BOUNDS ON MODIFIED STOP-LOSS PREMIUMS IN CASE OF KNOWN MEAN AND VARIANCE OF THE RISK VARIABLE

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ABSTRACT

In case of a stop-loss treaty the reinsurer takes over that part of the risk that exceeds a given amount y_1. We will deduce bounds on a modified stop-loss treaty where the liability of the reinsurer is limited to $y_2 - y_1$ in case the claim amount exceeds y_2. Upper and lower bounds of this modified stop-loss premium are obtained as a simple application of results obtained earlier by the first author.

INTRODUCTION

In case of a stop-loss treaty the insurer takes over that part of the risk that exceeds a given amount y_1. We now suppose that the stop-loss treaty is modified in such a way that the liability of the reinsurer is limited to $y_2 - y_1$ in case the claim amount exceeds the amount y_2. Hence, the risk of the reinsurer can be cast into the form

$$Y = \begin{cases} 0 & X \leqslant y_1 \\ X - y_1 & y_1 < X \leqslant y_2 \\ y_2 - y_1 & y_2 < X. \end{cases}$$

The net premium then equals:

$$E(Y) = \int_{y_1}^{y_2} (x - y_1)\, dF_X(x) + (y_2 - y_1) \int_{y_2}^{\infty} dF_X(x)$$

which can still be cast into the following form:

$$E(Y) = \int_a^b \max\{\min(x - y_1, y_2 - y_1), 0\}\, dF_X(x)$$

where $y_2 \geqslant y_1$, $F_X(a) = 0$, $F_X(b) = 1$.

Let $\psi(x) = \max\{\min(x - y_1, y_2 - y_1), 0\}$, then with y_1, y_2, m, m_2 real numbers, we have to consider the following primal problems:

$$p_1(m, m_2; y_1, y_2)$$
$$= \sup \left(\int_a^b \psi(x)\, dF(x) \,\middle|\, \int_a^b x\,dF(x) = m, \int_a^b x^2 dF(x) = m_2, \int_a^b dF(x) = 1 \right)$$

$$q_1(m, m_2; y_1, y_2) = \inf\left(\int_a^b \psi(x)\, dF(x) \,\middle|\, \int_a^b x\, dF(x) = m,\ \int_a^b x^2 dF(x) = m_2,\ \int_a^b dF(x) = 1\right)$$

where the supremum (infimum) is taken over the distributions F on $[a, b]$ satisfying the constraints indicated after the slash.

We remark that in case $y_1 < a$ or $y_2 > b$ the solution of the problem at hand coincides with the solution obtained in DE VYLDER and GOOVAERTS (1982*a*). This paper contains the basis for our present analysis and the same notation will be used.

Let us first consider the case $y_1 < a$. We have:

$$\int_a^b \psi(x)\, dF(x) = m - y_1 - \int_{y_2}^b (x - y_2)\, dF(x).$$

Consequently:

$$\sup \int_a^b \psi(x)\, dF(x) = m - y_1 - \inf \int_{y_2}^b (x - y_2)\, dF(x).$$

Hence:

$$p_1(m, m_2; y_1, y_2) = m - y_1 - q_1(m, m_2)$$

and

$$q_1(m, m_2; y_1, y_2) = m - y_1 - p_1(m, m_2)$$

where $q_1(m, m_2)$ and $p_1(m, m_2)$ are the values of the corresponding problems in DE VYLDER and GOOVAERTS (1982*a*), with e changed in y_2.

In case $y_2 > b$, on the other hand, we get:

$$\int_a^b \psi(x)\, dF(x) = \int_{y_1}^b (x - y_1)\, dF(x)$$

such that:

$$p_1(m, m_2; y_1, y_2) = p_1(m, m_2)$$

and

$$q_1(m, m_2; y_1, y_2) = q_1(m, m_2)$$

where $p_1(m, m_2)$ and $q_1(m, m_2)$ are the values of the corresponding problems in the cited reference, with e changed in y_1. Consequently, without loss of generality we can restrict ourselves to values y_1, y_2 such that:

$$a < y_1 < y_2 < b.$$

The distribution F for which the supremum and infimum are obtained are 3-atomic at most, see e.g., DE VYLDER (1982). If α and β are two different atoms of the 2-atomic probability distribution F satisfying the first-order moment constraint $\int x\, dF = m$, then the corresponding probability masses p_α, p_β must necessarily be:

$$p_\alpha = \frac{m-\beta}{\alpha-\beta}, \qquad p_\beta = \frac{m-\alpha}{\beta-\alpha}.$$

