last lemma, there ia a choice of co-ordinates* $x_1, x_2, \ldots, x_n$ *in* E_n *such that in some neighbourhood of each point* p *of* C, *the co-ordinates of points on* V *satisfy a set of equations of the form:*

$$f_i(x_1, x_2, \ldots, x_{r+i}) = 0, i = 1, 2, \ldots, n - r, \tag{1}$$

where f_i *is a polynomial in* x_{r+i} *with coefficients real analytic in* $x_1, x_2, x_3, \ldots, x_{r+i-1}$ *at* p, *the coefficient of the highest power of* x_{r+i} *being* 1.

It is to be understood in this statement that the set of f_i may change when the point p is changed. On the other hand, in the case which is to be considered later, it will be true that $\dim_p V = \dim V$ at each point p of C, and this will be precisely the value of r for each set of equations of the type (1).

A further adjustment to the co-ordinate system is necessary to enable the local equations of V to be brought into a certain canonical form. Let p be a point of V at which $\dim_p V = \dim V = r$, and let V_0 be an r-dimensional component of V in a neighbourhood of p. Taking p as origin let R_n denote the ring of power series in $x_1, x_2, \ldots, x_n$ with real coefficients convergent around p and let I be the ideal of V_0 in this ring. In the residue class ring R_n/I let ξ_i be the residue class of x_i, for each i. Then equations (1) are satisfied with $(x_1, x_2, \ldots, x_n)$ replaced by $(\xi_1, \xi_2, \ldots, \xi_n)$, from which it follows at once that the quotient field of R_n/I is a finite algebraic extension of that of R_r, the ring of convergent power series in $\xi_1, \xi_2, \ldots, \xi_r$ with real coefficients. In addition, the dimensional condition imposed at p implies that the ξ_i for $i = 1, 2, \ldots, r$ are independent indeterminates over the real numbers. The object of the next bit of working is to pick out a primitive root for this extension, and then to change the co-ordinates so that this root will be the residue class of one of the co-ordinates. That this can be done locally at the point p is, of course, a well-known result. But here the idea is to make the choice of co-ordinates in such a way as to bear the relation just described to each component of V in some neighbourhood of each point of a compact subset C.

Returning now to the notation just introduced, note that the ξ_{r+i} for $i = 1, 2, \ldots, n - r$ are integral over R_r. Let $u_1, u_2, \ldots, u_{n-r}$ be independent indeterminates over the quotient field of R_r and write $R_r' = R_r[u_1, u_2, \ldots, u_{n-r}]$. Then clearly

$$\xi = \sum_{i=1}^{n-r} u_i \xi_{r+i}$$

is integral over R_r'. Thus ξ will satisfy an equation of the type:

$$\xi^m + a_1\xi^{m-1} + \ldots + a_m = 0, \tag{2}$$

where each of the a_i is in R_r'. As regards the convergence conditions, each of the a_i is a polynomial in the u_j with coefficients in R_r, and so only finitely many power series are involved in the equation (2), each series being convergent for sufficiently small values of the ξ_i. (That these series are convergent at all can be seen, for example, from the fact that equation (2) can be derived by rational processes from equations satisfied by the individual ξ_{r+i}, $i = 1, 2, \ldots, n - r$, such as the equations (1), where convergence is known. Note that so far nothing is said or known about the reducibility or otherwise of (2).) Now the theorem of the primitive root for a finite algebraic extension **(3)** says that, provided the u_i do not satisfy a set of linear equations (which they do not, being independent indeterminates), ξ is a primitive root for the quotient field of $R_r'(\xi_{r+1}, \xi_{r+2}, \ldots, \xi_n)$. This means in particular that

$$\xi_i = F_i(u, \xi_1, \xi_2, \ldots, \xi_r, \xi)/G(u, \xi_1, \xi_2, \ldots, \xi_r) \tag{3}$$

