SOME SEMIGROUPS HAVING QUASI-FROBENIUS ALGEBRAS. II

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The investigation of finite semigroups S with quasi-Frobenius (q.-F.) algebras $F(S)$ over a field F was begun in (**7**; **8**). The problem for commutative semigroups was reduced (**7**, Theorem 3) to the study of semigroups of the form $S = G \cup S_1$, where G is a group and S_1 is either the null set or is a nilpotent ideal in S (i.e., $S_1^n = \{0\}$ for some positive integer n). Such semigroups were called "of type C". The question is "When does a semigroup of type C have a q.-F. algebra over a field?" (**7**, Theorem 4) shows that no distinction need be made between the properties q.-F. and Frobenius for commutative algebras.

In § 1, the J-class semigroup T is assigned to the commutative semigroup S under the homomorphism which assigns to each element of S its J-class. Theorem 1 concludes that $F(T)$ is q.-F. if $F(S)$ is q.-F. Theorem 2 provides a necessary and sufficient condition for a J-class semigroup (or a semigroup of type C′) to have a q.-F. algebra.

Theorems 3 and 4 in § 2 give necessary conditions for a semigroup of type C to have a q.-F. algebra. These conditions also describe the relationship between the principal indecomposable modules of $F(S)$ and those of $F(G)$. The last section provides a method by which some semigroups of type C can be constructed from semigroups of type C′ and subgroups of an arbitrary finite abelian group. Theorem 6 gives a characterization of semigroups constructed in this way which have q.-F. algebras.

The terminology is the same as that in (**1**; **7**; **8**). If S is of type C, then S has an identity (**4**) and S may be assumed to be a subsemigroup of $F(S)$.

1. Semigroups of type C and C′ and their algebras. Throughout this discussion, $S = S_0 \supset S_1 \supset \ldots \supset S_{r+1}$ will always denote a principal series for a semigroup S of type C. As a group ring over a field is always q.-F., $S_1 \neq \emptyset$ will be assumed so that $S_{r+1} = \{0\}$. The sets $J_i = S_i - S_{i+1}$ (the set complement of S_{i+1} in S_i), $i = 0, 1, \ldots, r$, are called the J-classes of S. If s_i is a fixed element of J_i, then $J_i = \{s \in S\colon sS = s_iS\}$. The identity of the group G of S is also the identity for S; therefore, $G = S - S_1$.

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LEMMA 1. (i) *If* $a, b \in J_i$, $0 \leqq i \leqq r + 1$, *and* $s \in S$ *such that* $a = sb$, *then* $s \in G$ *or* $a = b = 0$.

(ii) $S_iS_j \subseteq S_{j+1}$ *if* $1 \leqq i \leqq j < r + 1$.

Proof. (i) Let $a, b \in J_i$. Then $Sa = Sb$; hence, there are $s, s' \in S$ such that $a = sb$ and $s'a = b$. Then $a = sb = (ss')a$; thus, $a = (ss')^k a$ for each positive integer k. If either s or s' is in S_1, then $ss' \in S_1$; therefore, $a = b = 0$ as S_1 is nilpotent. This implies that $s, s' \in G$ or $a = b = 0$.

(ii) Let $a \in S_i$, $b \in S_j$, $r \geqq j \geqq i \geqq 1$. Then $ab \in S_j$ as S_j is an ideal of S. If b or ab is in S_{j+1}, then the result is true. If $b, ab \in J_j$, then $Sab = Sb$; thus, there exists an $s \in S$ such that $s(ab) = b$. Then $(sa)b = b$ and by the first part, either $sa \in G$ or $b = 0$, neither of which is true. Thus, either b or ab is in S_{j+1}, and the lemma follows.

The way in which the elements of G act on the elements of S is described in the next lemma. If X is a set, let $|X|$ denote its cardinality.

LEMMA 2. *Let* $g \in G$. *Then*

(i) $ga = gb$ *in* S *if and only if* $a = b$;

(ii) $gJ_i = J_i$ *for each* $i = 0, 1, \ldots, r + 1$;

(iii) *If* $a, b \in J_i$, *then* $ga = a$ *implies* $gb = b$;

(iv) G *is transitive as a permutation group on the set* J_i, $i = 0, 1, \ldots, r + 1$;

(v) *If* $G_i = \{g \in G: ga = a, a \in J_i\}$, *then*

$$|G| = |G_i|\ |J_i|, \qquad i = 0, 1, \ldots, r + 1;$$

(vi) *If* $a \in J_i$, $b \in J_j$, *and* $ab \in J_k$, *then* $G_iG_j \subseteq G_k$.