If α, β, γ are different atoms of the 3-atomic probability distribution F, satisfying the moment constraints $\int x\, dF = m$, $\int x^2\, dF = m_2$, then the corresponding probability masses can only be:

$$p_\alpha = \frac{s^2+(m-\beta)(m-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \quad p_\beta = \frac{s^2+(m-\alpha)(m-\gamma)}{(\beta-\alpha)(\beta-\gamma)}, \quad p_\gamma = \frac{s^2+(m-\alpha)(m-\beta)}{(\gamma-\alpha)(\gamma-\beta)}.$$

The domain of the parameters m and $m_2 = s^2+m^2$ is defined as:

$$C' = \{(m, m_2) | a \leqslant m \leqslant b, 0 \leqslant s^2 \leqslant (m-a)(b-m)\}.$$

2. DEMONSTRATION OF THE MAIN RESULT

Theorem

For (m, m_2) belonging to the domain C', defined above, the problems $p_1(m, m_2; y_1, y_2)$ and $q_1(m, m_2; y_1, y_2)$ with $a < y_1 < y_2 < b$ have the value and solution indicated in Table 1 and Table 2 at the end of this note.

Demonstration

Let E be the curve with parametric equations:

$$X = x, \qquad Y = x^2, \qquad Z = \max\{\min(x-y_1, y_2-y_1), 0\}, \qquad a \leqslant x \leqslant b.$$

The curve E is shown in fig. 1. She consists of three parts E_1, E_2, E_3. The parametric representation in each of the three indicated regions is the following:

$$\begin{array}{lllll} E_1 & X = x & Y = x^2 & Z = 0 & a \leqslant x \leqslant y_1 \\ E_2 & X = x & Y = x^2 & Z = x - y_1 & y_1 \leqslant x \leqslant y_2 \\ E_3 & X = x & Y = x^2 & Z = y_2 - y_1 & y_2 \leqslant x \leqslant b. \end{array}$$

As far as the problem $p_1(m, m_2; y_1, y_2)$ is concerned we get immediately three domains, namely D_1, D_2, D_3. We successively obtain:

(1) $\quad D_1 = \{(m, s^2) | 1)\ a \leqslant m \leqslant y_2, (m-a)(y_2-m) \leqslant s^2 \leqslant (m-a)(b-m)$

$\qquad\qquad 2)\ y_2 \leqslant m \leqslant b, (m-y_2)(b-m) \leqslant s^2 \leqslant (m-a)(b-m)\}$.

The equation of the plane through the three points A, P_2 and B enables us to construct an upper bound or a solution of the problem $p_1(m, m_2; y_1, y_2)$ in D_1:

$$Z = (y_2 - y_1) \frac{Y - (y_2+b)X + a(y_2+b-a)}{(a-b)(y_2-a)}.$$

Consequently in D_1:

$$p_1(m, m_2; y_1, y_2) = (y_2 - y_1)\frac{(m-a)(b+y_2-m-a)-s^2}{(b-a)(y_2-a)}$$

(2) $$D_2 = \{(m, s^2) | y_2 \leqslant m \leqslant b, 0 \leqslant s^2 \leqslant (m-y_2)(b-m)\}.$$

In this case it is readily seen that:

$$p_1(m, m_2; y_1, y_2) = y_2 - y_1$$

(3) $$D_3 = \{(m, s^2) | a \leqslant m \leqslant y_2, 0 \leqslant s^2 \leqslant (m-a)(y_2-m)\}.$$

We consider a point Q_1 on E_1 with coordinates $(x, x^2, 0)$ and a point Q_2 on E_2 with coordinates $(y, y^2, y-y_1)$ and determine the equation of the plane through Q_1 and Q_2, tangent on E_1 in Q_1 and tangent on E_2 in Q_2. The equation reads:

$$Z = z_1 X + z_2 Y + z_3$$

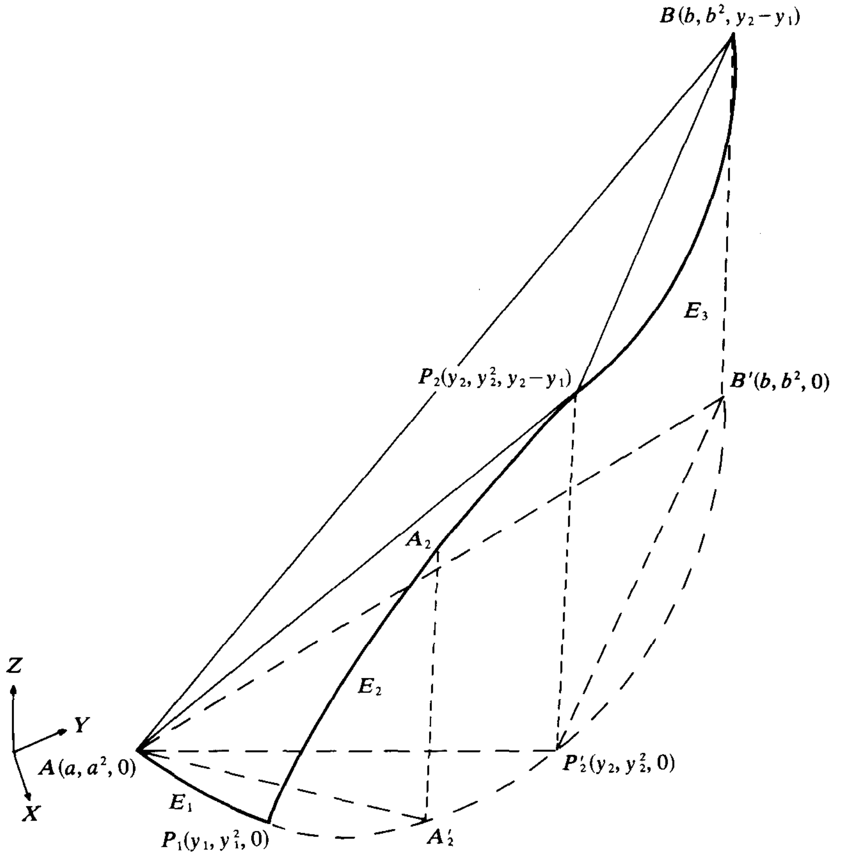

FIGURE 1. Curve E.

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with:

$$\begin{aligned} 0 &= z_1x + z_2x^2 + z_3 && Q_1 \in \text{plane} \\ y - y_1 &= z_1y + z_2y^2 + z_3 && Q_2 \in \text{ plane} \\ 1 &= z_1 + 2z_2y && \text{tangent in } Q_2 \\ 0 &= z_1 + 2z_2x && \text{tangent in } Q_1. \end{aligned}$$

Solving the first three equations of this system of equations with respect to z_1, z_2, z_3 gives:

$$z_2 = \frac{y_1 - x}{(x-y)^2}, \qquad z_1 = \frac{x^2 + y^2 - 2yy_1}{(x-y)^2}.$$

Of course z_1, z_2 still need to satisfy the last equation. This gives:

$$(y-x)(y+x-2y_1) = 0.$$

Hence with $Q_1(x, x^2, 0)$ on E_1 corresponds the point $Q_2(2y_1 - x, (2y_1 - x)^2, y_1 - x)$ on E_2.

Now we have to consider two cases according to the position of the point $A_2(2y_1 - a, (2y_1 - a)^2, y_1 - a)$ corresponding to $A(a, a^2, 0)$.

In case $y_1 - a \leqslant y_2 - y_1$, A_2 is lying under P_2, and we can consider a partition of D_3 in D_{31} and D_{32}.

In case $y_1 - a \geqslant y_2 - y_1$ the point A_1 on E_1 corresponding with P_2 is lying to the right of A and we have to consider a partition D'_{31} and D'_{32} as in fig. 3.

Let us examine now both cases separately.

(i) $2y_1 - a \leqslant y_2$

$$D_{31} = \{(m, s^2) | a < m \leqslant 2y_1 - a, 0 \leqslant s^2 \leqslant (2y_1 - a - m)(m - a)\}.$$

