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MULTI-SOLITONS FOR NONLINEAR KLEIN–GORDON EQUATIONS

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Abstract

In this paper, we consider the existence of multi-soliton structures for the nonlinear Klein–Gordon (NLKG) equation in $\mathbb{R}^{1+d}$. We prove that, independently of the unstable character of NLKG solitons, it is possible to construct a N-soliton family of solutions to the NLKG equation, of dimension $2N$, globally well defined in the energy space $H^1 \times L^2$ for all large positive times. The method of proof involves the generalization of previous works on supercritical Nonlinear Schrödinger (NLS) and generalized Korteweg–de Vries (gKdV) equations by Martel, Merle, and the first author [R. Côte, Y. Martel and F. Merle, Rev. Mat. Iberoam. 27 (1) (2011), 273–302] to the wave case, where we replace the unstable mode associated to the linear NLKG operator by two generalized directions that are controlled without appealing to modulation theory. As a byproduct, we generalize the linear theory described in Grillakis, Shatah, and Strauss [J. Funct. Anal. 74 (1) (1987), 160–197] and Duyckaerts and Merle [Int. Math. Res. Pap. IMRP (2008), Art ID rpn002] to the case of boosted solitons, and provide new solutions to be studied using the recent work of Nakanishi and Schlag [Zurich Lectures in Advanced Mathematics, vi+253 pp (European Mathematical Society (EMS), Zürich, 2011)] theory.

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1. Introduction

In this paper, we are interested in the problem of constructing multi-soliton solutions for some well-known scalar field equations. Let $f = f(s)$ be a real-valued $\mathscr{C}^1$-function. We consider the nonlinear Klein–Gordon (NLKG) equation in $\mathbb{R}^{1+d}$, $d \geqslant 1$,

$$\partial_{tt} u - \Delta u + u - f(u) = 0, \quad u(t,x) \in \mathbb{R}, \ (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{NLKG}$$

This equation arises in quantum field physics as a model for a self-interacting, nonlinear scalar field, invariant under Lorentz transformations (see below).

Let F be the standard integral of f:

$$F(s) := \int_0^s f(\sigma) d\sigma. \tag{1}$$

We will assume that, for some fixed constant $C > 0$, the following hold.

(A) If $d = 1$,

  (i) f is odd, and $f(0) = f'(0) = 0$, and

  (ii) there exists $s_0 > 0$ such that $F(s_0) - \frac{1}{2}s_0^2 > 0$.

(B) If $d \geqslant 2$, f is a pure power H^1-subcritical nonlinearity: $f(u) = \lambda |u|^{p-1} u$, where $\lambda > 0$, $p \in (1, 1 + 4/(d-2))$.

Prescribing f to the above class of focusing nonlinearities ensures that the corresponding Cauchy problem for (NLKG) is locally well posed in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, for any $s \geqslant 1$: we refer to Ginibre and Velo [12] and Nakamura and Ozawa [23] (when $d = 2$) for more details.

Also under the above conditions, the energy and the momentum (every integral is taken over $\mathbb{R}^d$)

$$E[u, u_t](t) = \frac{1}{2} \int \left[|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + |u(t,x)|^2 - 2F(u(t,x)) \right] dx, \tag{2}$$

$$P[u, u_t](t) = \frac{1}{2} \int \partial_t u(t,x) \nabla u(t,x) \, dx, \tag{3}$$

are conserved along the flow.

Another important feature of equation (NLKG), still under the previous conditions, is the fact that it admits stationary solutions of the form $u(t,x) = U(x)$ (that is, with no dependence on t). Among them, we are interested in the ground-state $Q = Q(x)$, where Q is a positive solution of the elliptic partial differential equation

$$\Delta Q - Q + f(Q) = 0, \quad Q > 0, \quad Q \in H^1(\mathbb{R}^d). \tag{4}$$

The existence of this solution goes back to Berestycki and Lions [1], provided that the above conditions (in particular (ii)) hold. Additionally, it is well known that Q is radial and exponentially decreasing, along with its first and second derivatives (Gidas, Ni, and Nirenberg [9]), and *unique* up to definition of the origin (see Kwong [14] and Serrin and Tang [27]).

In fact, our main result written below could be extended to more general nonlinearity under an additional assumption of spectral nature, namely that the linearized operator around Q has a standard simple spectrum. More precisely, Theorem 1 holds, as soon as f satisfies (i), (ii), and the following.