Proof. (i) is clear since the identity of G is the identity of S.

(ii) If $a \in J_i$ and $ga \in S_{i+1}$, then $a \in g^{-1}S_{i+1} \subseteq S_{i+1}$, a contradiction. Thus $ga \in J_i$. Then (i) implies $gJ_i = J_i$ since J_i is finite.

(iii) If $a, b \in J_i$, then $sa = b$ for some $s \in S$. If $ga = a$, then $gb = g(sa) = s(ga) = sa = b$.

(iv) G is a permutation group on the set J_i by (i) and (ii). If $a, b \in J_i$, then there is an $s \in S$ such that $sa = b$. Then $s \in G$ by Lemma 1, hence G is transitive on J_i.

(v) This follows from (iv) and a well-known theorem for permutation groups (**9**, p. 5, Theorem 3.2).

The subgroup G_i of G will be called the *fixing group* for the J-class J_i, $i = 0, 1, \ldots, r + 1$. Note that $G_0 = \{e\}$ if e is the identity of G, and $G_{r+1} = G$ since $S_{r+1} = \{0\}$. As S is commutative, the equivalence relation ρ, defined by $a\rho b$ if, and only if, a and b are in the same J-class, is a congruence. Then $S/\rho = \{J_i: i = 0, 1, \ldots, r + 1\}$ is a semigroup of type C with precisely one element in each J-class and with principal series of the same length as those of S (**1**, p. 16). A semigroup of type C with one element in each J-class will be called of type C′. It is convenient to construct a semigroup $T = \{t_0, t_1, \ldots, t_{r+1}\}$ isomorphic to S/ρ by defining $t_it_j = t_k$ if $J_iJ_j = J_k$ in S. Then $\phi'(a) = t_i$ for $a \in J_i$, $i = 0, 1, \ldots, r + 1$, is an epimorphism of S to T.

Extend ϕ' linearly to the algebra homomorphism ϕ: $F(S) \to F(T)$. The mapping ϕ will be called the *J-homomorphism* of $F(S)$. The kernel of ϕ is

$$K = \left\{ \sum_{s \in S} \alpha_s s : \sum_{s \in J_i} \alpha_s = 0, i = 0, 1, \ldots, r \right\}.$$

If $X \subseteq F(S)$, then $F[X]$ denotes the linear subspace of $F(S)$ which is spanned by X, $A(X) = \{y \in F(S): xy = 0 \text{ for each } x \in X\}$, and $A'(X) = \{s \in S: xs = 0$ for each $x \in X\}$. The characteristic of a "generic field" F will be represented by c or $c(F)$ for emphasis. Let $E_i = \sum_{s \in J_i} s$ and let $E = E_0 = \sum_{g \in G} g$ throughout this article. Note that if $g \in G$ and $s \in J_k$, then $gE_i = E_i$ and $Es = |G_k|E_k$ by Lemma 2. Theorem 1 will show that if $F(S)$ is q.-F., then $c \nmid |G_i|$, $k = 1, \ldots, r$, and $F(T)$ is q.-F. This will be done as follows. Refine the ideal series $S \supset A'(E)$ to a principal series (1): $S = S_0 \supset S_1 \supset \ldots \supset S_{r+1}$ for S with $A'(E) = S_{p+1}$, say. Let this be the series on which all the notation depends; i.e., J_k, G_k, E_k, etc. Lemma 4 will establish that $A(E_p) = A(E_r)$. If $F(S)$ is q.-F., then $A(E_p) = A(E_r)$ implies that $F(S)E_p = A(A(E_p)) = A(A(E_r)) = F(S)E_r$; thus, $p = r$ by Lemma 2 (ii). However, $p = r$ implies that $A'(E) = \{0\}$. As $Es_k = |G_k|E_k$, $A'(E) = \{0\}$ implies that $c \nmid |G_k|$ for each $k = 1, 2, \ldots, r$. In this case, Lemma 3 implies that $A(K) = F(S)E$; thus $F(S)/K \cong F(T)$ is q.-F. (**5**, Theorem 9). Thus, the lemmas which follow yield the proof of Theorem 1.

