SOME EXAMPLES OF NONCOMMUTATIVE LOCAL RINGS

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In this paper we construct examples which answer three questions in the general area of noncommutative Noetherian local rings and rings of finite global dimension. The examples are formed in the same basic way, beginning with a commutative polynomial ring A over a field k and a k-derivation δ of A, taking the skew polynomial ring $R = A[x;\delta]$ and localizing at a prime ideal of the form IR, where I is a prime ideal of A invariant under δ. The localization is possible by a result of Sigurdsson [**13**].

Walker [**14**] generalized the commutative regular local rings by defining a right Noetherian local ring to be n-dimensional regular if its Jacobson radical is generated by a regular normalizing sequence of n elements. By means of an example, he showed that, in contrast with the commutative case, such a regular local ring need not be a unique factorization domain in the sense of [**4**]. Recently there has been interest in the question of whether a regular local ring R must be a Noetherian UFR in the sense of [**3**], that is whether every non-zero prime ideal of R must contain a non-zero principal prime ideal. The general construction described above yields several examples of regular local rings which are not Noetherian UFRs. In our first examples of this kind the regular normalizing sequences of generators for the Jacobson radical were not centralizing. One such example can be formed by localizing the ring given as Example 2.10 by Bell and Sigurdsson [**1**]. In Section 2, we present an example of a regular local ring which is not a Noetherian UFR and in which the Jacobson radical is generated by a regular centralizing sequence of three elements. As P. F. Smith has pointed out to the author, three is the least possible dimension for such an example.

The rings which we discuss all have finite global dimension, usually two, and they share many of the properties of the three examples described by Brown, Hajarnavis and MacEacharn [**2**, Section 7] in their study of Noetherian rings of finite global dimension. In particular we construct, in Section 3, a polycentral Noetherian local ring S of global dimension two, finitely generated as a module over its centre, which is not regular in Walker's sense. This closes the gap between the positive results (for global dimension one) and negative results (for global dimension three) in [**2**, p. 368].

Our third example, described in Section 4, answers a question asked by Maury [**9**]. It is of a regular local ring with a height one prime ideal which is not completely prime.

The first section of the paper includes definitions and some general results applying to the examples given later. Any unexplained terminology is used as in [**11**].

I would like to thank P. F. Smith and A. W. Chatters for stimulating my interest in the problems considered here.

1. Generalities. The rings which we consider are either skew polynomial rings of the form $A[x;\delta]$, where δ is a derivation of a commutative ring A, or localizations of such rings. Since $A[x;\delta]$ has an involution mapping x to $-x$ and acting as the identity map on A, all our results and arguments are right-left symmetrical. Hence such terms as

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Noetherian, localizable, AR-property, global dimension, Krull dimension will be used without qualification by the words 'right' or 'left'.

The Jacobson radical of a ring R will be denoted $J(R)$ and R will be said to be *local* (resp. *scalar local*) if $R/J(R)$ is simple Artinian (resp. a division ring). This terminology follows [**2**] and the differences with that of [**14**] cause no difficulty here.

We shall assume that the reader is familiar with the terms *normal* element, *normalizing sequence, centralizing sequence, regular normalizing sequence* and *regular centralizing sequences*. References for these include [**2**], [**11**] and [**14**]. A ring R is *polycentral* if every ideal of R has a centralizing sequence of generators. An n-dimensional *regular* local ring is a Noetherian local ring R in which $J(R)$ has a regular normalizing sequence of n generators.

A Noetherian UFR [**3**] is a Noetherian prime ring in which every non-zero prime ideal contains a non-zero principal prime ideal. (For the purpose of this paper, an ideal of R is *principal* if it has the form $aR = Ra$ for some $a \in R$. In all instances where we use the term, this is equivalent to being of the form $aR = Rb$ for some a, $b \in R$, see [**3**, Remark 1].)