(iii) If $d = 2$, $|f'(s)| \leqslant C|s|^p e^{\kappa s^2}$, for some $p \geqslant 0$, $\kappa > 0$ and all $s \in \mathbb{R}$.

(iv) If $d \geqslant 3$, $|f'(s)| \leqslant C(1 + |s|^{p-1})$ for some $p < 1 + \dfrac{4}{d-2}$ and all $s \in \mathbb{R}$.

(v) $-\Delta z + z - f'(Q)z$ has a unique simple negative eigenvalue, and its kernel is given by $\{x \cdot \nabla Q | x \in \mathbb{R}^d\}$, and it is nondegenerate.

Assumption (v) has been checked in cases (A) and (B) (using ordinary differential equation analysis), and is believed to hold for a wide class of functions f. (See Lemma 4.)

Since (NLKG) is invariant under *Lorentz boosts*, we can define a *boosted ground state* (a *soliton* from now on) with relative velocity $\beta \in \mathbb{R}^d$. More precisely, let $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$, with $|\beta| < 1$ (we denote $|\cdot|$ the Euclidian norm on $\mathbb{R}^d$). The corresponding Lorentz boost is given by the $(d+1) \times (d+1)$ matrix

$$\Lambda_\beta := \begin{pmatrix} \gamma & -\beta_1\gamma & \cdots & -\beta_d\gamma \\ -\beta_1\gamma & & & \\ \vdots & & \mathrm{Id}_d + \dfrac{(\gamma-1)}{|\beta|^2}\beta\beta^T & \\ -\beta_d\gamma & & & \end{pmatrix} \quad \text{where} \quad \gamma := \frac{1}{\sqrt{1-|\beta|^2}} \tag{5}$$

($\beta\beta^T$ is the $d \times d$ rank-1 matrix with coefficient of index (i, j) $\beta_i\beta_j$). Then the boosted soliton with velocity β is

$$Q_\beta(x) := Q\left(\Lambda_\beta \begin{pmatrix} 0 \\ x \end{pmatrix}\right) = Q\left(x + \frac{\gamma - 1}{|\beta|^2}(\beta \cdot x)\beta\right), \tag{6}$$

where with a slight abuse of notation $Q(t, x) = Q(x)$ in the first equality (namely, we project on the last d coordinates). Also notice that (NLKG) is invariant by space translation (shifts). Hence the general family of solitons is parameterized by speed $\beta \in \mathbb{R}^d$ and shift (translation) $x_0 \in \mathbb{R}^d$; they are the travelling wave solution to (NLKG) defined by

$$R_{\beta,x_0}(t,x) := \begin{pmatrix} Q_\beta(x-\beta t-x_0) \\ -\beta\cdot\nabla Q_\beta(x-\beta t-x_0) \end{pmatrix}. \tag{7}$$

(Observe that the second component is the time derivative of the first one.) This family is the orbit of $\{Q\}$ under all the symmetries of (NLKG) (general Lorentz transformation, time and space shifts); in particular, it is invariant under these transformations: see Appendix A for further details.

In the rest of this work, it will be convenient to work with vector data $(u, \partial_t u)^T$. For notational purposes, we use upper-case letters to denote vector valued functions and lower-case letters for scalar functions (except for the scalar field Q_β).

We will work in the energy space $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ endowed with the following scalar product: denoting $U=(u_1,u_2)^T$, $V=(v_1,v_2)^T$, we define

$$\langle U|V\rangle = \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle := (u_1|v_1)+(u_2|v_2) = \int (u_1v_1+u_2v_2), \tag{8}$$
$$\text{where } (u|v) := \int uv,$$

and the energy norm

$$\|U\|^2 := \langle U|U\rangle + (\nabla u_1|\nabla u_1) = \|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2. \tag{9}$$

It is well known (see, e.g., Grillakis, Shatah, and Strauss [10]) that $(Q,0)$ is unstable in the energy space (This result is known in the physics literature as Derrick's theorem [5].). The instability properties of Q and the solution with energy slightly above $E[(Q,0)]$ have recently been further studied by Nakanishi and Schlag; see [24] and subsequent works. Their ideas are further developments of the primary idea introduced in Duyckaerts and Merle [6], in the context of the energy-critical nonlinear wave equation; there, the relevant nonlinear object is the ground state W, the unique positive function (up to symmetries) that solves $\Delta W + W^{1+4/(d-2)} = 0$ – a major difference is that W has polynomial decay, whereas Q has exponential decay, and so the interaction between two solitons with different speeds is exponentially small.