LEMMA 3. *The ideal* $A(K) = \sum_{i=0}^{r} F[E_i]$. *If* $(c, |G_i|) = 1$, $i = 1, 2, \ldots, r$, *then* $A(K)$ *is principal and* $A(K) = F(S)E$.

Proof. The same calculation as for groups, together with Lemma 2 (iv), shows that K is spanned by all elements of the form $(g - e)s$, $g \in G$, $s \in S$. Thus, $y \in A(K)$ if, and only if, $(g - e)y = 0$ for each $g \in G$. Let $y = \sum_{i=0}^{r} y_i$, $y_i \in F[J_i]$. Suppose that αs, $\alpha \in F$, is a non-zero summand in the unique expression for y_i as an F-linear combination of the elements of J_i. As $gy = y$ for each $g \in G$, $gy_i = y_i$ for each $g \in G$ by Lemma 2 (ii). Let $\beta s'$, $s' \in J_i$, $\beta \in F$, be another summand of y_i ($\beta \neq 0$ is not assumed). As $Gs = J_i$, there is a $g \in G$ such that $gs = s'$. Then $g(\alpha s) = \alpha s'$; thus, the coefficient of s' in gy_i is α. The coefficient of s' in y_i is β and as $gy_i = y_i$, one has $\alpha = \beta$. Thus, $y_i = \alpha E_i$ for some $\alpha \in F$ and $A(K) = \sum_{i=0}^{r} F[E_i]$.

As $Es_i = |G_i|E_i$ for each $i = 0, 1, \ldots, r$, if $(c, |G_i|) = 1$ for each i, then $E_i = |G_i|^{-1}s_iE \in F(S)E$; therefore $A(K) = F(S)E$.

If $c \mid |G|$, then the radical of $F(G)$ is $\sum (e - g)F(G)$, where the sum is taken over all elements $g \neq e$ in the c-Sylow subgroup of G (**2**, p. 435). Then, as $gE_i = E_i$ for each i and for each $g \in G$, E_i annihilates the radical of $F(G)$. As E_r also annihilates S_1 and rad $F(S) = F(S_1) + \text{rad } F(G)$ (**7**, Lemma 5), E_r annihilates rad $F(S)$.

LEMMA 4. *If* p *is as above, then* $E_p \in A(\text{rad } F(S))$. *Moreover,* $A(E_p) = A(E_r)$.

Proof. By the remarks above, it is sufficient to prove that $E_pS_1 = \{0\}$.

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As $Es = |G_p|E_p \neq \{0\}$ for $s \in J_p$, one has $(c, |G_p|) = 1$. Then $E_p = |G_p|^{-1}sE$. If $p > 0$ and $s \in J_p$, then $E_pS_1 = |G_p|^{-1}E_sS_1 = \{0\}$ as $sS_1 \subseteq A'(E) = S_{p+1}$. If $p = 0$, then $A'(E) = S_1$ and $E_pS_1 = EA'(E) = \{0\}$ also. Thus,

$$E_p \in A(\text{rad } F(S)).$$

For each k, $A(E_k) = (A(E_k) \cap F(G)) + (A(E_k) \cap F(S_1))$. Then $A(E_p) = (A(E_p) \cap F(G)) + F(S_1)$ and $A(E_r) = (A(E_r) \cap F(G)) + F(S_1)$. However, $x = \sum_{g \in G} \alpha_g g \in A(E_k)$ if, and only if, $xE_k = (\sum_{g \in G} \alpha_g)E_k = 0$; i.e., if, and only if, $\sum_{g \in G} \alpha_g = 0$. Thus, $A(E_p) \cap F(G) = A(E_r) \cap F(G)$; hence $A(E_p) = A(E_r)$.

Lemmas 3 and 4 together with preceding remarks yield the following theorem.

THEOREM 1. *If $F(S)$ is q.-F., and T is the J-class semigroup for S, then $c \nmid |G_k|$ for each $k = 0, 1, \ldots, r$, and $F(T)$ is q.-F.*

Note that if $(c, |G|) = 1$, the elements $n_i^{-1} E_i$, $i = 0, 1, \ldots, r + 1$, n_i the index of G_i in G, are F-independent and a simple calculation shows that they form a multiplicative semigroup which is isomorphic with $T(n_i^{-1}E_i \rightarrow t_i$ for each $i)$. Then $F(S)E \cong F(T)$; hence, $F(T)$ is actually isomorphic to a direct summand of $F(S)$.

