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ON RELATIVE BASE POINT FREENESS OF ADJOINT BUNDLE

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Abstract. We give an effective result on the relative base point freeness of an adjoint bundle for a pair of a projective morphism and a relatively ample line bundle.

§1. Introduction

Recently, Angehrn and Siu [AS] and Tsuji [Tj] independently obtained results on the following:

FUJITA'S FREENESS CONJECTURE OF ADJOINT BUNDLES. ([F]) *Let X be an n-dimensional projective manifold defined over $\mathbb{C}$ with an ample line bundle L. Then the adjoint bundle $\mathcal{O}_X(K_X \otimes L^{\otimes m})$ is generated by global sections for every $m > n$.*

Their effective bounds are $m > n(n+1)/2$. The basic ideas of their proofs from [AS] and [Tj] (use of Riemann-Roch theorem, Nadel's vanishing theorem, Ohsawa-Takegoshi's L^2-extension theorem and so on) are extremely simple and can be applied to a variety of contexts. In this note we would like to go into detail about the method and consider the following relative version:

MAIN THEOREM. *Let $f : X \longrightarrow Y$ be a projective morphism from a complex manifold X to a complex space Y, and let L be a relatively ample line bundle on X. Then $\mathcal{O}_X(K_X \otimes L^{\otimes m})$ is f-free, i.e., the natural sheaf homomorphism*

$$f^* f_* \mathcal{O}_X(K_X \otimes L^{\otimes m}) \longrightarrow \mathcal{O}_X(K_X \otimes L^{\otimes m}) \quad \textit{is surjective,}$$

for every

$$m > \frac{1}{2}\, d\,(d+1),$$

here d is the maximum dimension of the fibres of f.

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Our relative version has some applications to the classification theory of higher dimensional algebraic varieties. For example, we have the following (refer to [KMM] for terminologies):

COROLLARY. *Let X be a projective manifold defined over $\mathbb{C}$ and let $\varphi : X \longrightarrow X'$ be the contraction morphism of an extremal ray R of $\overline{NE}(X)$. Then the $d(d+1)/2$-th anti-pluri-canonical divisor $-d(d+1)/2K_X$ is φ-free, here d is the maximum dimension of the fibres of φ.*

This corollary is much helpful to the classification of the singular fibres of φ and contraction morphisms (cf. [K])

The reader should refer to [D2] (analytic approach) and [L] (algebraic approach) for the recent development of the theory of adjoint bundles.

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§2. Singular Hermitian metric and vanishing theorem

Our basic tool is singular Hermitian metrics as in [D1], [D2]. We use vector bundles and the associated locally free sheaves interchangeably. In this section we let X be a complex manifold.

2A. Singular Hermitian metric

Let L be a holomorphic line bundle on X. A metric h on L is called **singular Hermitian**, if there exist a function $\varphi \in L^1_{\text{loc}}(X)$ and a smooth Hermitian metric h_0 on L such that $h = e^{-\varphi} h_0$ holds. This defines a closed current

$$\operatorname{curv} h := \operatorname{curv} h_0 + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\operatorname{curv} h_0$ is the curvature form of the Hermitian metric h_0 and $\partial\bar{\partial}$ is taken in the sense of currents. The $(1,1)$-current $\operatorname{curv} h$ is said to be the **curvature current** of the singular Hermitian line bundle (L, h). It is easy to see that $\operatorname{curv} h$ is independent of the choices of h_0 and φ. $\sqrt{-1}\,\bar{\partial}\partial \log h$ is the formal expression of curv h. For a singular Hermitian line bundle (L, h) on a Hermitian manifold (X, ω). The L^2-sheaf $\mathcal{L}^2(L, h)$ is the sheaf defined by

$$\mathcal{L}^2(L,h)(U) = \{s \in \Gamma(U, L)\ ;\ h_0(s,s)e^{-\varphi} \in L^1_{\text{loc}}(U)\},$$

where $h = e^{-\varphi}h_0$ is a local expression of h as above. Similarly, the **multiplier ideal sheaf** $\mathcal{I}(h)$ of the singular Hermitian metric is defined by

$$\mathcal{I}(h)(U) := \{f \in \Gamma(U, \mathcal{O}_M) \ ; \ |f|^2 e^{-\varphi} \in L^1_{\mathrm{loc}}(U)\}.$$

These sheaves do not depend of the choices φ, h_0 and ω, and satisfy the following relation: $\mathcal{L}^2(L, h) = L \otimes \mathcal{I}(h)$.