In this paper, we want to understand the dynamics of large, quantized energy solutions. More precisely, we address the question whether is it possible to construct a multi-soliton solution for (NLKG), that is a solution u to (NLKG) defined on a semi-infinite interval of time, such that

$$(u,\partial_t u)(t,x) \sim \sum_{j=1}^{N} R_{\beta_j,x_j}(t,x) \quad \text{as } t\to+\infty.$$

Such solutions were constructed for the nonlinear Schrödinger (NLS) equation and the generalized Korteweg–de Vries (gKdV) equation, first in the L^2-critical and subcritical case by Merle [18], Martel [16], and Martel and Merle [19]. These results followed from the stability and asymptotic stability theory that these authors developed.

The existence of multi-solitons was then extended by Martel, Merle, and the first author [3] to the L^2 supercritical case: in this latter case, each single soliton is unstable, and hence the multi-soliton is a highly unstable solution. It turns out that this is also the case for scalar field equations such as (NLKG). We prove that, regardless of the instability of the soliton, one can construct large mass multi-solitons, on the whole range of parameters $\beta_1, \dots, \beta_N \in \mathbb{R}^d$ distinct, with $|\beta_j| < 1$ and $x_1, \dots, x_N \in \mathbb{R}^d$. More precisely, the main result of this paper is the following.

THEOREM 1. *Assume (A) or (B), let* $\beta_1, \beta_2, \dots, \beta_N \in \mathbb{R}^d$ *be a set of different velocities*

$$\forall i \neq j, \quad \beta_i \neq \beta_j, \quad \textit{and} \quad |\beta_j| < 1,$$

and let $x_1, x_2, \dots, x_N \in \mathbb{R}^d$ *be the shift parameters.*

Then there exist a time $T_0 \in \mathbb{R}$, *constants* $C > 0$, *and* $\gamma_0 > 0$, *only depending on the sets* $(\beta_j)_j$, $(x_j)_j$, *and a solution* $(u, \partial_t u) \in \mathscr{C}([T_0, +\infty), H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ *of (NLKG), globally defined for forward times, and satisfying*

$$\forall t \geqslant T_0, \qquad \left\| (u, \partial_t u)(t, x) - \sum_{j=1}^{N} R_{\beta_j, x_j}(t, x) \right\| \leqslant C e^{-\gamma_0 t}.$$

We remark that this is the first multi-soliton result for wave-type equations. Although the nonlinear object under consideration is the same as for NLS, for example, the structure of the flow is different (recall that all solitons are unstable for (NLKG), irrespective of the nonlinearity). Hence we need to work in a more general framework, the one given by a matrix description of (NLKG).

Let us describe the main steps of the proof. We first revisit the standard spectral theory of linearized operators around the soliton, and the second-order derivative of the energy–momentum functional (see H in (15)) [10]. Since solitons are unstable objects, it is clear that such a theory will not be enough to describe the dynamics of several solitons. However, a slight variation of this functional (see $\mathcal{H}$ in (25)) turns out to be the key element to study. We describe its spectrum in great detail; in particular, we prove that this operator has three eigenvalues: the kernel zero, and two opposite-sign eigenvalues, with associated eigenfunctions $Z_\pm$. After some work we are able to prove a coercivity property for the operator $\mathcal{H}$ modulo the two directions Z_+ and Z_-. This analysis was first conducted by Pego and Weinstein [25] in the context of gKdV equations.

The rest of the work is devoted to the study of the dynamics of small perturbations of the sum of N solitons, and in particular how the two directions associated to $Z_\pm$ evolve. Using a topological argument, we can show the existence of suitable initial data for (NLKG) such that both directions remain controlled for all large positive time, proving the main theorem. We remark that this method is general and does not require the study of the linear evolution at large, only a deep understanding of suitable alternative directions of the linearized operator. A nice open question should be the extension of this result to the nonlinear wave case, where the soliton decays polynomially.

For the sake of easiness and clarity, we present the detailed computations in the one-dimensional case $d = 1$. This case encompass all difficulties, the higher-dimension case adding only indices and notational inconvenience: we will briefly describe the corresponding differences at the end of each section.