The equation of the plane through $(x, x^2, 0)$ tangent to E_1 and also tangent to E_2 is:

$$Z = \frac{1}{(x-y)^2}[(x^2 + y^2 - 2yy_1)(X - x) + (y_1 - x)(Y - x^2)]$$

of course with $y = 2y_1 - x$, or:

$$Z = \frac{1}{4(y_1 - x)}(-2xX + Y + x^2).$$

Hence the equation of the envelope of this set of planes reads:

$$4(y_1 - X + 2Z)Z = -2(X - 2Z)X + Y + (X - 2Z)^2.$$

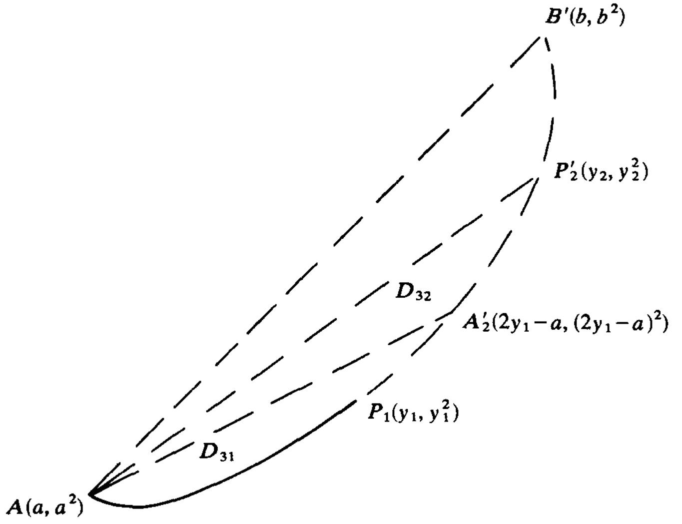

FIGURE 2. Partition of D_3 in case $2y_1-a \leq y_2$.

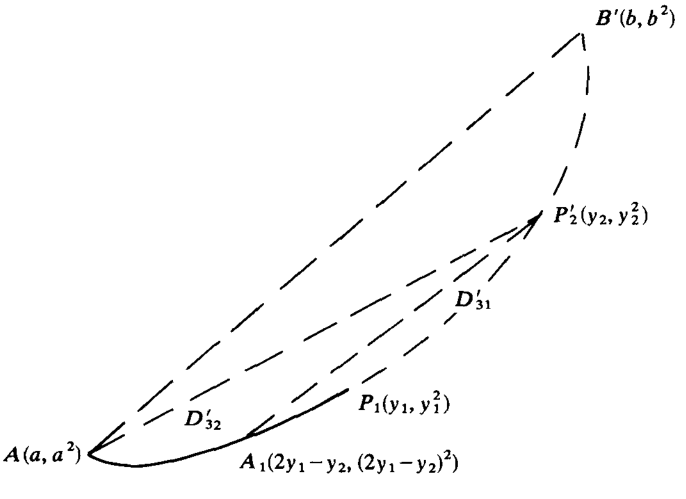

FIGURE 3. Partition of D_3 in case $2y_1-a \geq y_2$.

Consequently:

$$p_1(m, m_2; y_1, y_2) = \tfrac{1}{2}(m-y_1) + \tfrac{1}{2}s_{my_1}$$

with

$$s^2_{my_1} = (y_1-m)^2 + s^2.$$

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Let us consider now:

$$D_{32}=\{(m,s^2)\,|\,1)\ a\leqslant m\leqslant 2y_1-a,\ (m-a)(2y_1-a-m)\leqslant s^2\leqslant(m-a)(y_2-m)$$
$$2)\ 2y_1-a\leqslant m\leqslant y_2,\ 0\leqslant s^2\leqslant(m-a)(y_2-m)\}.$$

The equation of the plane through $A(a, a^2, 0)$ and through Q_2 and tangent on E_2 in Q_2 is obtained by eliminating z_1, z_2, z_3 from the following system of equations:

$$Z=z_1X+z_2Y+z_3$$
$$0=z_1a+z_2a^2+z_3$$
$$y-y_1=z_1y+z_2y^2+z_3$$
$$1=z_1+2z_2y.$$

This gives:

$$(Z-X+a)(Y-2Xa+a^2)+(y_1-a)(X-a)^2=0.$$

And consequently:

$$p_1(m,m_2;y_1,y_2)=m-a-\frac{(y_1-a)(m-a)^2}{s^2+(m-a)^2}.$$

(ii) $2y_1-a\geqslant y_2$

The point A_1 corresponding to P_2 has the following set of coordinates:

$$(2y_1-y_2,(2y_1-y_2)^2,0).$$

Consequently in:

$$D'_{31}=\{(m,s^2)\,|\,2y_1-y_2\leqslant m\leqslant y_2,\ 0\leqslant s^2\leqslant(m-2y_1+y_2)(y_2-m)\}$$

we obtain the same upper bound as in D_{31}.