2B. Vanishing theorem

We recall the following:

NADEL'S COHERENCE AND VANISHING THEOREM 2.1. ([N], [D1, §4]) *Let (X, ω) be a complete Kähler manifold and let (L, h) be a singular Hermitian line bundle on X. Assume that there exists a real number c such that* $\operatorname{curv} h \geq c\,\omega$ *on X. Then*

(1) *the sheaf $\mathcal{I}(h)$ is a coherent ideal sheaf of $\mathcal{O}_X$, and*

(2) *if c is positive, the q-th L^2-cohomology group*

$$H^q_{(2)}(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0$$

for every $q \geq 1$.

As a simple application of the above theorem, we have

PROPOSITION 2.2. *Let (X, ω) be a complete Kähler manifold, x be a point of X, and let L be a holomorphic line bundle on X. Assume that L admits a singular Hermitian metric h_x such that*

(1) *there exists a positive constant c such that* $\operatorname{curv} h_x \geq c\,\omega$*, and that*

(2) *x is isolated in the zero complex space $V\mathcal{I}(h_x)$.*

Then there exists a holomorphic section of $K_X \otimes L$ which does not vanish at x.

We will need the following Serre type vanishing theorem:

PROPOSITION 2.3. ([F, Theorem N′]) *Let L be a positive line bundle on a weakly 1-complete manifold X, i.e., a complex manifold with a smooth plurisubharmonic exhaustion function $\Phi : X \longrightarrow \mathbb{R}$. Then for every coherent analytic sheaf $\mathcal{F}$ on X and for every $c < \sup_X \Phi$, there exists a positive integer m_0 such that*

$$H^q(X_c, \mathcal{F} \otimes L^{\otimes m}) = 0$$

for any $q \geq 1$ and for any $m \geq m_0$, where $X_c := \{x \in X \ ; \ \Phi(x) < c\}$ is a sublevel set of (X, Φ).

2C. Singular Hermitian metric with analytic singularities [AS, §2]

In this subsection we explain how to construct a singular Hermitian metric and how to control the multiplier ideal sheaf. The standard method of the construction is to use holomorphic sections; such metrics are said to be singular Hermitian metrics **with analytic singularities**.

It is convenient to introduce the notion of rational coefficient geometry as follows. We consider a family of local holomorphic functions $s = \{s_\lambda; s_\lambda \in H^0(U_\lambda, \mathcal{O}_X)\}_{\lambda\in\Lambda}$ for some locally finite open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X. For a positive rational number q and for any smooth Hermitian holomorphic line bundle (L, h_0) on X, the family of local holomorphic functions s is said to be a **multivalued holomorphic section** of $L^{\otimes q}$ over X, if there exists a positive integer p such that pq being an integer and that $s^p := \{s_\lambda^p\}_{\lambda\in\Lambda}$ defines an element of $H^0(X, L^{\otimes pq})$. We denote

$$\begin{aligned} |s| \ &:= (h_0^{\otimes pq}(s^p, s^p))^{1/(2p)}: \quad \text{the pointwise length,} \\ (s)_0 &:= \{x \in X \ ; \ s_\lambda(x) = 0 \quad \text{for some } \lambda \in \Lambda\}. \end{aligned}$$

We just consider $(s)_0$ as a set of zeros. We also define a singular Hermitian metric h of $L^{\otimes q}$ by a family of local real valued measurable functions such that h^p defines a singular Hermitian metric of $L^{\otimes pq}$. We can also define the curvature current and the multiplier ideal sheaf.

Let $s_1, \ldots, s_k$ be a finite number of multivalued holomorphic sections $L^{\otimes q}$ such that $(s_i)^p$ $(1 \le i \le k)$ is a holomorphic section of $L^{\otimes pq}$ for some positive integer p with pq being an integer. Then we can define a singular Hermitian metric of $L^{\otimes q}$ by

$$h := \frac{h_0^q}{\sum_{i=1}^k |s_i|^2}.$$

The curvature current is a closed positive current on X. Indeed, for local expressions $s_i = \{s_{i\lambda}\}_{\lambda\in\Lambda}$, we see

$$\operatorname{curv} h = \sqrt{-1}\partial\bar\partial \log \sum_{i=1}^k |s_{i\lambda}|^2$$

on every open set U_λ. We note that both the positivity of the curvature current and the multiplier ideal sheaf do not depend on the smooth Hermitian metric h_0. Let $\mathcal{J}$ be the sheaf of ideal of $\mathcal{O}_X$ generated locally by

$\{(s_i)^p\}_{i=1}^k$. We assume that the support of $\mathcal{O}_X/\mathfrak{J}$ is compact. We take a modification $\pi : \widetilde{X} \longrightarrow X$ by a finite number of successive monoidal transforms with nonsingular centers and a family of smooth divisors E_i in $\widetilde{X}$ with only simple normal crossing so that the following three consitions hold:

(0) For every i, $\pi(E_i) \subset \text{ supp } \mathcal{O}_X/\mathfrak{J}$.