for each $i = 1, 2, \ldots, n - r$, where each F_i is a polynomial in the u_j and ξ, with coefficients in R_r, and G is a polynomial in the u_j with coefficients in R_r. Then in all the equations (3) there are only finitely many power series in $\xi_1, \xi_2, \ldots, \xi_r$, all convergent for sufficiently small values of the ξ_i. Identifying ξ_i with x_i for each $i = 1, 2, \ldots, r$, it follows that there is a neighbourhood $U(p)$ of p in which all the series (in $x_1, x_2, \ldots, x_r$) appearing in the equations (2) and (3) are convergent. Also denote by $L(p)$ the set of linear equations in the u_i which must not be satisfied if ξ is to be a primitive root as just described. Then if C is a compact set on V it can be covered by a finite number of neighbourhoods of the type $U(p)$, and the u_i can be given real values such that none of the corresponding sets of equations $L(p)$ are satisfied, and such that the rational functions appearing in the equations of the type (3) corresponding to each of these neighbourhoods are defined. At least one of the u_i will be non-zero, say u_1. Then take as a new set of co-ordinates in E_n

$$x_1, x_2, \ldots, x_r, \sum_{i=1}^{n-r} u_i x_{r+i}, x_{r+2}, \ldots, x_n.$$

Changing the notation so that these co-ordinates are again written as $x_1, x_2, \ldots, x_n$ the result obtained can be summed up as follows:

LEMMA 3.3. *Let C be a compact subset of the real analytic variety V such that, at each point p of C, $\dim_p V = \dim V = r$. Then co-ordinates in E_n can be chosen in such a way that C is covered by a finite number of neighbourhoods in each of* *which the points of each r-dimensional local component of V satisfy equations of the form*

$$(4)\qquad \begin{aligned} F(x_{r+1}) &= 0 \\ x_i &= F_i(x_{r+1})/G, \qquad i = 2, \ldots, n-r, \end{aligned}$$

where F and the F_i are polynomials in x_{r+1} with coefficients which are power series in $x_1, x_2, \ldots, x_r$, and G is a power series in $x_1, x_2, \ldots, x_r$, all the series being convergent in the relevant neighbourhood.

(It is assumed in speaking of these power series that the origin has been shifted to a certain point of the neighbourhood in question.)

Finally take any point p on the compact set C as origin. p will lie in some neighbourhood of the covering of C described in the last lemma, and so the points of the r-dimensional components of V at p will satisfy equations of the type (4), with G and the coefficients of the F_i and of F real analytic at p. The irreducible factors of F corresponding to these components can now be picked out, and will have coefficients which can be written as power series in $x_1, x_2, \ldots, x_r$ convergent in some neighbourhood of p; the co-ordinates $x_{r+2}, x_{r+3}, \ldots, x_n$ for points of these components will still be given by (4), convergence of the series involved holding in some neighbourhood of p. These remarks enable a refinement of Lemma 3.3. to be stated:

LEMMA 3.4. *C being as in the last lemma each point p of C has a neighbourhood $U(p)$ in which each r-dimensional component of V is exactly the set of points satisfying equations of the type* (4) *with F and the F_i polynomials in x_{r+1} and* (*with p as origin*) *G and the coefficients of F and the F_i power series in $x_1, x_2, \ldots, x_r$ convergent in $U(p)$. In addition, C, being compact, can be covered by a finite number of neighbourhoods of the type $U(p)$.*

At this stage it is convenient to make a definition in preparation for the next section. In the notation of the last lemma, cover C by a finite number of the neighbourhoods $U(p)$, and set up equations of the type (4) for each r-dimensional component of V in each such neighbourhood. Let D be the discriminant of F; it will be a series convergent in $U(p)$. Then the set of points in $(x_1, x_2, \ldots, x_r)$-space defined by $G = D = 0$ is a local analytic variety in the projection of $U(p)$. The union of all the local varieties obtained in this way from all the r-dimensional local components of V in all the $U(p)$ of the finite covering of C described in Lemma 3.4. will be called the branch locus of V relative to the sets of local equations described in that lemma (or simply branch locus if the context is clear). For brevity a set of points in a Euclidean space which, like the branch locus just introduced, in the union of a finite number of local analytic varieties, each defined in some neighbourhood, will be called an open variety.

4. Displacement of an arc from an open variety. As already pointed out, one of the difficulties presented by real analytic varieties is that a curve, although not lying entirely in the singular locus, may nevertheless have an arc in common with that locus. The lemma about to be proved is the main step towards resolving this difficulty.