Characterizations of semigroups T of type C′ which have q.-F. algebras are thus of interest. A general result about commutative q.-F. rings is needed. Kupisch (**3**) has proved that a commutative ring R with minimum condition is q.-F. if, and only if, each ideal Rf, with f a primitive idempotent, has a simple socle; i.e., a unique simple R-submodule. If T is of type C′, then a principal series for T is of the form $T = T_0 \supset T_1 \supset \ldots \supset T_{r+1}$, where $T_i - T_{i+1} = \{t_i\}$, $i = 0, 1, \ldots, r + 1$, $T_{r+1} = \{0\}$, t_0 is the identity of T, and T_1 is nilpotent. This notation will be used in what follows. Note that $T_1 t_r \subseteq \{t_r, 0\}$.

LEMMA 5. *If T is of type C′, then $F(T)$ is q.-F. if, and only if, x divides t_r for each non-zero $x \in F(T)$.*

Proof. The only non-zero idempotent of $F(T)$ is t_0 (**7**, Lemma 5); thus, t_0 is the only primitive idempotent of $F(T)$ and $F(T)t_0 = F(T)$. As $F(T)t_r$ is simple, $F(T)$ is q.-F. if, and only if, $F(T)x \supseteq F(T)t_r$ for each non-zero $x \in F(T)$, by the result of Kupisch. This completes the proof.

A more useful characterization can be obtained by using the remarks in (**7**, § 4; **5**, § 2). Suppose that $S = \{s_0, s_1, \ldots, s_{r+1}\}$ is an arbitrary finite semigroup with $s_{r+1} = 0$ if $0 \in S$ (if $0 \notin S$, let $S = \{s_0, s_1, \ldots, s_r\}$). Only the case with $0 \in S$ will be treated, as the remaining case is similar. Let $\lambda_0, \lambda_1, \ldots \lambda_{r+1}$ be parameters representing elements of F with $\lambda_{r+1} = 0$. Let $\Lambda = [\alpha_{ij}]$ be the $(r + 1) \times (r + 1)$ matrix with $\alpha_{ij} = \lambda_k$ if $s_i s_j = s_k$. Then $F(S)$ is Frobenius if, and only if, the parameters λ_k can be chosen so that the corresponding "intertwining" matrix Λ is non-singular. The next theorem uses the matrix Λ which corresponds to a semigroup T of type C′. This theorem will give a constructive and intrinsic method for deciding whether a semigroup of type C′ has a q.-F. algebra without considering the algebra itself. As before, let $T = T_0 \supset T_1 \supset \ldots \supset T_{r+1} = \{0\}$ be a principal series for T with $T_i - T_{i+1} = \{t_i\}$. Then $T_1 t_r \supseteq \{t_r, 0\}$.

THEOREM 2. *If T is of type C′, then $F(T)$ is q.-F. if, and only if, the matrix Λ of parameters for T is non-singular when $\lambda_r = 1$ and $\lambda_k = 0$ if $k \neq r$.*

Proof. If such a matrix exists, then $F(T)$ is Frobenius (hence, q.-F.) by the remarks above. Thus, suppose that $F(T)$ is q.-F. (hence, Frobenius, as $F(T)$ is commutative (**7**, Theorem 4)). Let $\Lambda = [\alpha_{ij}]$ be the $(r+1) \times (r+1)$ matrix described in the theorem. Note that Λ is symmetric by the commutativity of T. Suppose that the rows R_k of Λ are dependent, say $\sum_{k=0}^{r} \beta_k R_k$ is a zero row vector. As $t_0 t_k = t_r$ if, and only if, $k = r$, $\alpha_{k,0} = \alpha_{0,k} = 0$ if $k \neq r$ and $\alpha_{r,0} = \alpha_{0,r} = 1$. Thus, $\beta_0 = \beta_r = 0$. Let $x = \sum_{k=1}^{r-1} \beta_k t_k \in F(T)$. If $t_j \in T$ and $x t_j = \sum_{k=1}^{r-1} \beta_k (t_k t_j) = \sum_{q=1}^{r} \gamma_q t_q$, then $\gamma_r = 0$ will be proved. One has that $\gamma_r = \sum \beta_k$, if this summation is taken over all k such that $t_k t_j = t_r$. However, this sum is also the entry in the jth position of the zero row vector $\sum_{k=0}^{r} \beta_k R_k$, as $\alpha_{kj} = 1$, if, and only if, $t_k t_j = t_r$. This is true for each $t_j \in T$, hence x does not divide $F(T)$. If $x \neq 0$, Lemma 5 yields a contradiction. Thus, $x = 0$, hence $\beta_k = 0$ for each k and the rows of Λ are independent, as desired.