(1) The sheaf $\pi^{-1}\mathfrak{J}\cdot\mathcal{O}_{\widetilde{X}}$ which is the image of $\pi^*\mathfrak{J}$ under the natural map $\pi^*\mathfrak{J} \longrightarrow \mathcal{O}_{\widetilde{X}}$ is equal to the ideal sheaf $\mathcal{O}(-\sum r_i' E_i)$ for some non-negative integers r_i'.

(2) $K_{\widetilde{X}} = \pi^* K_X \otimes \mathcal{O}(\sum b_i E_i)$ for some non-negative integers b_i. In other words, the holomorphic Jacobian determinant of the map $\pi : \widetilde{X} \longrightarrow X$ vanishes precisely of order b_i along E_i and vanishes nowhere on $\widetilde{X} - \bigcup_i E_i$. Let $r_i := r_i'/p$. For every $t \geq 0$, we set

$$\mathcal{I}(t) := \mathcal{L}^2(\mathcal{O}_X, (\sum_{i=1}^k |s_i|^2)^{-t}).$$

We see that every $\mathcal{I}(t)$ is a coherent ideal sheaf by 2.1(1) and that $\mathcal{I}(1) = \mathcal{I}(h)$. Then a point x on X belongs to the zero complex subspace $V\mathcal{I}(t)$ if and only if there exists an index i such that E_i intersects $\pi^{-1}(x)$ and that $tr_i - b_i \geq 1$. For every point $x \in V\mathcal{I}(1)$, we see that

$$\begin{aligned} &\sup\{t \geq 0 \ ; \ \mathcal{I}(t)_x = \mathcal{O}_{X,x}\} \\ &\qquad = \min\{t \geq 0 \ ; \ tr_i - b_i \geq 1 \text{ for } i \text{ such that } E_i \text{ intersects } \pi^{-1}(x)\}. \end{aligned}$$

Note that the quantity, say $\alpha(x)$, is always a rational number and $0 < \alpha(x) \leq 1$.

§3. Preliminary lemma

3A. Reduction and non-vanishing with a parameter space

Let $f : X \longrightarrow Y$ and L be as in Main Theorem. We fix a point x on X. The situation is local on Y, so we may assume that Y is a closed complex subspace of the unit ball $\mathbb{B}^M$ in $\mathbb{C}^M$ with the global coordinate $(y_1, \ldots, y_M)$ and that $f(x) = 0$. Since L is f-ample, restricting Y on a smaller ball $\mathbb{B}^M$ if necessary, we may assume L admits a smooth Hermitian metric h whose curvature form is positive on X.

In the local situation as above, we show the following non-vanishing lemma with a parameter space. It is important to handle the case that the zero complex subspace of the multiplier ideal sheaf has singularities.

Lemma 3.1. ([AS, Lemma 4.1]) *Let Z be a closed subvariety (reduced and irreducible) in X of positive dimension d such that $x \in Z$ and $f(Z) = f(x)$, B_0 and B_1 be smaller balls centered at $0 \in \mathbb{B}^M$ with $B_0 \subsetneqq B_1 \subsetneqq \mathbb{B}^M$, $Y_i := Y \cap B_i$ and $X_i := f^{-1}(Y_i)$ for $i = 0, 1$, and let N be a positive integer. Let Δ' be a local holomorphic curve in Z passing through x with x as the only singularity such that the normalization $\sigma : \Delta \longrightarrow \Delta'$ is a one to one holomorphic map from the open unit disk Δ in $\mathbb{C}$ with $\sigma(0) = x$. Then, replace Δ with a smaller disk if necessary, there exist a positive integer m and a finite number of holomorphic sections $\{\tilde{\tau}_j\}_{j=1}^K \subset H^0(X_1 \times \Delta, pr_X^* L^{\otimes m(N+1)})$, where $pr_X : X \times \Delta \to X$ is the first projection, such that*

$$\tilde{\tau}_j|_{Z\times u} \in H^0(Z, L^{\otimes m(N+1)} \otimes \mathcal{M}_{Z,\sigma(u)}^{mN}) \quad \text{for every } u \in \Delta - 0$$

and for every j, and that their common zeros satisty

$$x \in X_0 \cap \bigcap_{j=1}^K (\tilde{\tau}_j|_{X_1\times 0})_0 \subsetneqq Z.$$