Organization of this paper. In Section 2, we develop spectral aspects of the linearized flow around Q_β, which are more subtle than in the NLS or gKdV case. In Section 3, we construct approximate N-soliton solutions in Proposition 3, which we do by estimation backward in time as in [16, 18, 19]. There we present the nonlinear argument, relying *in fine* on a topological argument as in [3]. The Lyapunov functional has to be chosen carefully, as we cannot allow mixed derivatives of the form $\partial_{tx} u$. Finally, in Section 4, we prove Theorem 1, relying on the previously proved Proposition 3 and a compactness procedure.

2. Spectral theory

In this section, we describe and solve two spectral problems related to (NLKG). We will work with functions independent of time, unless specified explicitly. The main result of this section is Proposition 2.

2.1. Coercivity of the Hessian.

First of all, we recall the structure of the Hessian of the energy around Q. Given $Q = Q(x)$ the ground state of (4) and $Q_\beta(x) = Q(\gamma x)$, where $\gamma = (1-\beta^2)^{-1/2}$, we define the operators

$$L^+ := -\partial_{xx} + \mathrm{Id} - f'(Q), \quad \text{and} \quad L^+_\beta = -\gamma^{-2}\partial_{xx} + \mathrm{Id} - f'(Q_\beta), \tag{10}$$

Note that L^+_β is a rescaled version of L_+:

$$L^+_\beta(v(\gamma x)) = (L^+ v)(\gamma x).$$

As a consequence of the Sturm–Liouville theory and the previous identity, we have the following spectral properties for L^+, and therefore for L_β^+.

LEMMA 1. *The unbounded operator* L^+, *defined in* $L^2(\mathbb{R})$ *with domain* $H^2(\mathbb{R})$, *is self-adjoint, has a unique negative eigenvalue* $-\lambda_0 < 0$ *(with corresponding* L^2*-normalized eigenfunction* Q^-*), and its kernel is spanned by* $\partial_x Q$*. Moreover, the continuous spectrum is* $[1, +\infty)$*, and* 0 *is an isolated eigenvalue.*

We recall that, from standard elliptic theory, Q^- is smooth, even, and exponentially decreasing in space: there exists $c_0 > 0$ such that

$$\forall\, k \in \mathbb{N},\ \forall\, x \in \mathbb{R},\ \exists\, C_k, \quad |\partial_x^k Q^-(x)| \leqslant C_k e^{-c_0|x|}. \tag{11}$$

It is not difficult to check that one can take any c_0 satisfying $0 < c_0 \leqslant \sqrt{1+\lambda_0}$.

Another consequence of Lemma 1 is the following fact: L_β^+ has a unique negative eigenvalue $-\lambda_0$ with (even) eigenfunction $Q_\beta^-(x) := Q^-(\gamma x)$, and its kernel is spanned by $\partial_x Q_\beta$ and has continuous spectrum $[1, +\infty)$. Additionally, we have the following.

COROLLARY 1. *There exists* $\nu_0 \in (0, 1)$ *such that, if* $v \in H^1(\mathbb{R})$ *satisfies* $(v|Q_\beta^-) = (v|\partial_x Q_\beta) = 0$*, then* $(L_\beta^+ v|v) \geqslant \nu_0 \|v\|_{H^1}^2$.

We now introduce suitable matrix operators associated to the dynamics around a soliton. These operators will be dependent on the velocity parameter β, but, for simplicity of notation, we will omit the subscript β when there is no ambiguity. Define (Do not confuse with the transpose symbol $(\cdot)^T$.)

$$T = T_\beta := -\partial_{xx} + \mathrm{Id} - f'(Q_\beta) = L_\beta^+ - \beta^2 \partial_{xx}, \tag{12}$$

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{13}$$

$$L := \begin{pmatrix} T & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \tag{14}$$

and

$$H := L - J \begin{pmatrix} \beta\partial_x & 0 \\ 0 & \beta\partial_x \end{pmatrix} = \begin{pmatrix} T & -\beta\partial_x \\ \beta\partial_x & \mathrm{Id} \end{pmatrix}. \tag{15}$$

The operator H is the standard second-order derivative of the functional for which the vector soliton $R = (Q_\beta, -\beta\partial_x Q_\beta)^T$ is an associated local minimizer. Later, we will discuss this assertion in detail. The following proposition describes the main spectral properties of H. Recall that $\langle\cdot|\cdot\rangle$ and $(\cdot|\cdot)$ denote the symmetric bilinear forms on $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively, introduced in (8), and $\|\cdot\|$ is the energy norm defined in (9).