LEMMA 4.1. *Let V be an open variety in Euclidean n-space E_n, and let C be an analytic arc contained in V. Then there exists in E_n an analytic arc C', which is an arbitrarily good approximation of C and which meets V only at finitely many points.*

Proof. Clearly no generality is lost by assuming that all the components of the various local analytic varieties of which V is composed are of dimension $n - 1$; this can always be arranged if necessary by enlarging V. The discussion of the last section can now be applied to V. The argument is not affected by the fact that V is now an open variety rather than a real analytic variety, since at each stage only local properties are used. It then follows that co-ordinates can be chosen in such a way that C is covered by a finite number of neighbourhoods in each of which V is given by equations of the type (4). In this case these equations reduce to a single polynomial equation in x_n with coefficients which are power series in the remaining variables convergent in the neighbourhood in question, a point in that neighbourhood being taken as origin. Let the parametric equations of C be $x_i = f_i(t)$, $i = 1, 2, \ldots, n$, where the f_i are real analytic functions of t, and t varies over some finite interval, say $0 \leqslant t \leqslant 1$. Then there is an analytic mapping of the (t, x_n)-plane into E_n given by $f(t, x_n) = (f_1(t), f_2(t), \ldots, f_{n-1}(t), x_n)$. The image of f in E_n is a piece of analytic surface S containing C, and in particular C is the image of the curve $\bar{C}$ in the (t, x_n)-plane with the equation $x_n = f_n(t)$. If $F(x_1, x_2, \ldots, x_n) = 0$ is the equation defining one of the local varieties of which V is composed, then the equation

$$F(f_1(t), f_2(t), \ldots, f_{n-1}(t), x_n) = 0$$

defines a local variety in a neighbourhood of some point of the (t, x_n)-plane, and the union of all the local varieties obtained in this way forms an open variety in this plane. Denote this variety by $\bar{V}$. It is clear that the image of $\bar{V}$ under f is the intersection of S and V, and, moreover, the points of $\bar{V}$ are the only ones mapped into $S \cap V$. And so if the lemma can be proved for the curve $\bar{C}$ relative to the open variety $\bar{V}$ in the (t, x_n)-plane, giving an approximation $\bar{C}'$ of $\bar{C}$ meeting $\bar{V}$ at only finitely many points, then the curve $C' = f(\bar{C}')$ will satisfy the requirements of the lemma. In order to prove the lemma in the plane it is clearly sufficient to take $\bar{C}'$ to be any sufficiently good approximation of $\bar{C}$ in the plane. But in view of the applications to be made of this result, it is necessary to make the approximation in a particular way, as will now be explained. The open variety $\bar{V}$ consists of a finite number of curve branches, each defined in some neighbourhood; let the finite collection of points p_i denote the centres of these branches along with a finite set of points arbitrarily chosen on $\bar{C}$. Let t_i be the value of t at p_i. Let $g(t)$ be a polynomial in t vanishing only at the t_i, to an order at least r. For example

$g(t) = \Pi(t - t_i)^r$ will do. Then, remembering that $\bar{C}$ has the equation $x_n = f_n(t)$ define $\bar{C}'$ as the curve with equation $x_n = f_n(t) + \lambda g(t)$, where λ is a real parameter. Note that, in a neighbourhood of each p_i the power series expressions of the parametric equation of $\bar{C}$ and of $\bar{C}'$ differ only by high power of the parameter if r is taken large, and also that, as the parameter λ tends to zero, the approximation of $\bar{C}$ by $\bar{C}'$ can be made arbitrarily close. Since one of the branches making up $\bar{V}$ at each p_i is part of $\bar{C}$, and since $\bar{C}$ and $\bar{C}'$ meet only at the p_i, it is not hard to see that, for λ samll enough, $\bar{C}'$ will meet $\bar{V}$ only at the p_i. Applying the mapping f to the curve $\bar{C}'$ constructed in this way, the following corollary of the above lemma is obtained:

COROLLARY. *In the above lemma, C' can be constructed in such a way that, at each of the finitely many points where it meets V, each of its branches is associated with a branch of C, and, with a suitable choice of parameter, the parametric equations of these branches differ only by terms of high order with coefficients depending analytically on a parameter λ and tending to zero as λ tends to zero.*

The proof of Theorem 1 can now be undertaken, the following lemma giving the discussion of the most difficult step in the proof.