Proof. We denote the ideal sheaf of the graph $\sigma \times 1 : \Delta \longrightarrow Z \times \Delta$ by $\mathcal{I}_\Gamma \subset \mathcal{O}_{Z\times\Delta}$. Since Z is compact, the direct image sheaf $pr_{\Delta *}(pr_X^* L^{\otimes m(N+1)} \otimes \mathcal{O}_{Z\times\Delta} \otimes \mathcal{I}_\Gamma^{mN})$ is a coherent sheaf on Δ, where $pr_\Delta : Z \times \Delta \longrightarrow \Delta$ be the projection. The sheaf $pr_{\Delta *}(pr_X^* L^{\otimes m(N+1)} \otimes \mathcal{O}_{Z\times\Delta} \otimes \mathcal{I}_\Gamma^{mN})$ is generically locally free with $H^0(Z \times u, L^{\otimes m(N+1)} \otimes \mathcal{M}_{Z\times u,\sigma(u)\times u}^{mN})$ as the generic fibre. We see that the latter space is non-zero for every large m and for every $u \in \Delta - 0$ by the following asymptotic dimension compairing:

$$\dim H^0(Z, L^{\otimes m(N+1)}) = (N+1)^d (L^d \cdot Z)(d!)^{-1} m^d + O(m^{d-1});$$
$$\operatorname{rank} \mathcal{O}_Z/\mathcal{M}_{Z,\sigma(u)}^{mN} = \binom{mN+d-1}{d} = N^d (d!)^{-1} m^d + O(m^{d-1}).$$

Then by Theorem A of Cartan-Serre we see that, for every large m, there exists a section $\tilde{\tau} \in H^0(Z \times \Delta, pr_X^* L^{\otimes m(N+1)} \otimes \mathcal{O}_{Z\times\Delta} \otimes \mathcal{I}_\Gamma^{mN})$ such that $\tilde{\tau}|_{Z\times 0}$ is not identically zero.

For a smaller disk Δ_1, we take a sublevel set W of a weakly 1-complete manifold $X \times \Delta$ for an appropriate smooth plurisubharmonic exhaustion function which contains $X_1 \times \Delta_1$. By Proposition 2.3, for every $x_0 \times u_0 \in X \times \Delta - Z \times \Delta$, there exists a positive integer m_0 such that

$$H^1(W, pr_X^* L^{\otimes m} \otimes \mathcal{I}_{Z\times\Delta} \otimes \mathcal{M}_{X\times\Delta, x_0\times u_0}) = 0$$

for any $m \geq m_0$. Hence we can extend $\tilde{\tau}$ as sections $\tilde{\tau}_1, \ldots, \tilde{\tau}_K \in H^0(X_1 \times \Delta_1, pr_X^* L^{\otimes m(N+1)})$ which satisfy the desired properties. □

3B. Calculus Lemma

The following simple calculus lemma on non-integrability will be used later to locate the zero-set of the multiplier ideal sheaf of a singular metric.

LEMMA 3.2. ([AS, Lemma 3.1]) *Let m and N be positive integers and $0 < a < 1$. Let $f_1, \dots, f_k$ be holomorphic functions on the unit polydisk Δ^n on $\mathbb{C}^n$ with coordinates $z, w_1, \dots, w_{n-1}$. Let $H := \{z = 0\}$ and let V be the subset of $H \cap \Delta^n$ where the vanishing order of $f_j|_{H\cap\Delta^n}$ is at least mN for any j. Let d be the codimension of V in $H \cap \Delta^n$ at the origin. Then $|z|^{-2a}(\sum |f_j|^2)^{-t/(mN)}$ is not locally integrable at the origin for $t \geq d + mN(1-a)$.*

Proof. By slicing and Fubini's theorem, we may assume $d = n-1$. Then $\sum |f_j|^2 \leq C_1(|z|^2 + |w|^{2mN})$ for some positive constant C_1, where we set $|w|^2 := \sum_{i=1}^{n-1} |w_i|^2$. The non-integrability of $|z|^{-2a}(\sum |f_j|^2)^{-t/(mN)}$ follows from that of $|z|^{-2a}(|z|^2 + |w|^{2mN})^{-t/(mN)}$. Then we see the non-integability by direct calculation by using polar coordinates for z and w with $x = |z|^2$ and $y = |w|$. □