For the remainder of this section, let A be a commutative Noetherian ring with a derivation δ and a prime ideal I such that $\delta(I) \subseteq I$ and let R be the skew polynomial ring $A[x; \delta]$. Thus R consists of polynomials $a_n x^n + \ldots + a_1 x + a_0$ over A with $xa = ax + \delta(a)$ for all $a \in A$ and is Noetherian. Let $P = IR$ which, by [**6**, Lemma 1.3], is a prime ideal of R. Sigurdsson [**13**, Theorem 2.7] has shown that P is a localizable prime ideal. Since A/I and R/P are domains, the localization $S = R_P$ is Noetherian scalar local with $J(S) = PS$ and $S/J(S)$ isomorphic to the quotient division ring of R/P.

PROPOSITION 1.1. *Let A, I, R, P and S be as above. Suppose further that A is a UFD and that I is generated by two non-associate irreducible elements t, w of A. Then*:

(i) *S has global dimension two*;

(ii) *$J(S)$ is not principal.*

Proof. (i) It is easily checked that $J(S) = wS + tS$ and that $wS \cap tS = wtS$. It follows, by [**11**, 7.3.16], that $S/J(S)$ has projective dimension two as a right S-module. Combining this with the left-sided version of [**11**, 7.3.14], we see that S has global dimension two.

(ii) This follows from (i) using [**11**, 7.3.7]. Alternatively, it can be checked directly.

PROPOSITION 1.2. *Let S be a Noetherian local ring. If either S is regular or S has the form R_P, $R = A[x; \delta]$, described above then $J(S)$ has the AR-property.*

Proof. Since the Jacobson radical of any ring is invariant under all automorphisms of the ring, [**11**, 4.2.7(i)] deals with the regular case. In the other case, P has the AR-property by [**13**, Proposition 2.3] and hence $J(S) = PS$ also has the AR-property.

It follows from Proposition 1.2 and [**2**, Proposition 5.3] that a 2-dimensional regular local ring must be a Noetherian UFR. This fact was pointed out to the author by P. F. Smith who gave a more direct proof.

2. A regular local ring which is not a UFR. We shall present an example of a regular local ring S of dimension 3 which is not a UFR. In this example, $J(S)$ has a regular centralizing sequence u, w, t of generators and w and t generate a non-principal height one prime ideal.

Let k be a field of characteristic zero, let A be the commutative polynomial ring $k[u, w, t]$ in three indeterminates, let δ be the k-derivation $ut\,\partial/\partial w + (w^2 + wt)\partial/\partial t$ of A and let R be the skew polynomial ring $A[x; \delta]$. Let I and H be the prime ideals $uA + wA + tA$ and $wA + tA$ respectively. Note that both I and H are δ-invariant. Let $P = IR$ and $Q = HR$. These are completely prime ideals of R and both are localizable by [**13**, Theorem 2.7]. Let $S = R_P$, as in Section 1, and let $T = R_Q$. By [**11**, 2.1.16(vii)], QS is a prime ideal of S. Since P and Q are completely prime with $Q \subseteq P$, QS is localizable with $S_{QS} = T$.

Since $\delta(u) = 0$, $\delta(w) \in uA$ and $\delta(t) \in uA + wA$, u, w, t is a centralizing sequence in S. It is clearly regular and generates $J(S) = PS$. Thus S is a 3-dimensional regular local ring.

Proposition 2.1. *The prime ideal QS of S is not principal.*

Proof. If QS were principal then so too would be the maximal ideal $QT = J(T)$ of T. But $J(T)$ is not principal by Proposition 1.1(ii).

Proposition 2.2. *The prime ideal QS of S has height one.*

Proof. Suppose that there is a prime ideal Q' of S such that $0 \subset Q' \subset QS$. Then, by [**11**, 2.1.16(vii)], $Q' \cap R$ is a prime ideal of R and $0 \subset Q' \cap R \subset Q$. By [**7**, Lemma 1], $Q' \cap A$ is non-zero and, by [**6**, Lemma 1.3, Lemma 2.1 and Theorem 2.2], $Q' \cap A$ is a prime ideal of A. Since $Q = HR$, $0 \subset Q' \cap A \subset H$. By [**8**, Theorem 5], $Q' \cap A$, being a height one prime ideal, is principal generated by, say, f. But then $\delta(f) \in fA$ and, by Lemma 2.3 below, $f \in k[u] \cap H = 0$. Hence Q' does not exist and so QS has height one.