LEMMA 4.2. *Let V be a real analytic variety in E_n and let C be an analytic arc on V with at least one of its end-points regular on V. Let C_0 be the set of points on C for which the local dimension of V is $r = \dim V$, and let co-ordinates be chosen in E_n in accordance with Lemma* 3.4, *relative to the compact set C_0 on V. Let B be the branch locus in $(x_1, x_2, \ldots, x_r)$-space E_r corresponding to this co-ordinate choice, and to the choice of local equations for V around a finite number of points of C_0. Then in the ϵ-neighbourhood of C on V, for preassigned ϵ, there is an analytic arc C' whose projection in E_r meets B at only finitely many points. And in particular the end-points of C' will lie in ϵ-neighbourhoods of those of C.*

Proof. Let K be the projection of C in E_r. If $K \cap B$ already consists of finitely many points there is nothing to be done, for C' can be taken equal to C. But it may be that subarcs of K lie in B. In this case a set of points p_i is to be defined on C, along with their projections q_i on K (the q_i being not necessarily all distinct). The q_i are to include all singularities of K and all isolated points of $K \cap B$, and the p_i are to include all points of C projecting on these. Using local equations of the type (4) for V in neighbourhoods of a finite number of points of C, it may turn out that certain arcs on K can be lifted to give several copies on V, apart from those which form part of C. The p_i are to include the intersections of C with these other copies (these points will clearly be finite in number) and the q_i are to include their projections. Finally, approximate K by an arc K' in E_r as in Lemma 4.1 and its corollary. According to these results $K' \cap B$ consists of finitely many points, and it is clear from the proof of the corollary that these can be assumed to include the q_i already defined. Any additional intersections of K' and B are now to be included also among the q_i, and as before all points of C projecting on them are to be taken among the p_i. It will be remembered that according to the corollary to Lemma 4.1, K' depends on a parameter λ and that, around the q_i, K' takes the limiting position K as λ tends to zero. It is now to be shown that, if the approximation of K by K' is close enough and if λ is small enough there is an arc C' on V satisfying the requirements of this lemma and projecting on K'.

To establish the existence of this arc C', sets of neighbourhoods covering C and K will now be constructed. K' will then be made to lie in the union of these neighbourhoods, and will be divided into arcs each lying in one of the neighbourhoods. These arcs will then be lifted into V, and it will be shown that some of them can be joined end to end to form the required analytic arc C'. First define $U(q_i)$ as a neighbourhood of q_i such that the parametric equations of each branch of K' at q_i, when expressed in terms of a suitable parameter, give each co-ordinate as a power series convergent in $U(q_i)$, and in fact uniformly convergent with respect to λ, and also such that, if these equations are substituted into the equations of the type (4) for V around p_i, the resulting equations can be solved for $x_{r+1}, x_{r+2}, \ldots, x_n$ as fractional power series in the parameter, convergent for values of the parameter corresponding to points of K' in $U(q_i)$. If several q_i coincide, take $U(q_i)$ as the smallest of the corresponding neighbourhoods. $U(p_i)$ is then to be a neighbourhood of p_i projecting onto $U(q_i)$. In addition $U(p_i)$ is assumed to be taken so small that no two curve branches on V at p_i projecting into branches of K' at q_i have any point in common other than p_i. It should be noted that, as the results of § 3 are being used here, what has just been said makes sense only around the points of C_0 and their projections. It will appear presently, however, that $C = C_0$. The curves C, K, K' will now be split up into a number of arcs for which it is convenient to introduce some terminology now. These arcs will be called C-arcs, K-arcs, or K'-arcs according to the curve they lie on. C-arcs lying in the neighbourhoods $U(p_i)$ will be said to be of the first kind. When these arcs are removed the remainder of C consists of finitely many disjoint non-singular arcs. Those whose projections are contained in B will be said to be of the second kind, and those whose projections do not meet B of the third kind. There is a certain amount of arbitrariness in the definition of B, depending as it does in the choice of particular neighbourhoods; it is not hard to see that, by shrinking certain of these neigbourhoods if necessary, it can be arranged that the whole of C is a union of arcs of the three kinds described. The projections of these arcs will be called K-arcs of the first, second, and third kinds, respectively.