§4. Proof of Theorem

We fix a point x on X. We let $f : X \longrightarrow Y \subset \mathbb{B}^M \subset (\mathbb{C}^M; y_1, \dots, y_M)$ and (L, h) be the local reduction around $f(x)$ as in 3A. We take smaller balls B_0 and B_1 centered at $0 \in \mathbb{B}^M$ with $B_0 \subsetneq B_1 \subsetneq \mathbb{B}^M$. We set $Y_i := Y \cap B_i$ and $X_i := f^{-1}(Y_i)$ for $i = 0, 1$. With these notations, our Main Theorem follows from the following

THEOREM 4.1. *For every $m > d_0(d_0+1)/2$, there exists a holomorphic section $\tau \in H^0(X_1, \mathcal{O}_X(K_X \otimes L^{\otimes m}))$ such that $\tau(x) \neq 0$, where d_0 is the dimension of a maximum dimensional irreducible component of the fibre $f^{-1}(f(x))$ which contains x.*

By Proposition 2.2, all we have to do is to show the following

PROPOSITION 4.2. *For every $m > d_0(d_0+1)/2$, $L|_{X_1}^{\otimes m}$ admits a singular Hermitian metric H_m such that*

(1) *the curvature current dominants a complete Kähler form on X_1, and that*

(2) *x is isolated in the zero subspace $V\mathcal{I}(H_m)$.*

If f is constant, then X is a projective manifold with an ample line bundle L, that is (a part of) the statement of [AS], [Tj]. Hence we assume that f is non-constant.

4A. Statement of the induction step

We formulate an induction statement for the proof of Proposition 4.2. Let $m_d := \sum_{n=d+1}^{d_0} n$ for $0 \leq d < d_0$ and let $m_{d_0} := 0$. We take rational numbers $0 = \varepsilon(d_0+1) < \varepsilon(d_0) < \varepsilon(d_0-1) \cdots < \varepsilon(0) < 1$. For every positive rational number q and for every multivalued holomorphic section s of $L^{\otimes q}$ on an open set of X, we denote $|s|$ the length with respect to the smooth Hermitian metric h. For every d with $0 \leq d \leq d_0$, we consider the following

INDUCTION STATEMENT $(*)_d$. There exist a rational number $\varepsilon(d+1) < \varepsilon_d < \varepsilon(d)$ and a finite number of multivalued holomorphic sections $s_1^{(d)}, \ldots, s_{k_d}^{(d)}$ of $L^{\otimes(m_d+\varepsilon_d)}$ on X_1 such that

(i) $(\bigcap_{i=1}^{k_d}(s_i^{(d)})_0 \cap X_0) \subset f^{-1}(f(x))$,

(ii) $x \in (Z_d(1) \cap X_0) \subset f^{-1}(f(x))$,

(iii) $x \notin Z_d(t)$ for $t < 1$, and that

(iv) The dimension of $Z_d(1)$ at x is at most d,

where

$$\mathcal{I}_d(t) := \mathcal{L}^2(\mathcal{O}_{X_1}, (\sum_{i=1}^{k_d} |s_i^{(d)}|^2)^{-t}) \quad \text{for every } t \geq 0$$

be the multiplier ideal sheaf and where $Z_d(t) := V\mathcal{I}_d(t)$ be the complex subspace of X_1 defined by the ideal sheaf $\mathcal{I}_d(t)$. □

We note that, by the vanishing theorem: Proposition 2.3, there exist a finite number of multivalued holomorphic sections $\{t_i\}_{i=1}^{K}$ of L on X_1 such that $X_0 \cap \bigcap_{i=1}^{K}(t_i)_0$ is empty. We verify the first step:

LEMMA 4.3. *$(*)_{d_0}$ holds.*

Proof. We set

$$\mathcal{I}_*(t) := \mathcal{L}^2(\mathcal{O}_X, (\sum_{i=1}^{M} |f^* y_i|^2)^{-t}) \quad \text{for every } t \geq 0;$$
$$\alpha_* := \sup\{t \geq 0 \; ; \; \mathcal{I}_*(t)_x = \mathcal{O}_{X,x}\}.$$