$$p_1(m,m_2;y_1,y_2)=\tfrac{1}{2}(m-y+s_{my_1}).$$

Let us consider next:

$$D'_{32}=\{(m,s^2)\,|\,1)\ 2y_1-y_2\leqslant m\leqslant y_2,\ (y_2-m)(m-2y_1+y_2)\leqslant s^2$$
$$\leqslant(m-a)(y_2-m)$$
$$2)\ a\leqslant m\leqslant 2y_1-y_2,\ 0\leqslant s^2\leqslant(m-a)(y_2-m)\}.$$

We then have to determine the equation of the plane going through $P_2(y_2, y_2^2, y_2-y_1)$, through $Q_1(x, x^2, 0)$ and tangent on E_1 in Q_1. This equation is obtained by eliminating z_1, z_2 and z_3 from the following system of equations:

$$Z=z_1X+z_2Y+z_3$$
$$y_2-y_1=z_1y_2+z_2y_2^2+z_3$$
$$0=z_1x+z_2x^2+z_3$$
$$0=z_1+2z_2x.$$

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This gives:

$$Z(2y_2X - Y - y_2^2) = X^2(y_2 - y_1) - Y(y_2 - y_1)$$

and of course:

$$p_1(m, m_2; y_1, y_2) = (y_2 - y_1)\frac{s^2}{s^2 + (m - y_2)^2}.$$

As far as the atoms of the extremal distributions are concerned the solution can be obtained, completely similar to the solutions obtained in DE VYLDER and GOOVAERTS (1982*a*).

Now we come to the solution of the problem $q_1(m, m_2; y_1, y_2)$. In this case we have to consider the following three domains.

(1) $D_4 = \{(m, s^2) |\ 1)\ a \le m \le y_1, (m-a)(y_1 - m) \le s^2 \le (m-a)(b-m)$

$2)\ y_1 \le m \le b, (m - y_1)(b-m) \le s^2 \le (m-a)(b-m)\}.$

In order to obtain the solution of the problem $q_1(m, m_2; y_1, y_2)$ we have to determine the equation of the plane through A, P_1 and B. The equation reads:

$$Z = (y_2 - y_1)\frac{Y - (y_1 + a)X + ay_1}{(b-a)(b-y_1)}.$$

Hence:

$$q_1(m, m_2; y_1, y_2) = (y_2 - y_1)\frac{s^2 + (m-a)(m-y_1)}{(b-a)(b-y_1)}$$

(2) $D_5 = \{(m, s^2) | a \le m \le y_1, 0 \le s^2 \le (m-a)(y_1 - m)\}.$

In this case it is readily seen that:

$$q_1(m, m_2; y_1, y_2) = 0$$

(3) $D_6 = \{(m, s^2) | y_1 \le m \le b, 0 \le s^2 \le (m - y_1)(b-m)\}.$

We have to determine the equation of the plane through $Q_2(y, y^2, y - y_1)$ tangent on E_2 in Q_2 and through $Q_3(z, z^2, y_2 - y_1)$ tangent on E_3. The equation of this plane reads:

$$Z = z_1X + z_2Y + z_3$$

where:

$$y - y_1 = z_1y + z_2y^2 + z_3$$
$$y_2 - y_1 = z_1z + z_2z^2 + z_3$$
$$0 = z_1 + 2zz_2$$
$$1 = z_1 + 2z_2y.$$

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Solving this equation with respect to z_1 and z_2 gives:

$$z_2 = \frac{y_2 - z}{(y-z)^2}, \qquad z_1 = \frac{y^2 + z^2 - 2yy_2}{(y-z)^2}.$$

These solutions have to satisfy $0 = z_1 + 2zz_2$, hence:

$$z = 2y_2 - y.$$

Consequently with the point $Q_2(y, y^2, y - y_1)$ on E_2 corresponds the point $Q_3(2y_2 - y, (2y_2 - y)^2, y_2 - y_1)$ on E_3. We have to consider two cases, namely $2y_2 - y_1 \leqslant b$ and $2y_2 - y_1 \geqslant b$.

(i) $2y_2 - y_1 \leqslant b$

In the present situation we consider a partition of D_6 as shown in fig. 4.