Lemma 2.3. *Let $f \in A$ be such that $\delta(f) \in fA$. Then $f \in k[u]$.*

Proof. Note that if $a \in A$ is homogeneous of degree m then $\delta(a)$, if non-zero, is homogeneous of degree $m + 1$. It follows, by considering the homogeneous components of f of least and greatest degree, that $\delta(f) = bf$, where b has the form $\alpha u + \beta w + \gamma t$, α, β, $\gamma \in k$, and that $\delta(f_i) = bf_i$ for each of the homogeneous components f_i of f. Thus there is no loss of generality in assuming that f is homogeneous of degree m.

If $\alpha \neq 0$ then the leading term of bf in the lexicographic ordering with $u > w > t$ cannot appear in $\delta(f)$. Thus $\alpha = 0$. Similarly, using the lexicographic ordering with $t > w > u$, we can see that $\gamma = 0$. Thus $\delta(f) = \beta wf$.

For $i \geqq 0$, let V_i be the k-subspace of homogeneous polynomials in u, w, t of degree i and let $d_i = (i + 1)(i + 2)/2$, the dimension of V_i. For $i \geqq 0$ and $\lambda \in k$, let $\theta_{i,\lambda} : V_i \to V_{i+1}$ be the linear transformation such that $\theta_{i,\lambda}(v) = \delta(v) - \lambda wv$ for all $v \in V_i$. Thus f is in the kernel of $\theta_{m,\beta}$. We shall show, inductively, that the rank of $\theta_{i,\lambda}$ is d_i if $\lambda \neq 0$ and $d_i - 1$ if $\lambda = 0$. Consequently the kernel of $\theta_{m,\beta}$ is either zero or is spanned by u^m. Thus $f \in k[u]$, as required.

For each V_i we take the basis consisting of all monomials in u, w, t of degree i, ordered lexicographically with $u > w > t$. Let $A_{i,\lambda}$ be the d_{i+1} by d_i matrix representing $\theta_{i,\lambda}$ relative to these bases. For $v \in V_{i-1}$, $\theta_{i,\lambda}(uv) = u\theta_{i-1,\lambda}(v)$ and, for $0 \leqq j \leqq i$,

$$\theta_{i,\lambda}(w^j t^{i-j}) = (i - j)w^{j+2}t^{i-j-1} + (i - j - \lambda)w^{j+1}t^{i-j} + juw^{j-1}t^{i-j+1}.$$

Thus, for $i > 0$, $A_{i,\lambda}$ has the form

$$\left[\begin{array}{c|c} A_{i-1,\lambda} & \begin{array}{c} 0 \\ \hline B_i \end{array} \\ \hline 0 & C_{i,\lambda} \end{array}\right] \begin{array}{l} \text{row } d_{i-1} \\ \\ \text{row } d_i \\ \\ \text{row } d_{i+1} \end{array},$$

where B_i is the $i+1$ by $i+1$ matrix

$$\begin{bmatrix} 0 & & & & & \\ i & 0 & & & \mathbf{0} & \\ & i-1 & \ddots & & & \\ & & \ddots & 0 & & \\ & \mathbf{0} & & 2 & 0 & \\ & & & & 1 & 0 \end{bmatrix}$$

and $C_{i,\lambda}$ is the $i+2$ by $i+1$ matrix

$$\begin{bmatrix} -\lambda & 1 & & & & \\ 0 & 1-\lambda & 2 & & \mathbf{0} & \\ & 0 & 2-\lambda & \ddots & & \\ & & 0 & \ddots & i-1 & \\ & & & \ddots & i-1-\lambda & i \\ & \mathbf{0} & & & 0 & i-\lambda \\ & & & & & 0 \end{bmatrix}.$$

If $\lambda \notin \{0, 1, 2, \ldots, i\}$ then it is clear that

$$\operatorname{rank} A_{i,\lambda} = \operatorname{rank} A_{i-1,\lambda} + (i+1). \tag{*}$$