Suppose that S is of type C with J-class semigroup T. Let J_r be the J-class such that $\phi: J_r \to t_r$ as before. A corollary can be stated using this notation.

COROLLARY 1. *If S is of type C and $F(S)$ is q.-F., then for each non-zero $b \in S$ and for each $a \in J_r$, b divides a.*

Proof. Theorem 1 shows that $F(S)$ q.-F. implies $F(T)$ q.-F. Then t divides t_r for each non-zero $t \in T$ (Lemma 5). If $\phi': S \to T$ is the J-class homomorphism, then $a \in J_r$ implies that $\phi'(a) = t_r$. If $b \neq 0$ in S, then $\phi'(b) \neq 0$ in T; hence, there exists an element $t = \phi'(c) \in T$ such that $\phi'(a) = \phi'(b)\phi'(c) = \phi'(bc)$. The definition of ϕ' implies that $(bc)S = aS$, therefore there is an $s \in S$ such that $b(cs) = a$.

2. Primitive idempotents and the fixing groups.

In this section, more necessary conditions for $F(S)$ to be q.-F. are found using the relationship between the primitive idempotents of $F(S)$ and the fixing groups G_i of the J-classes of S. All non-zero idempotents of $F(S)$ are in $F(G)$ (**7**). The result of Kupisch will be used. Let f be an idempotent of $F(S)$. The next theorem shows that an irreducible $F(S)$-submodule of $F(S)f$ can be constructed from an irreducible $F(G)$-submodule of $F(G)f$ by multiplying by an appropriate element of S.

THEOREM 3. *Let f be a primitive idempotent in $F(S)$. If $F(S)$ is q.-F., then the following conditions hold.*

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(i) *There is an* a $(\neq 0)$ *in* S *such that if* $s \in S$ *and* $s \notin A'(f)$, *then* s *divides* a *in* S.

(ii) *If* $F(G)u$ *is the unique* $F(G)$*-irreducible submodule of* $F(G)f$, *then there exists a* $b \in S$, *such that* $F(G)ub$ *is the unique* $F(S)$*-irreducible submodule of* $F(S)f$ (*clearly,* b *divides* a *in* S). *Moreover, if* $s \in S$ *and* $s \notin A'(u)$, *then* s *divides* b *in* S.

Proof. Refine the ideal series $S \supset A'(f)$ to a principal series $S = S_0 \supset S_1 \supset \ldots \supset S_{r+1}$, with $S_{p+1} = A'(f)$ and let $J_p = S_p - S_{p+1}$. Let M be the unique irreducible $F(S)$-submodule of $F(S)f$. If $a \in J_p$, then $F[J_p] = F[Ga] \supseteq F(S)af \supseteq M$, by Lemma 2(iv), the choice of p, and the uniqueness of M. If $s \in A'(f)$, then $F(S)sf \supseteq M$ also. Thus, $F[J_p] \cap F(S)sf \supseteq M \neq \{0\}$. Then $Ss \cap J_p \neq \{0\}$ and, as $a \in J_p$, Lemma 2(iv) implies that s divides a.

Next refine $S \supset A'(u)$ to a principal series $S = S_0 \supset \ldots \supset S_{r+1}$ for S with $S_{q+1} = A'(u)$. Let $b \in J_q = S_q - S_{q+1}$. Since $S_1 b \subseteq A'(u)$, $S_1 ubf = \{0\}$; hence, $F(S)ub = [F(G) + F(S_1)]ufb = F(G)ub$. However, u annihilates the radical of $F(G)$ and fb annihilates $F(S_1)$; therefore, $F(G)ub$ annihilates the radical of $F(S)$. Thus, $F(G)ub$ is a sum of irreducible $F(S)$-submodules of $F(S)f$. As $F(S)f$ contains precisely one such submodule, $F(G)ub$ must be $F(S)$-irreducible.

Note that if $(c, |G|) = 1$, then $F(G)$ is semisimple and u may be set equal to f and a set equal to b.

The result of Kupisch makes it clear that if $e = e_1 + \ldots + e_n$ is a decomposition of the identity of $F(S)$ into a sum of pairwise orthogonal primitive idempotents and if condition (ii) of Theorem 3 holds for each e_i, then $F(S)$ is q.-F.