Construct now an open covering of C by sets in V. Let γ be a C-arc of the second or third kind, and let κ be its projection in E_r. Let $U(\gamma)$ be a neighbourhood of γ not meeting C except in the points of an arc obtained by extending γ slightly in each direction (not far enough to reach any of the p_i). Also $U(\gamma)$ is to be chosen so that its projection $U(\kappa)$ meets K in the arc κ slightly extended at each end. The set of $U(p_i)$ and $U(\gamma)$ covers C while the $U(q_i)$ and $U(\kappa)$ cover K. Now in each $U(q_i)$ there will be a set of branches of K' approximating K-arcs of the first kind, and if the approximation of K by K' is sufficiently close each $U(\kappa)$ will contain exactly one non-singular arc of K' approximating κ. These arcs of K' lying in the $U(q_i)$ and the $U(\kappa)$ will be called K'-arcs of the first, second, or third kind according to the kind of K-arc they approximate.

C'-arcs, some of which will make up the required curve C', will now be defined. To obtain a C'-arc of the first kind, take a K'-arc κ' of the first kind, through q_i, say, and change the parameter on it so that it is zero at q_i. Substitute these parametric equations into the appropriate equations of the type (4) for V, and calculate all the corresponding roots x_{r+1} as fractional power series in the parameter t. By the choice of the $U(p_i)$ and $U(q_i)$ convergence holds for these series for all t corresponding to points on κ', and the resulting curve branches in E_n will all actually lie in $U(p_i)$. If the fractional power series corresponding to one of these branches involve an even root of t, the end-points of that branch will lie in the same set $U(\gamma)$ for some γ whenever the parameter λ on which K' depends is taken small enough. On the other hand, if an odd root of t is taken, the end-points of the branch will lie in two different $U(\gamma)$'s for λ small enough. The set of all branches on V obtained in this way will be called C'-arcs of the first kind. Consider now a K'-arc κ' of the second kind approximating a K-arc κ. κ' is to be lifted into V in a similar way to that applied to the arcs of the first kind. κ' in this case may pass through several neighbourhoods in each of which a different set of equations of the type (4) for V must be used, and so κ' must be lifted in sections. A number of copies will be obtained of κ' lifted in this way into V, and, taking κ' to be a compact arc, it is clear that the points of certain of the lifted arcs will converge to the points of some C-arc lying over κ, as the parameter λ on which K' depends tends to zero, and the convergence is uniform with respect to the variable point on κ'. If γ' is one of these lifted arcs, converging to the arc γ as λ tends to zero, γ' will lie in $U(\gamma)$ for λ sufficiently small. Assume now that λ is so small that this happens for all arcs like γ' obtained in this way; these arcs will be called C-arcs of the second kind. C'-arcs of the third kind are obtained in the same way from the K'-arcs of the third kind. The only difference is that in this case there is a unique C'-arc over a given K'-arc.

Suppose that the family of C'-arcs has been constructed and that λ is so small that the conditions mentioned relative to the arcs of the second and third kinds and the end-points of those of the first kind are satisfied. A maximal C'-arc can now be defined as a maximal union of C'-arcs which is itself an analytic arc. Since two distinct analytic arcs cannot have a subarc in common, it is clear that each C'-arc belongs to exactly one maximal arc, and also the C'-arcs forming a maximal arc follow one another in a well-defined sequence (defined by the variation of an analytic parameter on the maximal arc) in which each C'-arc is traced out exactly once, unless the maximal arc is closed.