If $\lambda = i$ then (*) is seen to hold on interchanging rows d_i and $d_{i+1} - 1$ of $A_{i,\lambda}$ and then adding row $d_{i+1} - 2$ to row $d_{i+1} - 1$. Finally suppose that $\lambda \in \{0, 1, 2, \ldots, i-1\}$. In this

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case

$$C_{i,\lambda}=\begin{bmatrix} -\lambda & 1 & & & & & & & \\ 0 & 1-\lambda & \ddots & & & & & 0 & \\ & 0 & \ddots & \lambda-1 & & & & & \\ & & \ddots & -1 & \lambda & & & & \\ & & & 0 & 0 & \lambda+1 & & & \\ & & & & 0 & 1 & \lambda+2 & & \\ & & & & & 0 & 2 & \ddots & \\ & 0 & & & & & 0 & \ddots & i \\ & & & & & & & \ddots & i-\lambda \\ & & & & & & & & 0 \end{bmatrix}, \text{ row } \lambda+2$$

$$B=\begin{bmatrix} 0 & & & & & & \\ i & 0 & & & 0 & & \\ & i-1 & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & i-\lambda & \ddots & & \\ & 0 & & & \ddots & 0 & \\ & & & & & 1 & 0 \end{bmatrix} \text{row } \lambda+2$$

and $C_{i-1,\lambda}$ is obtained from $C_{i,\lambda}$ by deleting the last row and column. Observe that row $\lambda+2$ of $C_{i-1,\lambda}$ is a linear combination of rows $\lambda+1$, $\lambda+3$, $\lambda+4, \ldots, i$. Hence the row space of $A_{i-1,\lambda}$ has a basis X excluding row $d_{i-1}+\lambda+2$. It follows that the row space of $A_{i,\lambda}$ has a basis consisting of the rows which are extrapolations of the elements of X, row $\lambda+2$ of B_i and rows $1, 2, \ldots, \lambda, \lambda+2, \ldots, i+1$ of $C_{i,\lambda}$. Thus (*) holds in this case also. Since $d_{i-1}+i+1=d_i$ and $\theta_{0,\lambda}$ has rank 0 if $\lambda=0$ and rank 1 otherwise, it follows, inductively, that for $i \geqq 0$ and $\lambda \in k$, $\theta_{i,\lambda}$ has rank d_i-1 if $\lambda=0$ and rank d_i otherwise. As indicated above, this completes the proof.

The above results establish that S has the properties claimed for it. There are some other features of the rings S and T which may be of interest.

REMARK 2.4. The Noetherian local ring T, which is the localization of the regular local ring S at a height one prime ideal, is not itself regular. It has classical Krull dimension one and, by Proposition 1.1, global dimension 2. Its Krull dimension is also 2, as can be deduced from [**14**, Theorem 2.7] and [**11**, 6.5.9 and 6.5.3 Proposition].

REMARK 2.5. The regular local ring S has saturated chains of prime ideals, $0 \subset uS \subset uS+wS \subset PS$ and $0 \subset QS \subset PS$, of different lengths and so does not have the catenary property.

REMARK 2.6. There are two questions in the literature concerning reflexive prime ideals in maximal orders upon which these examples have some bearing. Maury [**9**] proved that regular local rings are maximal orders and, as Question 1, asked whether the reflexive prime ideals of a regular local ring must be principal. The ring S does not answer this question as the prime ideal QS is not reflexive. If it were then, by [**10**, IV.2.15], QT would be principal which, by Proposition 1.1, it is not.

In [**10**, p. 181], Maury and Raynaud asked whether there exists a maximal order in a simple Artinian ring having a height one prime which is not reflexive. Hajarnavis and Williams [**5**] have pointed out that the ring of [**2**, Example 7.2] is such a ring. The above comments show that the regular local ring S is another such example. So too is the local ring T, which is a maximal order by [**10**, IV.2.1] and in which the unique non-zero prime ideal QT is not reflexive.