Some additional information is needed concerning idempotents in a group ring. Certain subgroups of G will be associated with idempotents in $F(G)$. If $x = \sum_{g \in G} \alpha_g g$, let $||x|| = \sum_{g \in G} \alpha_g$. Clearly, if $x, y \in F(G)$, then $||x + y|| = ||x|| + ||y||$.

LEMMA 6. *If* f *is an idempotent in* $F(G)$, *then* $||f||$ *is zero or one.*

Proof. All summations run over the elements of G. Let $f = \sum_g \alpha_g g$. Then

$$\sum_g \alpha_g g = \Big(\sum_g \alpha_g g\Big)\Big(\sum_h \alpha_h h\Big) = \sum_g \sum_h \alpha_g \alpha_h gh.$$

If $k = gh$, then $\sum_g \alpha_g g = \sum_{g^{-1}k} (\sum_g \alpha_g \alpha_{g^{-1}k}) k$; hence, $\sum_g \alpha_g \alpha_{g^{-1}k} = \alpha_k$ for each $k \in G$. Summing on k, one has that

$$||f|| = \sum_k \alpha_k = \sum_k \sum_g \alpha_g \alpha_{g^{-1}k} = \sum_g \alpha_g \Big(\sum_k \alpha_{g^{-1}k}\Big) = \Big(\sum_g \alpha_g\Big)^2 = ||f||^2.$$

If $e = e_1 + \ldots + e_n$ is a decomposition of the identity e of $F(G)$ into a sum of idempotents e_i, the lemma implies that $1 = ||e|| = ||e_1|| + \ldots + ||e_n||$; thus, for exactly one i, say $i = 1$, $||e_1|| = 1$, and $||e_i|| = 0$ for $i = 2, \ldots, n$.

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If $e_i = \sum_g \alpha_g g$, let $H_i = \{H: H$ is a subgroup of G and $\sum_{g \in kH} \alpha_g = 0$ for each $k \in G\}$. Let G be the group associated with a semigroup S of type C. For $s \in S$, let G_s be the subgroup of G that is the fixing group for the J-class that contains s. If R is a complete set of coset representatives for G_s in G, then $sh = sh'$, $h, h' \in R$, if, and only if, $h = h'$. Since

$$se_i = \sum_{g \in G} \alpha_g sg = \sum_{h \in R} \left(\sum_{g \in hG_s} \alpha_g \right) sh,$$

$s \in A'(e_i)$ if, and only if, $G_s \in H_i$. Lemma 6 implies that $G \in H_i$ if $i \geqq 2$ and $H_1 = \emptyset$. Note also that $A'(e_1) = \{0\}$. Assume that the e_i are pairwise orthogonal primitive idempotents and for each $i = 1, \ldots, n$, let e_i and b_i be related as are f and b in Theorem 3. The following necessary condition for $F(S)$ to be q.-F. can be stated with this notation.

THEOREM 4. *Let $F(S)$ be q.-F. and let $s \in S$. Then $G_s \in \bigcap H_i$, if the intersection is taken over all i such that s does not divide b_i in S.*

Proof. Suppose that $G_s \notin H_i$ and s does not divide b_i in S. Then $se_i \neq 0$; hence, $F(S)e_is \neq \{0\}$ and $F(S)e_is \not\supseteq F(S)u_ib_i$, contradicting the uniqueness of $F(S)u_ib_i$ in Theorem 3.

3. Semigroups of type C obtained from semigroups of type C′. In the preceding discussion, a semigroup S of type C was given and from it a group G and the J-class semigroup T of type C′ were obtained. This can be reversed. It will be described in a more general context first. Let T be an arbitrary finite (not necessary) semigroup, say $T = \{t_i: i = 0, 1, \ldots, r+1\}$ and let G be an arbitrary finite (not necessary) group. A collection of normal subgroups $\{G_i: i = 0, 1, \ldots, r+1\}$ of G is said to be *admissible relative to* T if $G_iG_j \subseteq G_k$ whenever $t_it_j = t_k$. Let $(G, T) = \{(g, t): g \in G, t \in T\}$ be the direct product of G and T. In (G, T) define the congruence σ as $(g, t_i)\sigma(h, t_j)$ if, and only if, $i = j$ and $g \in hG_i$. Then $S = (G, T)/\sigma$ is said to be the semigroup constructed from T and the admissible collection $\{G_i\}$. Note that if S' is the collection of equivalence classes with representatives (e, t_i), $i = 0, 1, \ldots, r+1$, then $S' \cong T$ and the intersection of S' with each J-class of S contains precisely one element. The next theorem shows that these conditions are also sufficient for a semigroup S of type C to be constructed in this way.