Now one end-point p of C is assumed, in the hypothesis of the lemma, to be regular on V. If γ_1 is the C-arc on which p lies, it follows that there is exactly one C'-arc γ_1' which has γ_1 as its limit when λ tends to zero, and exactly one point p' of that arc will have p as limit. The last statement rules out the possibility that γ_1' is an arc of the first kind with parameter obtained from that of γ_1 by taking an even root. Now define C' to be the unique maximal C'-arc starting off with γ_1'. C' is certainly not closed, for it has p' as an end-point. The other end of γ_1' will be joined to a second C'-arc γ_2', that to a third γ_3', and so on until the other end of C' is reached. This must happen after a finite number of steps, since there are only finitely many C'-arcs, no one of which can be used twice. Also, it has been arranged that each C'-arc of the second or third kinds will lie in one of the $U(\gamma)$ and so can be assumed to end in one of the $U(p_i)$, so that it will join up to an arc of the first kind; and similarly each C'-arc of the first kind will join up to one of the second or third kind; all this with the exception of arcs which end near the end-points of C. It follows that the second end-point of C' will lie in a preassigned neighbourhood of the second end-point of C, if λ is small enough. Also, the definition of the C'-arcs has ensured that, if λ is small enough, C' will lie in a preassigned neighbourhood of C, and so the requirements of the lemma are satisfied by C', whose projection K' meets B at only finitely many points.

COROLLARY. *With the notations of the last lemma, C_0, the set of points on C where the local dimension of V is $r = \dim V$, coincides with C.*

Proof. The proof of this is implicit in what has gone before. If the result is not true there will be a point p_0 different from the initial point p of C such that at all points of the arc pp_0 the local dimension of V is r, but following p_0 and arbitrarily close to it there will be points of lower local dimension. p_0 will in this case necessarily be among the points p_i defined above. Let γ be the C-arc which passes beyond p_0 into the region of lower local dimension, κ its projection, and κ' the K'-arc which approximates it. Then every C'-arc lying in $U(p_0)$ over κ' must correspond to taking an even root of the parameter on κ', since the second half of κ' has no points of V over it near p_0. It would follow that C' could not have a second end-point; for such an end-point could not lie over any point of K' preceding q_0, and yet it follows from what has just been said that a moving point on C', starting at p, must always double back when it reaches p_0, always projecting on a point of K' which precedes q_0. But since there are only finitely many C'-arcs and C' cannot be closed a contradiction is thus obtained which proves the corollary.

The proof of Theorem 1 will now be completed. C is now to be a piece-wise analytic curve on V with all the joints at regular points of V. Co-ordinates in E_n are to be chosen as in Lemma 3.4, the curve C being taken as the compact set; this choice of compact set is admissible in view of the corollary of the last lemma which shows that the local dimension of V is r at all points of C. Apply Lemma 4.2 to each of the analytic arcs of which C is composed. The result is a collection of analytic arcs $\bar{C}_i$ lying in a preassigned neighbourhood of C, with end-points lying arbitrarily close to the end-points and joints of C, and with projections in E_r meeting the branch locus B in finitely many points. In addition, since the joints of C are regular on V, it can be assumed that the end-points of the $\bar{C}_i$ lie in cellular neighbourhoods of these points, and so they can be joined up by analytic arcs within these cells. And these new analytic arcs will have projections in E_r not meeting B. Thus, given the piecewise analytic curve C, a new piecewise analytic curve $\bar{C}$ has been constructed in a preassigned neighbourhood of C, with its end-points in preassigned neighbourhoods of those of C, and such that the projection K of $\bar{C}$ in E_r meets the branch locus B in at most finitely many points. Thus to complete the proof of Theorem 1 it is only necessary to prove the theorem for $\bar{C}$. And the major difficulty has now been removed, namely that caused by subarcs of the given curve projecting into the branch locus. Using Theorem 3 of **(4)**, quoted above as Lemma 2.1, let K' be an approximation of K, with analytic equivalence at all singularities of K and at points of $K \cap B$, but smoothing K at the joints. In addition, the results leading to Theorem 3 of **(4)** imply that K' can be assumed to depend on a parameter λ in such a way that, as λ tends to zero, K' takes the limiting position K in a neighbourhood of each of the singularities and points of $K \cap B$. K' is now to be lifted into V to give the analytic curve C' required by Theorem 1. The simplest way of doing this is to repeat the argument of Lemma 4.2, with the simplification here that there are no arcs of the second kind. Alternatively the lifting can be done as in **(4)**, using sets of local equations like (4) for V instead of the global equation which was available here. Note that if this second method is used, it is possible to make C' and $\bar{C}$ analytically equivalent at the singularities of the latter. This may be of interest if $\bar{C}$ happens to be the curve which is given. However if the given curve is such that the initial adjustment replacing C by $\bar{C}$ (to avoid having arcs projecting into B) it necessary, any property of analytic equivalence is liable to be lost in the process. It is not hard to see that this adjustment will be necessary if and only if C has arcs in common with the singular locus of V, a situation which has been shown to be possible by the example of § 1.