We see that every $\mathcal{I}_*(t)$ is a coherent ideal sheaf and that α_* is a positive rational number. We consider the complex subspace $V\mathcal{I}_*(\alpha_*)$ defined by the ideal sheaf $\mathcal{I}_*(\alpha_*)$. This space $V\mathcal{I}_*(\alpha_*)$ is compact and $x \in V\mathcal{I}_*(\alpha_*) \subset f^{-1}(f(x))$. We choose a positive rational number $0 < \varepsilon_{d_0} < \varepsilon(d_0)$ and set the multivalued holomorphic sections

$$\{s_i^{(d_0)}\}_{i=1}^{k_{d_0}} := \{f^* y_i^{\alpha_*} \times t_j^{\varepsilon_{d_0}}\}_{i,j}$$

of $L^{\otimes \varepsilon_{d_0}}$ on X_1. Then we can verify $(*)_{d_0}$ by the following relation on X_0:

$$\mathcal{I}_{d_0}(t) := \mathcal{L}^2(\mathcal{O}_{X_1}, (\sum_{i=1}^{k_{d_0}} |s_i^{(d_0)}|^2)^{-t}) = \mathcal{I}_*(t\alpha_*).$$

□

4B. Concentration of the singularity

In this subsection we verify the induction step. We assume $(*)_d$ with $d > 0$. Let p be a positive integer such that $p(m_d + \varepsilon_d)$ being integer and that $(s_i^{(d)})^p$ $(1 \le i \le k_d)$ is a holomorphic section of $L^{\otimes p(m_d+\varepsilon_d)}$ on X_1. Let $\mathfrak{J}'$ be the sheaf of ideal of $\mathcal{O}_{X_0}$ generated locally by $\{(s_i^{(d)})^p|_{X_0}\}_{i=1}^{k_d}$. By the assumption $(*)_d$ (i), we can extend $\mathfrak{J}'$ as a coherent ideal sheaf $\mathfrak{J}$ of $\mathcal{O}_X$ by setting $\mathfrak{J} = \mathcal{O}_X$ on $X - X_0$. We take a modification $\pi : \widetilde{X} \longrightarrow X$ by a finite number of successive monoidal transforms with nonsingular centers and a family of smooth divisors E_i in $\widetilde{X}$ with only simple normal crossing so that the following three consitions hold:

(0) $\pi(E_i) \subset f^{-1}(f(x))$ for every i.

(1) The sheaf $\pi^{-1}\mathfrak{J}\cdot\mathcal{O}_{\widetilde{X}}$ which is the image of $\pi^*\mathfrak{J}$ under the natural map $\pi^*\mathfrak{J} \longrightarrow \mathcal{O}_{\widetilde{X}}$ is equal to the ideal sheaf $\mathcal{O}(-\sum r_i' E_i)$ for some non-negative integers r_i'.

(2) $K_{\widetilde{X}} = \pi^* K_X \otimes \mathcal{O}(\sum b_i E_i)$ for some non-negative integers b_i.

Let $r_i := r_i'/p$. The three conditions (ii)–(iv) in the statement $(*)_d$ can now be rewritten as the condition (ii)′–(iv)′ below. Let Λ be the set of all i so that E_i intersects $\pi^{-1}(x)$ and $r_i - b_i \ge 1$.

(ii)′ Λ is not empty;

(iii)′ $i \in \Lambda$ then $r_i - b_i = 1$;

(iv)′ $i \in \Lambda$ then $\dim \pi(E_i) \le d$.

We may assume that the index $i = 0$ is an element of Λ and that $\dim \pi(E_0) = \max\{\dim \pi(E_i) \ ; \ i \in \Lambda\}$.

We choose ε_{d-1} such that $\varepsilon(d) < \varepsilon_{d-1} < \varepsilon(d-1)$. Let $Z := \pi(E_0) \subset f^{-1}(f(x))$. If the dimension of Z is less than d, then for $(*)_{d-1}$ we simply choose multivalued holomorphic sections

$$\{s_i^{(d-1)}\}_{i=1}^{k_{d-1}} := \{s_i^{(d)} \times t_j^{(m_{d-1}+\varepsilon_{d-1})-(m_d+\varepsilon_d)}\}_{i,j}$$

of $L^{\otimes(m_{d-1}+\varepsilon_{d-1})}$ on X_1. We now assume without loss of generality that the dimension of Z is precisely d.

We take a local smooth holomorphic curve Γ in $E_0 - \bigcup\{E_i \ ; \ i \notin \Lambda\}$ with the following three properties:

(3) Γ intersects $\pi^{-1}(x)$ at one point.

(4) Γ either does not intersect $\bigcup_{i \neq 0} E_i$ or intersects $\bigcup_{i \neq 0} E_i$ only at the point $\Gamma \cap \pi^{-1}(x)$.