THEOREM 5. *A semigroup S of type* C *can be constructed from a semigroup T of type* C′ *and admissible subgroups of an abelian group G if, and only if, there exists a monomorphism μ: $T \rightarrow S$ such that $\phi\mu$ is an isomorphism of T onto the J-class semigroup of S.*

Proof. Suppose that $\phi\mu$ is an isomorphism of $T = \{t_i: i = 0, 1, \ldots, r+1\}$ onto the J-class semigroup of S. Then $\phi\mu t_i = \phi\mu t_j$ if, and only if, $i = j$; hence, μt_i and μt_j are in the same J-class of S only if $i = j$. The fixing groups G_i are determined by S and $g\mu t_i = g'\mu t_j$ if, and only if, $i = j$ and $g \in g'G_i$, as desired.

In the following discussion, S will be a semigroup of type C with group G and with a subsemigroup $S^* = \{s_i: i = 0, 1, \ldots, r+1\}$ such that S^* contains precisely one element of each J-class of S. Let $T = \{t_i: i = 0, 1, \ldots, r+1\}$ again denote the J-class semigroup of S, where

$$T = T_0 \supset T_1 \supset \ldots \supset T_{r+1}, \qquad T_i - T_{i+1} = \{t_i\},$$

is a principal series for T. Then $S^* \cong T$ and one may assume that the s_i's are labeled so that $\phi|_{S^*}: s_i \to t_i$ is the isomorphism. If $(c, |G|) = 1$, Theorem 6 will characterize semigroups S of this type such that $F(S)$ is q.-F. by decomposing $F(S)$ into a direct sum of semigroup rings which are formed from certain homomorphic images of T. Note that $F(S) = \sum_{i=0}^{r} F(G)s_i$; thus, for $x \in F(S)$,

$$F(S)x = \sum_{i=0}^{r} F(G)xs_i = \sum_{s_i \in S - A'(x)} F(G)xs_i.$$

LEMMA 7. *Let S be a semigroup of type* C *with group G and with a subsemigroup S^* such that S^* contains precisely one element from each J-class of S. Suppose that $(c(F), |G|) = 1$. If f is a primitive idempotent in $F(G)$, let $L = F(G)f$. Then $F(S)f \cong L(S^*/A'(f) \cap S^*)$, the semigroup ring for $S^*/A'(f) \cap S^*$ over the field L.*

Proof. Let $S^* = \{s_i: i = 0, 1, \ldots, r+1\}$, $s_0 = e$, $s_{r+1} = 0$, with $s_i \to \bar{s}_i$ under the natural mapping of S^* onto $S^*/A'(f) \cap S^*$. Define

$$\Psi: F(S)f \to L(S^*/A'(f) \cap S^*)$$

as

$$\Psi\Big(\sum_{s_i \in S - A'(f)} k_i s_i\Big) = \sum_{s_i \in S^* - A'(f)} k_i \bar{s}_i, \qquad k_i \in L.$$

If

$$\sum_{s_i \in S - A'(f)} k_i s_i = 0,$$

then as $k_i \in F(G)$, $k_i s_i = 0$ for each i by Lemma 2(ii). Since $L = F(G)f$ is $F(G)$-irreducible, if $k_i \neq 0$, there is a $y \in F(G)$ such that $yk_i = f$. Then $k_i s_i = 0$ implies that $0 = yk_i s_i = s_i f$, contradicting $s_i \in S - A'(f)$. Thus, $k_i = 0$ for each i; hence, Ψ is a function. Clearly, Ψ preserves sums, and, by the preceding remark, is one-to-one. Furthermore, $\Psi(s_i f)\Psi(s_j f) = f\bar{s}_i f\bar{s}_j = f\bar{s}_i\bar{s}_j = \Psi(s_i s_j f) = \Psi(s_i f s_j f)$ as $f \in L$; thus, by linearity, products are preserved and Ψ is an isomorphism.

This lemma, together with (**7**, Lemma 1), provides the proof of the next theorem.