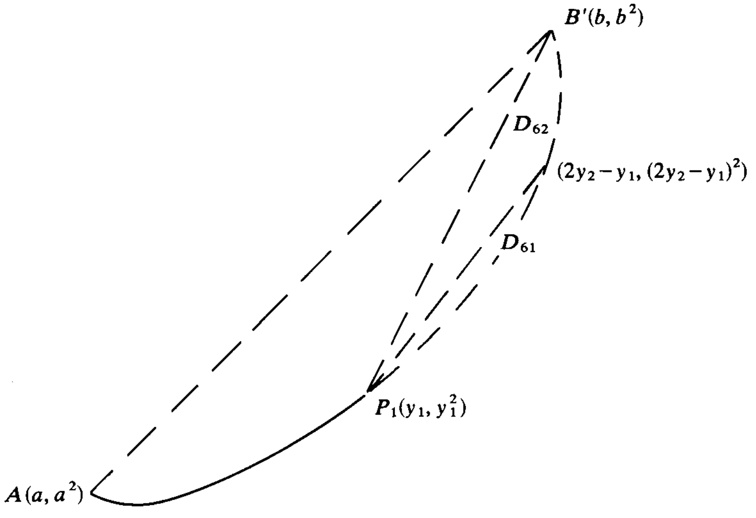

FIGURE 4. Partition of D_6 in case $2y_2 - y_1 \leqslant b$.

We have:

$$D_{61} = \{(m, s^2) | y_1 \leqslant m \leqslant 2y_2 - y_1, 0 \leqslant s^2 \leqslant (m - y_1)(2y_2 - y_1 - m)\}.$$

Next we have to deduce the equation of the envelope of the set of planes:

$$Z = y_2 - y_1 + \frac{y^2 + z^2 - 2yy_2}{(y-z)^2}(X - z) + \frac{y_2 - z}{(y-z)^2}(Y - z^2)$$

with $z = 2y_2 - y$.

Substitution gives:

$$4(y - y_2)(Z - y_2 + y_1) = 2(y - 2y_2)(X - (2y_2 - y)) + Y - (2y_2 - y)^2.$$

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The equation of the envelope is obtained by eliminating y between this equation and the next one, obtained by taking the derivative with respect to y in the preceeding equation

$$y = 2Z + 2y_1 - X.$$

Hence the equation of the envelope becomes:

$$(2Z + 2y_1 - X - y_2)^2 = (y_2 - X)^2 + Y - X^2.$$

Finally

$$q_1(m, m_2; y_1, y_2) = \tfrac{1}{2}(y_2 + m - 2y_1 - s_{my_2}).$$

Next we consider:

$$D_{62} = \{(m, s^2) \mid 1)\; y_1 \leqslant m \leqslant 2y_2 - y_1, (m - y_1)(2y_2 - y_1 - m) \leqslant s^2 \leqslant (m - y_1)(b - m)$$
$$2)\; 2y_2 - y_1 \leqslant m \leqslant b, 0 \leqslant s^2 \leqslant (m - y_1)(b - m)\}.$$

In the present situation the envelope is obtained by considering a set of planes through $(y_1, y_1^2, 0)$ and tangent on E_3. We get:

$$\begin{aligned} Z &= z_1 X + z_2 Y + z_3 \\ 0 &= z_1 y_1 + z_2 y_1^2 + z_3 \\ y_2 - y_1 &= z_1 z + z_2 z^2 + z_3 \\ 0 &= z_1 + 2z_2 z. \end{aligned}$$

Eliminating z_1, z_2 and z_3 gives:

$$Z = 2z \frac{y_2 - y_1}{(z - y_1)^2}(X - y_1) - \frac{y_2 - y_1}{(z - y_1)^2}(Y - y_1^2).$$

Consequently the envelope of this set of planes depending on z is obtained by eliminating z between this equation and the derivative with respect to z.

$$(z - y_1)\, Z = (y_2 - y_1)(X - y_1).$$

This results in

$$-Z(2y_1 X - y_1^2 - Y) = (y_2 - y_1)(X - y_1).$$

Hence

$$q_1(m, m_2; y_1, y_2) = (y_2 - y_1)\frac{(m - y_1)^2}{s^2 + (m - y_1)^2}.$$

(ii) $2y_2 - y_1 \geqslant b$

In this case we have to consider a partition of D_6 in D'_{61} and D'_{62}, as indicated in fig. 5.