One further remark must be made concerning Theorem 1. It will be noticed that the approximating curve C' given by that theorem passes through all the points p_i constructed in the course of the proof of Lemma 4.2, and that along with the points which must belong to this set any finite collection of points on C can be included. This can be stated as a corollary which strengthens the result of the theorem:

COROLLARY 2. *In Theorem* 1, *C' can be constructed so as to pass through each of a finite set of points arbitrarily given on C. In particular the end-points of C' can be made to coincide with those of C.*

5. Sheets of an analytic variety. Let V be a real analytic variety in E_n. A subset S of V is said to be analytically connected if every pair of points of S can be joined by an analytic arc lying in S. A sheet of V is an analytically connected subset not contained in any larger analytically connected subset of V. The sheet S is said to be proper if it contains a point p with a neighbourhood U in E_n such that $V \cap U = S \cap U$. This terminology agrees with that of **(4)**. The following results correspond to some of the properties derived for sheets of real algebraic varieties in **(4)**.

LEMMA 5.1. *Let p, q, r be three points of a real analytic variety V and let q be regular on V. Then, if there are analytic arcs on V joining p to q and joining q to r, there is an analytic arc on V joining p to r and meeting a preassigned neighbourhood of q.*

Proof. The proof is as for Lemma 17.1 of **(4)**, using here Theorem 1 and its Corollary 2 to approximate the union of the given arcs pq and qr by an analytic arc from p to r.

THEOREM 2. *Let p be a regular point of the real analytic variety V and let S be the set of all points of V which can be joined to p by analytic arcs on V. Then S is a sheet of V, and every sheet of V containing a regular point can be constructed in this way.*

Proof. The proof, using Lemma 5.1, is essentially the same as that of Theorem 13 of **(4)**.

The following two corollaries correspond similarly to the corollaries of Theorem 13 in **(4)**.

COROLLARY 1. *Each regular point of V belongs to exactly one sheet.*

COROLLARY 2. *Each sheet of V containing a regular point of V is proper.*

The next theorem corresponds to the dimensional homogeneity established in **(4)** for sheets of real algebraic varieties:

THEOREM 3. *Let S be a sheet of a real analytic variety V containing a regular point p of V. Then for any q on S, every neighbourhood of q contains a point q' of S which is regular on V.*

Proof. Let C be an analytic arc joining p and q. Set up a co-ordinate system as for Lemma 4.2 relative to C, and apply that lemma, along with the Corollary 2 at the end of § 4. This gives an analytic arc C' joining p and q and with its projection K' in E_r meeting the branch locus B in only finitely many points. In particular, C' itself can meet the singular locus of V in at most finitely many points, and so any neighbourhood of q must contain a regular point q' of V lying on C'. Since q' is joined to the regular point p by an analytic arc on V, Theorem 2 along with its first corollary implies that q' is on S as required.

A sheet S containing a regular point of V will be called r-dimensional. The local cellular decomposition described in § 18 of **(4)** carries over to the real analytic case, since the whole construction depends only on the setting up of local equations. The result is:

LEMMA 5.2. *Let p be a point of a real analytic variety of dimension r and let W be a subvariety containing p (W in fact need only be defined in a neighbourhood of p). Then in any preassigned neighbourhood of p there is a neighbourhood U of p such that $V \cap U$ is the union of the closures of a set of disjoint open cells U_i of dimensions $\leqslant r$ such that:*

(1) $\bigcup \mathrm{Fr}\, U_i = U \cap W'$ *where W' is an analytic subvariety of V, defined at least around p, such that $U \cap W' \supset U \cap W$.*

(2) *Each r-cell in the decomposition of $V \cap U$ is contained in exactly one proper sheet (note that here, unlike the algebraic case, this statement is only made for the cells of highest dimension).*

(3) $p \in \bar{U}_i$ *for each i.*

(4) *The neighbourhood U of p can be chosen so that all points of $U \cap (V - W)$ can be joined to p by analytic arcs on V meeting W only at p.*