THEOREM 6. *Let S be a semigroup of type* C *with a subsemigroup S^* as in Lemma 7. Suppose that $(c(F), |G|) = 1$ and $e = e_1 + \ldots + e_n$ is a decomposition of the identity e of $F(S)$ into pairwise orthogonal primitive idempotents. Let $L_i = F(G)e_i$. Then $F(S)$ is q.-F. if, and only if, $L_i(S^*/A'(e_i) \cap S^*)$ is q.-F. for each $i = 1, \ldots, n$.*

As $S^* = T$, the semigroup $S^*/A'(e_i) \cap S^*$ can be obtained from T. If F

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is a splitting field for G and $(c, |G|) = 1$, then

$$F(S) \cong F(S^*/A'(e_1) \cap S^*) \oplus \ldots \oplus F(S^*/A'(e_n) \cap S^*);$$

hence, $F(S)$ is q.-F. if, and only if, each $F(S^*/A'(e_i) \cap S^*)$ is q.-F. As $S^*/A'(e_i) \cap S^*$ is of type C′ for each i, the algebra $F(S)$ is a direct sum of semigroup algebras for semigroups of type C′. This theorem implies one of the conclusions of Theorem 1 in this more restrictive context. If $e_1 = |G|^{-1} \sum_{g \in G} g$, then $A'(e_1) = \{0\}$; thus, $F(S^*/A'(e_1)) \cong F(T)$ is q.-F. if $F(S)$ is q.-F. Theorem 6 also has the following corollaries.

COROLLARY 2. *Let G be a finite abelian group and let T be a semigroup of type* C′. *Let S be the semigroup constructed from T and the admissible collection $G_i = G$, $i = 0, 1, \ldots, r + 1$. If $(c, |G|) = 1$, then $F(S)$ is q.-F. if, and only if $F(T)$ is q.-F.*

Proof. That $F(S)$ q.-F. implies $F(T)$ q.-F. has already been proved. Let $e = e_1 + \ldots + e_n$ with the e_j pairwise orthogonal primitive idempotents and $e_1 = |G|^{-1} \sum_{g \in G} g$. Then $A'(e_1) = \{0\}$ and $A'(e_j) = S_1$ for $j > 1$. Then $S^*/A'(e_1) \cap S^* \cong T$ and $S^*/A'(e_j) \cap S^*$ is a one-element group with zero for $j > 1$. As all L_i's are fields and $L_1 \cong F$, $L_i(S^*/A'(e_i) \cap S^*)$ is q.-F. for each $i = 1, \ldots, n$; thus, $F(S)$ is q.-F. by the theorem.

COROLLARY 3. *Let G be a finite abelian group and let T be of type* C′ *with T_1 cyclic. Suppose that $(c, |G|) = 1$ and that G_i, $i = 1, 2, \ldots, r$, is any collection of subgroups of G which is admissible with respect to T. If S is constructed from T and these groups, then $F(S)$ is q.-F.*

Proof. Suppose that T_1 is generated by t, and r is the minimal positive integer such that $t^{r+1} = 0$. First note that if L is a field and T is a semigroup of the given type, then $L(T)$ is q.-F. This follows from the fact that the matrix of parameters for T is non-singular if a one is placed in positions which correspond to t^r and all other entries are zero. As every homomorphic image of a semigroup of this type is again of this type, we have that $S^*/A'(e_i) \cap S^*$ is of this form for $i = 1, \ldots, n$; hence, $L_i(S^*/A'(e_i) \cap S^*)$ is q.-F. for each i. The theorem implies that $F(S)$ is q.-F.

An example of a semigroup of type C which cannot be constructed from a semigroup of type C′ and admissible subgroups of some group follows.

s	e	g	a	b	c	d	f	h	0
e	e	g	a	b	c	d	f	h	0
g	g	e	b	a	d	c	h	f	0
a	a	b	h	f	0	0	0	0	0
b	b	a	f	h	0	0	0	0	0
c	c	d	0	0	f	h	0	0	0
d	d	c	0	0	h	f	0	0	0
f	f	h	0	0	0	0	0	0	0
h	h	f	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

It has no subsemigroup which contains one element from each J-class of S and is isomorphic to the J-class semigroup of S as $a^2 = b^2 = h$ and $c^2 = d^2 = f$ are in the same J-class. The algebra $F(S)$ is q.-F. for arbitrary fields as the matrix Λ of parameters obtained by replacing e and h by one and all other elements by zero in the table above (ignore the zero row and column) is non-singular.

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