(5) Γ either does not intersect $\pi^{-1}(\mathrm{Sing}\ Z)$ or intersects $\pi^{-1}(\mathrm{Sing}\ Z)$ only at the point $\Gamma \cap \pi^{-1}(x)$.

The image $\pi(\Gamma)$ is a local holomorphic curve in Z. Let Δ be the unit disk in $\mathbb{C}$. By replacing Γ by a relatively compact open neighborhood of $\Gamma \cap \pi^{-1}(x)$ in Γ, we may assume that there is a nomalization $\sigma : \Delta \longrightarrow \pi(\Gamma)$ of $\pi(\Gamma)$ which is one to one and $\sigma(0) = x$.

We take a positive integer N such that $d/N < \varepsilon_{d-1} - \varepsilon(d)$. By Lemma 3.1, replace Δ with a smaller disk if necessary, there exist a positive integer m and a finite number of holomorphic sections $\{\widetilde{\tau}_j\}_{j=1}^{K_d} \subset H^0(X_1 \times \Delta,\ pr_X^* L^{\otimes m(N+1)})$ such that

$$\tau_{j,u}|_Z \in H^0(Z, L^{\otimes m(N+1)} \otimes \mathcal{M}_{Z,\sigma(u)}^{mN}) \quad \text{for every } u \in \Delta - 0$$

and for every j, and that

$$x \in X_0 \cap \bigcap_{j=1}^{K_d} (\tau_j)_0 \subsetneqq Z,$$

where $\tau_{j,u} := \widetilde{\tau}_j|_{X_1 \times u}$ (we regard $\tau_{j,u} \in H^0(X_1, L^{\otimes m(N+1)})$) and $\tau_j := \tau_{j,0}$.

Then we take a positive rational number ε such that $\varepsilon m r_0(1 + N) + d/N < \varepsilon_{d-1} - \varepsilon(d)$. For every $u \in \Delta$ and for every $t \geq 0$, we set

$$\mathcal{I}(u,t) := \mathcal{L}^2\left(\mathcal{O}_{X_1}, \left(\sum |s_i^{(d)}|^2\right)^{-(1-\varepsilon)} \left(\sum |\tau_{j,u}|^2\right)^{-t/(mN)}\right);$$
$$\alpha(u) := \sup\{t \geq 0 \ ; \ \mathcal{I}(u,t)_{\sigma(u)} = \mathcal{O}_{X,\sigma(u)}\}.$$

We see that every $\alpha(u)$ is a positive rational number and that

$$x \in (V\mathcal{I}(0, \alpha(0)) \cap X_0) \subsetneqq Z \subset f^{-1}(f(x)).$$

We would like to estimate $\alpha(0)$ by d and $O(\varepsilon)$ (Lemma 4.6 below). The following semicontinuity lemma of multiplier ideal sheaves due to Angehrn and Siu is the key step to reduce the case $x \in \mathrm{Sing}\ Z$ to the case $x \in \mathrm{Reg}\ Z$.

LEMMA 4.4. ([AS Lemma 6.1]) *Let t_0 be a positive number. Assume that $\alpha(u) < t_0$ for almost all $u \in \Delta - 0$ with respect to the 2-dimensional Lebesgue measure on Δ. Then $\alpha(0) \leq t_0$ holds.*

The outline of the proof is as follows: Assume that $\alpha(0) > t_0$. Then the following theorem of Ohsawa and Takegoshi shows that $\alpha(u) \geq t_0$ for almost all $u \in \Delta - 0$ which is a contradiction.

OHSAWA-TAKEGOSHI'S L^2-EXTENSION THEOREM 4.5. ([OT]) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinate $z_1, \ldots, z_n, w$. Let H be a complex hyperplane defined by $w = 0$, and let ϕ be a plurisubharmonic function on Ω. Then there exists a constant C_Ω depending only on the diameter of Ω such that; for any holomorphic function f on $\Omega \cap H$ satisfying*

$$\int_{\Omega\cap H} |f|^2 e^{-\phi} dV_n < \infty,$$

where dV_n denotes the $2n$-dimensional Lebesgue measure, there exists a holomorphic function F on Ω satisfying $F|_{\Omega\cap H} = f$ and

$$\int_\Omega |F|^2 e^{-\phi} dV_{n+1} \leq C_\Omega \int_{\Omega\cap H} |f|^2 e^{-\phi} dV_n.$$

Then we have

LEMMA 4.6. $\alpha(0) \leq d + mNr_0\varepsilon$.