PROPOSITION 1. *Let $\beta \in \mathbb{R}$, $|\beta| < 1$. The matrix operator H, defined in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with dense domain $H^2(\mathbb{R}) \times H^1(\mathbb{R})$, is a self-adjoint operator. Furthermore, there exist $\alpha_0 > 0$, $\Phi_0 = \Phi_{0,\beta}$, and $\Phi_- = \Phi_{0,\beta} \in \mathscr{S}(\mathbb{R})^2$ (with exponential decay, along with their derivatives) such that*

$$H\Phi_0 = 0, \quad \langle \Phi_0 | \Phi_- \rangle = 0, \tag{16}$$

$$\langle H\Phi_- | \Phi_- \rangle < 0, \tag{17}$$

and the following coercivity property holds. Let $V = (v_1, v_2)^T \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Then,

$$\textit{if} \quad \langle V | \Phi_0 \rangle = \langle V | H\Phi_- \rangle = 0, \quad \textit{one has } \langle HV | V \rangle \geqslant \alpha_0 \|V\|^2. \tag{18}$$

For simplicity of notation, we drop the index β in this section, but we will write for example $\Phi_{0,\beta}$ in the subsequent sections of the paper.

A stronger version of this result was stated by Grillakis, Shatah, and Strauss in [**10**, Lemma 6.2], but the proof given there contained a gap, as noted in the errata at the end of [**11**, page 347]. As a replacement, the proposition above (weaker than the original Grillakis–Shatah–Strauss result, but adequate for our purposes) was proposed in the errata [**11**], without proof. We have not found a clear definition and meaning of the function Φ_- in [**11**], so therefore, for the convenience of the reader, we write the details of the proof in the following.

Proof of Proposition 1. It is easy to check that H is indeed a self-adjoint operator. On the other hand, let $V = (v_1, v_2)^T$. We have, from (15),

$$\begin{aligned}\langle HV | V \rangle &= \left\langle \begin{pmatrix} Tv_1 - \beta\partial_x v_2 \\ \beta\partial_x v_1 + v_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle \\ &= (Tv_1|v_1) - \beta(\partial_x v_2|v_1) + \beta(\partial_x v_1|v_2) + (v_2|v_2) \\ &= (L_\beta^+ v_1|v_1) + \beta^2(\partial_x v_1|\partial_x v_1) + 2\beta(v_2|\partial_x v_1) + (v_2|v_2) \\ &= (L_\beta^+ v_1|v_1) + (\beta\partial_x v_1 + v_2|\beta\partial_x v_1 + v_2). \end{aligned} \tag{19}$$

Recalling the notation of Corollary 1, we define

$$\Phi_0 := \begin{pmatrix} \partial_x Q_\beta \\ -\beta\partial_{xx} Q_\beta \end{pmatrix}, \quad \Phi_- := \begin{pmatrix} Q_\beta^- \\ -\beta\partial_x Q_\beta^- \end{pmatrix}. \tag{20}$$

One can check from (19) that $\langle \Phi_0 | \Phi_0 \rangle \neq 0$ and $\langle H\Phi_0 | \Phi_0 \rangle = 0$, since $L_\beta^+ \partial_x Q_\beta = 0$. Note additionally that by parity $\langle \Phi_- | \Phi_0 \rangle = 0$. Therefore, (16) is directly satisfied.

Also notice that

$$H\Phi_- = -\lambda_0 \begin{pmatrix} Q_\beta^- \\ 0 \end{pmatrix}. \tag{21}$$

We now prove (18). Let $V = (v_1, v_2)^T \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfy the orthogonality properties

$$\langle V|\Phi_0\rangle = \langle V|H\Phi_-\rangle = 0.$$

Let us decompose v_1 in terms of the nonpositive spectral elements of L_β^+, and L^2-orthogonally:

$$v_1 = aQ_\beta^- + b\partial_x Q_\beta + q, \quad (q|Q_\beta^-) = (q|\partial_x Q_\beta) = 0.$$

From the orthogonality conditions in (18), we have

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} Q_\beta^- \\ 0 \end{pmatrix} \right\rangle = 0,$$

so that $a = 0$, and hence, from Corollary 1,

$$\langle HV|V\rangle = (L_\beta^+ q|q) + (\beta\partial_x v_1 + v_2|\beta\partial_x v_1 + v_2) \geqslant \nu_0 \|q\|_{H^1}^2 \geqslant 0. \tag{22}$$