3. A polycentral scalar local ring of global dimension two which is not regular. Let k be a field of characteristic 5, let A be the commutative polynomial ring $k[w, t]$ in two indeterminates, let δ be the k-derivation $t\,\partial/\partial w + w^2\,\partial/\partial t$ and let R be the skew polynomial ring $A[x; \delta]$. Let I be the maximal ideal $wA + tA$ and observe that $\delta(I) \subseteq I$. As in Section 1, let P be the localizable prime ideal IR of R and let S be the local ring R_P. Then S is Noetherian scalar local with $J(S) = PS$ and with $S/J(S)$ isomorphic to the rational function field $k(x)$. It can be checked that $\delta^5(w) = 0 = \delta^5(t)$ and it follows, since δ^5 is a derivation, that $\delta^5 = 0$. The ring R is finitely generated as a module over the Noetherian subring $k[w^5, t^5, x^5]$ of its centre and hence as a module over its Noetherian centre Z. It follows, by standard arguments (see [**12**, Section 3.1] for more general arguments), that S is the localization of R at the central set $Z\backslash P$. Hence S is also finitely generated as a module over its centre. The next result shows that R is polycentral and it follows easily that S is also polycentral. A derivation δ of a ring A is *locally nilpotent* if, for all $r \in A$, there exists n such that $\delta^n(r) = 0$.

PROPOSITION 3.1. *Let A be a commutative Noetherian ring with a locally nilpotent derivation δ. Then $A[x; \delta]$ is polycentral.*

Proof. Let $R = A[x; \delta]$. Since R is Noetherian, it suffices to show that for all ideals I, J of R with $I \subset J$, J/I has non-zero intersection with the centre of R/I. Let $f = f_n x^n + \ldots + f_0$ be of minimal degree in $J\backslash I$. The natural extension of δ to R is an inner derivation of R; so $\delta^i(f) \in J$ for all i. Since δ is locally nilpotent on A, it is locally nilpotent on R; so, for some i, $\delta^i(f) = 0 \in I$. Let $j \geqq 0$ be maximal such that $\delta^j(f) \in J\backslash I$. We may assume that $j = 0$ and that $\delta(f) \in I$. Thus, in R/I, $\bar{f}$ commutes with $\bar{x}$. Now let $r \in A$. Then $rf - fr \in J$ and has degree less than n. By the choice of f, $rf - fr \in I$. Thus, in R/I, $\bar{f}$ commutes with $\bar{r}$. Since R/I is generated by $\bar{x}$ and $\bar{A}$, $\bar{f}$ is central in R/I.

When applied to the rings R and S under consideration in this section, the above argument produces the centralizing sequence of generators $t^2 + w^3$, wt, w^2, t, w for P in R and for PS in R and for PS in S. Note that $t^2 + w^3$ generates prime ideals P_1 of R and P_1S of S; so that P and PS have height at least 2. Also P is minimal over $(t^2 + w^3)R + (wt)R$; so, by [**11**, 4.1.13], P and PS have height 2. Thus the Krull dimension and classical Krull dimension of S, which are equal by [**11**, 13.6.6], are 2. By Proposition 1.1, the global dimension of S is also 2. The next step is to show that S is not regular.

If S is regular then, by [**14**, Theorem 2.7], it is 2-dimensional; so $J(S)$ is generated by a regular normalizing sequence p, q. By [**14**, Lemma 2.6], pS is a prime ideal of S, necessarily of height one.

PROPOSITION 3.2.

(i) *Every normal element of R in P is in P^2.*

(ii) *Every height one prime ideal of R is generated by a normal element.*

(iii) *Every height one prime ideal of S is generated by a normal element of S belonging to $J(S)^2$.*