To make the statement of part (2) of the above lemma complete, note that each s-cell (for $s \leqslant r$) in the decomposition described there consists of regular points of a real analytic variety of dimension s defined in a neighbourhood of p. Such a cell is thus analytically connected, and so is contained in a maximal analytically connected subset of V, namely a sheet. But a sufficiently small neighbourhood of a point of the cell in question will meet V only at points of the cell, and so the sheet so obtained is proper. A slight strengthening of this statement gives the following theorem:

THEOREM 4. *Every point of a real analytic variety V belongs to a proper sheet of V.*

Proof. Take any point p on V, and construct a cellular decomposition of V around p as in Lemma 5.2. Let M be one of the cells, say of dimension $s \leqslant r$. As already pointed out, M is analytically connected. And so, by part (4) of Lemma 5.2, the set consisting of M along with the point p is analytically connected, and so is contained in some sheet. But a small neighbourhood of a point of M meets V only at points of M, and so this sheet is proper.

It is worth noting that, in the notation of this theorem, the closure $\bar{M}$ of the cell M is analytically connected. For M is part of a real analytic variety V_0 defined at least in a neighbourhood of p. Now take two points q_1 and q_2 in $\bar{M}$. If they are in M then they certainly be can joined by an analytic arc in $\bar{M}$. Suppose $q_1 \in \mathrm{Fr}M$. Then, applying Lemma 5.2 to V_0 around q_1, it follows that q_1 can be joined by an analytic arc γ_1 in $\bar{M}$ to a point q_3 of M, and if $q_2 \in M$ then q_3 and q_2 can be joined by an analytic arc γ_2 in M. Then, applying Lemma 5.1 to V_0 (the fact that V_0 may be only locally defined makes no difference), it follows that the union of γ_1 and γ_2 can be replaced by an analytic arc in $\bar{M}$ joining q_1 and q_2. A similar argument can be used if both q_1 and q_2 are in FrM.

The following result corresponds to Theorem 14 of **(4)**; note, however, the restriction as to dimension.

THEOREM 5. *Let V be a real analytic variety in E_n of dimension r. Then each r-dimensional sheet of V is a closed set.*

Proof. Let S be an r-dimensional sheet of V, and let p be in the closure S of S. Then a neighbourhood U of p contains a point q of S, which, by Theorem 3, can be assumed to be regular on V. By part (4) of Lemma 5.2 if U is small enough p and q can be joined by an analytic arc on V, and so by Theorem 2 and its Corollary 1, p belongs to the unique sheet determined by the regular point q, namely S. S is thus closed.

In contrast to the algebraic case, the properties of the lower dimensional sheets of a real analytic variety are somewhat elusive. For although such sheets are contained in the singular locus of V, this locus may not be an analytic variety. On the other hand, the points of V at which the local dimension is less than the maximum form a subset of the singular locus and this subset is (as will be shown in a future paper) part of a real analytic subvariety. Consequently, the results proved above will all extend to the proper sheets of V, regardless of their dimension. The proposed proof of the assertion just made depends on attaching some sort of multiplicity to each singular point of V. This is done with reference to local systems of equations. Then, attempting to build up a variety of singular points, one fits the p-fold locus in one neighbourhood to the q-fold locus in an adjoining one, with p not necessarily equal to q. p and q are necessarily both even or both odd, and the process only breaks down if one of them is equal to 1. This cannot happen if one has started at a point of lower local dimension, where the multiplicity attached is always even.

REFERENCES

1. S. Bochner and W. T. Martin, *Several complex variables* (Princeton, 1946).

2. H. Cartan, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Scc. Math. France, *85* (1957) 77–99.

3. B. L. van der Waerden, *Moderne algebra* (Berlin, 1931).

4. A. H. Wallace, *Algebraic approximation of curves*, Can. J. Math. *10* (1958), 242–278.

5. H. Whitney, *Analytic coordinate systems and arcs in a manifold*, Ann. Math., *38* (1937), 809–818.

6. H. Whitney and F. Bruhat, *Quelques propriétés fondamentales des ensembles analytiques-réels*, Comm. Math. Helvet., *33* (1959) 132–160.

University of Toronto