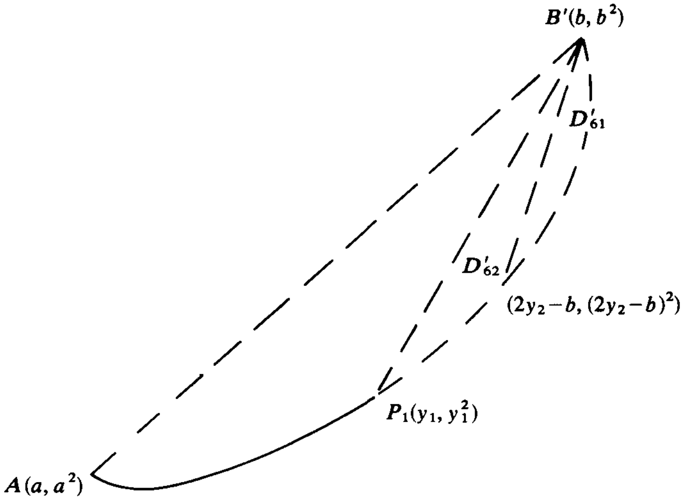

FIGURE 5. Partition of D_6 in case $2y_2-y_1 \geq b$.

In the domain

$$D'_{61}=\{(m, s^2)|2y_2-b \leq m \leq b, 0 \leq s^2 \leq (m-2y_2+b)(b-m)\}$$

the same result as in the case D_{61} applies.

Hence

$$q(m, m_2; y_1, y_2)=\tfrac{1}{2}(y_2+m-2y_1-s_{my_2}).$$

On the other hand we have:

$$D'_{62}=\{(m, s^2)|\ 1)\ 2y_2-b \leq m \leq b, (m-2y_2+b)(b-m) \leq s^2 \leq (m-y_1)(b-m)$$
$$2)\ y_1 \leq m \leq 2y_2-b, 0 \leq s^2 \leq (m-y_1)(b-m)\}.$$

We have to examine the set of planes through $B(b, b^2, y_2-y_1)$, through a point of E_2 and tangent on E_2 in that point.

These planes are determined by the following equations:

$$Z=z_1X+z_2Y+z_3$$
$$y_2-y_1=z_1b+z_2b^2+z_3$$
$$y-y_1=z_1y+z_2y^2+z_3$$
$$1=z_1+2z_2y.$$

Hence the parametric representation of these planes reads:

$$Z=y_2-y_1+\frac{y^2+b^2-2yy_2}{(y-b)^2}(X-b)-\frac{b-y_2}{(y-b)^2}(Y-b^2).$$

Taking the derivative with respect to y gives:

$$(y-b)(Z-y_2+y_1)=(y-y_2)(X-b).$$

Hence, the following equation is obtained for the envelope:

$$\left(\frac{Y-b^2}{X-b}-2b\right)(Z-y_2+y_1)=\left(\frac{Y-b^2}{X-b}-b-y_2\right)(X-b)$$

such that:

$$q_1(m, m_2; y_1, y_2)=y_2-y_1+\frac{s^2+(m-y_2)(m-b)}{s^2+(m-b)^2}(m-b).$$

TABLES 1 AND 2

VALUE AND SOLUTIONS OF THE PRIMAL PROBLEM

Abbreviation: $s_{my}^2=s^2+(m-y)^2$

Domain of the parameters: $a\leq m\leq b$, $0\leq s^2\leq(m-a)(b-m)$

Maximization Conditions	Value of the problem	Atoms
$a\leq m\leq y_2$ $(m-a)(y_2-m)\leq s^2\leq(m-a)(b-m)$	$(y_2-y_1)\dfrac{(m-a)(b+y_2-m-a)-s^2}{(b-a)(y_2-a)}$	a, y_2, b
$y_2\leq m\leq b$ $(m-y_2)(b-m)\leq s^2\leq(m-a)(b-m)$	$(y_2-y_1)\dfrac{(m-a)(b+y_2-m-a)-s^2}{(b-a)(y_2-a)}$	a, y_2, b
$y_2\leq m\leq b$ $0\leq s^2\leq(m-y_2)(b-m)$	y_2-y_1	y_2, m, b
$a\leq m\leq y_2$ $0\leq s^2\leq(m-a)(y_2-m)$		
(i) $2y_1-a\leq y_2$ $a\leq m\leq 2y_1-a$ $0\leq s^2\leq(2y_1-a-m)(m-a)$	$\frac{1}{2}(m-y_1+s_{my_1})$	$y_1-s_{my_1}, y_1+s_{my_1}$
$a\leq m\leq 2y_1-a$ $(m-a)(2y_1-a-m)\leq s^2$ $\leq(m-a)(y_2-m)$	$m-a-\dfrac{(y_1-a)(m-a)^2}{s^2+(m-a)^2}$	$a, m+\dfrac{s^2}{m-a}$
$2y_1-a\leq m\leq y_2$ $0\leq s^2\leq(m-a)(y_2-m)$	$m-a-\dfrac{(y_1-a)(m-a)^2}{s^2+(m-a)^2}$	$a, m+\dfrac{s^2}{m-a}$
$2y_1-y_2\leq m\leq y_2$ $0\leq s^2\leq(m-2y_1+y_2)(y_2-m)$	$\frac{1}{2}(m-y_1+s_{my_1})$	$y_1-s_{my_1}, y_1+s_{my_1}$
(ii) $2y_1-a\geq y_2$		
$2y_1-y_2\leq m\leq y_2$ $(y_2-m)(m-2y_1+y_2)\leq s^2$ $\leq(m-a)(y_2-m)$	$(y_2-y_1)\dfrac{s^2}{s^2+(m-y_2)^2}$	$m-\dfrac{s^2}{y_2-m}, y_2$
$a\leq m\leq 2y_1-y_2$ $0\leq s^2\leq(m-a)(y_2-m)$	$(y_2-y_1)\dfrac{s^2}{s^2+(m-y_2)^2}$	$m-\dfrac{s^2}{y_2-m}, y_2$

Minimization Conditions	Value of the problem	Atoms
$a \leqslant m \leqslant y_1$ $(m-a)(y_1-m) \leqslant s^2 \leqslant (m-a)(b-m)$	$(y_2-y_1)\dfrac{s^2+(m-a)(m-y_1)}{(b-a)(b-y_1)}$	a, y_1, b
$y_1 \leqslant m \leqslant b$ $(m-y_1)(b-m) \leqslant s^2 \leqslant (m-a)(b-m)$	$(y_2-y_1)\dfrac{s^2+(m-a)(m-y_1)}{(b-a)(b-y_1)}$	a, y_1, b
$a \leqslant m \leqslant y_1$ $0 \leqslant s^2 \leqslant (m-a)(y_1-m)$	0	a, m, y_1
$y_1 \leqslant m \leqslant b$ $0 \leqslant s^2 \leqslant (m-y_1)(b-m)$		
(i) $2y_2-y_1 \leqslant b$ $y_1 \leqslant m \leqslant 2y_2-y_1$ $0 \leqslant s^2 \leqslant (m-y_1)(2y_2-y_1-m)$	$\frac{1}{2}(y_2+m-2y_1-s_{my2})$	y_2-s_{my2}, y_2+s_{my2}
$y_1 \leqslant m \leqslant 2y_2-y_1$ $(m-y_1)(2y_2-y_1-m) \leqslant s^2$ $\leqslant (m-y_1)(b-m)$	$(y_2-y_1)\dfrac{(m-y_1)^2}{s^2+(m-y_1)^2}$	$y_1, m+\dfrac{s^2}{m-y_1}$
$2y_2-y_1 \leqslant m \leqslant b$ $0 \leqslant s^2 \leqslant (m-y_1)(b-m)$ (ii) $2y_2-y_1 \geqslant b$	$(y_2-y_1)\dfrac{(m-y_1)^2}{s^2+(m-y_1)^2}$	$y_1, m+\dfrac{s^2}{m-y_1}$
$2y_2-b \leqslant m \leqslant b$ $0 \leqslant s^2 \leqslant (m-2y_2+b)(b-m)$	$\frac{1}{2}(y_2+m-2y_1-s_{my2})$	y_2-s_{my2}, y_2+s_{my2}
$2y_2-b \leqslant m \leqslant b$ $(m-2y_2+b)(b-m) \leqslant s^2$ $\leqslant (m-y_1)(b-m)$	$y_2-y_1+\dfrac{s^2+(m-y_2)(m-b)}{s^2+(m-b)^2}(m-b)$	$m-\dfrac{s^2}{b-m}, b$
$y_1 \leqslant m \leqslant 2y_2-b$ $0 \leqslant s^2 \leqslant (m-y_1)(b-m)$	$y_2-y_1+\dfrac{s^2+(m-y_2)(m-b)}{s^2+(m-b)^2}(m-b)$	$m-\dfrac{s^2}{b-m}, b$

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