Proof. (i) It can be checked that a non-zero element $r_nx^n + \ldots + r_1x + r_0$ of R is normal if and only if 5 divides i for all i such that $r_i \neq 0$ and there exists $b \in A$ such that $\delta(r_i) = r_ib$ for all i. It therefore suffices to show that $I \cap N \subseteq I^2$, where $N = \{a \in A : \delta(a) \in aA\}$ and $I = wA + tA$. Suppose that $a \in I \cap N \backslash I^2$. Then a has the form $\lambda w + \mu t + \alpha w^2 + \beta wt + \gamma t^2 + b$, where $b \in I^3$, λ, μ, α, β, $\gamma \in k$, with λ, μ not both zero. Also $\delta(a) = a(a_0 + a_1t + a_2w + c)$, where $c \in I^2$. Comparing coefficients of t, w and w^2, gives us successively $\lambda = a_0\mu$, $0 = a_0\lambda$ (whence $\lambda = 0$ and $a_0 = 0$) and $\mu = 0$, a contradiction.

(ii) This is a consequence of [**3**, Theorem 5.4].

(iii) Let P' be a height one prime of S. Then $P' \cap R$ is a height one prime of R with $(P' \cap R)S = P'$ so that, by (i) and (ii), $P' = pS$ for some $p \in P^2$. By symmetry, $P' = Sq$ for some $q \in S$ and, by [**3**, Remark 1], $P' = pS = Sp$.

COROLLARY 3.3. *The ring S is not a regular local ring.*

Proof. Let $J = J(S) = PS$ and suppose that S is regular. By the remarks preceding Proposition 3.2, $J = pS + qS$ for some regular normalizing sequence p, q in S and pS is a height one prime of S. By Proposition 3.2(iii), $p \in J^2$. Hence $J = J^2 + qS$ and, by Nakayama's Lemma, $J = qS$. By symmetry, $J = Sq$. But then S would be 1-dimensional which is false. Thus S is not regular.

REMARK 3.4. The local ring S shares with Example 7.3 of [**2**] the property that 0, S and $S/J(S)$ are the only factor rings of S of finite global dimension. The argument given below to show this is similar to that used in [**2**, Example 7.3] but one difference between the two is that S does have a completely prime height one prime $(t^2 + w^3)S$.

Let $I \neq 0$, and let $J(S)$ be a proper ideal of S such that S/I has finite global dimension. The ring S/I is polycentral and so satisfies the hypothesis of [**2**, Corollary 4.3]. Hence S/I is a domain and I is a prime ideal of S of height one. By Proposition 3.2(iii), $I = aS = Sa$ for some $a \in J(S)$ and, by [**11**, 7.3.7], S/I has global dimension one. But then [**2**, Theorem 5.2] applies to show that S/I is 1-dimensional regular local and this would imply that S is 2-dimensional regular local, which is false. Thus I does not exist.

4. An incompletely prime height one prime ideal in a regular local ring. Maury [**9**, Question 2] asked whether, in a regular local ring, the reflexive prime ideals must be completely prime. We shall present a regular local ring S with a non-zero principal ideal which is not completely prime. The ring S is scalar local, hence local in the sense of [**9**] and [**14**].

Let k be a field of non-zero characteristic p, let A be the commutative polynomial ring $k[w, t]$ in two indeterminates, let δ be the k-derivation $w\,\partial/\partial t$ of A and let R be the skew polynomial ring $A[x; \delta]$. Let I be the maximal ideal $wA + tA$ of L, let $P = IR$ and, as in Section 1, let S be the local ring R_P.

PROPOSITION 4.1.
(i) *S is a 2-dimensional regular local ring.*
(ii) *S has a non-zero principal prime ideal which is not completely prime.*

Proof. (i) The sequence w, t is a regular centralizing sequence of generators for $J(S)$(see [**14**, Lemma 3.6]).

(ii) The prime ideal tA is not δ-invariant; so, by [**3**, Proposition 5.3(b)], t^pA is the largest δ-invariant ideal contained in tA. By [**6**, Theorem 2.2 and Lemma 1.3], t^pR is a prime ideal of R. Consequently t^pS is a prime ideal of S [**11**, 2.1.16(vii)]. It is clear that $t \notin t^pS$; so t^pS is not completely prime.

REMARK 4.2. The element t^p above is central in S. By methods used elsewhere in this paper, it can be checked that S is polycentral, a finitely generated module over its Noetherian centre and that S/t^pS has infinite global dimension.

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