Proof. By Lemma 4.4, it is enough to estimate $\alpha(u)$ for $u \in \Delta - 0$. We show $(\sum |s_i^{(d)}|^2)^{-(1-\varepsilon)}(\sum |\tau_{j,u}|^2)^{-t/(mN)}$ is not locally integrable at $\sigma(u)$ for $t \geq d + mNr_0\varepsilon$.

We take $u \in \Delta - 0$ and a point $\widetilde{x}$ in $\pi^{-1}(\sigma(u)) \cap \Gamma$. We see $\widetilde{x} \in E_0$ and $\widetilde{x} \notin \bigcup_{i\neq 0} E_i$. Let W be an open neighborhood of $\widetilde{x}$ in $\widetilde{X} - \bigcup_{i\neq 0} E_i$ so that a local coordinate $z, w_1, \ldots, w_{n-1}$ on W with $E_0 \cap W$ defined by $\{z = 0\}$, where n is the dimension of X. Since π maps the $(n-1)$-dimensional manifold E_0 onto the irreducible d-dimensional subvariety Z, it follows that the codimension of $\pi^{-1}(x) \cap E_0$ in E_0 is at most d at $\widetilde{x}$. The restriction $(\pi^*\tau_{j,u})|_{E_0}$ vanishes to order at least mN at $\widetilde{x}$. Let $\mathrm{Jac}(\pi)$ be the holomorphic Jacobian determinant of the map π. On W the divisor of $\mathrm{Jac}(\pi)$ is presicely b_0E_0. To conclude the local non-integrability of $(\sum |s_i^{(d)}|^2)^{-(1-\varepsilon)}(\sum |\tau_{j,u}|^2)^{-t/(mN)}$ at $\sigma(u)$, it suffices to prove the local non-integrability of

$$|z|^{-2(1-\varepsilon)r_0}(\sum |\pi^*\tau_{j,u}|^2)^{-t/(mN)}|z|^{2b_0} \quad \text{at } \widetilde{x}.$$

Since $r_0 - b_0 = 1$, by Lemma 3.2, $t \geq d + mNr_0\varepsilon$ implies the non-integrability. Hence $\alpha(u) < d + mNr_0\varepsilon$ for $u \in \Delta - 0$. ∎

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We set $\mathcal{I}(t) := \mathcal{I}(0,t)$ and $\alpha := \alpha(0)$. Then by Lemma 4.6, we see $(d+\varepsilon_{d-1}) - \alpha(1+1/N) > \varepsilon(d)$. We set the multivalued holomorphic section

$$\{s_i^{(d-1)}\}_{i=1}^{k_{d-1}} := \{s_i^{(d)(1-\varepsilon)} \times \tau_j^{\alpha/(mN)} \times t_k^{\varepsilon(m_d+\varepsilon_d)+(d+\varepsilon_{d-1})-\alpha(1+1/N)}\}_{i,j,k}$$

of $L^{\otimes(m_{d-1}+\varepsilon_{d-1})}$ on X_1. We can verify $(*)_{d-1}$ by noting that $\mathcal{I}_{d-1}(t) \supset \mathcal{I}(t\alpha)$ for $0 \le t \le 1$ and that $\mathcal{I}_{d-1}(1) = \mathcal{I}(\alpha)$, where

$$\mathcal{I}_{d-1}(t) := \mathcal{L}^2(\mathcal{O}_{X_1}, (\sum |s_i^{(d-1)}|^2)^{-t}).$$

4C. Completion of the proof

In 4A and 4B, we showed that $(*)_0$ hold. Therefore there exist a rational number ε_0 with $0 < \varepsilon_0 < 1$ and a finite number of multivalued holomorphic sections $\{s_i^{(0)}\}_{i=1}^{k_0}$ of $L^{\otimes(m_0+\varepsilon_0)}$ on X_1 such that x is isolated in $V\mathcal{L}^2(\mathcal{O}_{X_1}, (\sum |s_i^{(0)}|^2)^{-1})$. We take a smooth Hermitian metric h_1 of $L|_{X_1}$ such that the curvature form gives a complete Kähler form on X_1. Then we get the desired singular Hermitian metric

$$H_m := h_1^{m-(m_0+\varepsilon_0)} \frac{h_1^{m_0+\varepsilon_0}}{\sum |s_i^{(0)}|_1^2}$$

for every integer $m > m_0 = d_0(d_0+1)/2$, where $|s_i^{(0)}|_1$ is the length with respect to h_1.

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