We now argue by contradiction. Assume that there exists a normalized sequence $V^n = (v_1^n, v_2^n)^T \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ that satisfies the orthogonality properties

$$\langle V^n|\Phi_0\rangle = \langle V^n|H\Phi_-\rangle = 0, \quad \|V^n\|^2 = 1, \quad \text{and such that} \quad \langle HV^n|V^n\rangle \to 0. \tag{23}$$

Let us write the L^2-orthogonal decomposition for each v_1^n:

$$v_1^n = b_n \partial_x Q_\beta + q_n, \qquad (q_n|\partial_x Q_\beta) = 0.$$

Then, in view of (22) and (23) applied this time to the sequence V^n, $q_n \to 0$ in H^1 and $\beta\partial_x v_1^n + v_2^n \to 0$ in L^2. Now, we compute

$$\begin{aligned} 0 = \langle V^n|\Phi_0\rangle &= \int (v_1^n \partial_x Q_\beta - v_2^n \beta \partial_{xx} Q_\beta) \\ &= \int v_1^n \partial_x Q_\beta + \beta \int (\beta \partial_x v_1^n + o_{L^2}(1)) \partial_{xx} Q_\beta \\ &= b_n \|\partial_x Q_\beta\|_{L^2}^2 + \beta^2 \int (b_n \partial_{xx} Q_\beta + \partial_x q_n) \partial_{xx} Q_\beta + o(1) \\ &= b_n (\|\partial_x Q_\beta\|_{L^2}^2 + \beta^2 \|\partial_{xx} Q_\beta\|_{L^2}^2) - \beta^2 \int q_n \partial_{xxx} Q_\beta + o(1). \end{aligned}$$

Now $q_n \to 0$ in L^2, so that $(q_n|\partial_{xxx}Q_\beta) \to 0$, and hence $b_n \to 0$ as $n \to +\infty$. But, in this case, $v_1^n = b_n\partial_x Q_\beta + q_n \to 0$ in H^1 and $v_2^n = \beta\partial_x v_1^n + o_{L^2}(1) \to 0$ in L^2. Hence $\|V^n\|^2 = \|v_1^n\|_{H^1}^2 + \|v_2^n\|_{L^2}^2 \to 0$, a contradiction to (23).

It remains to show that $\langle H\Phi_-|\Phi_-\rangle < 0$, namely (17). Indeed,

$$\langle H\Phi_-|\Phi_-\rangle = -\lambda_0\left\langle\begin{pmatrix} Q_\beta^- \\ 0\end{pmatrix}\middle|\begin{pmatrix} Q_\beta^- \\ \partial_x Q_\beta^-\end{pmatrix}\right\rangle = -\lambda_0\|Q_\beta^-\|_{L^2}^2 < 0. \qquad \square$$

2.2. Eigenfunctions of the linearized flow and Hessian.

It is still unclear whether or not the coercivity property (18)—a key point in the proof of any stability result—is useful for us, since solitons are actually unstable. It turns out that, for our purposes, we need a different version of Proposition 1, for the linearized operator of the **flow around** Q. In order to state such a result, we introduce some additional notation.

Let $\beta \in \mathbb{R}$, $|\beta| < 1$ be a Lorentz parameter, and consider the operators T, J, L, and H defined in (12)–(15). Let

$$\mathscr{L} = \mathscr{L}(\beta) = JL = \begin{pmatrix} 0 & \mathrm{Id} \\ -T & 0\end{pmatrix}, \tag{24}$$

and

$$\mathscr{H} = \begin{pmatrix} -\beta\partial_x & -T \\ \mathrm{Id} & -\beta\partial_x\end{pmatrix} = -HJ. \tag{25}$$

Concerning this last operator, we prove the following result.

Lemma 2. *Let $\beta \in \mathbb{R}$, $|\beta| < 1$, $\gamma = (1-\beta^2)^{-1/2}$ be fixed parameters, and λ_0 from Lemma 1. There are functions $Z_0 = Z_{0,\beta}$, and $Z_\pm = Z_{\pm,\beta}$, with components exponentially decreasing in space, satisfying the spectral equations*

$$\mathscr{H}Z_0 = 0, \quad \text{and} \quad \mathscr{H}Z_\pm = \pm\frac{\sqrt{\lambda_0}}{\gamma}Z_\pm. \tag{26}$$

Moreover, by the nondegeneracy of the kernel spanned by Φ_0, we can assume that $\Phi_0 = JZ_0$.

Proof. The proof is similar to that of [10]. In particular, we obtain explicit expressions for Z_0 and $Z_\pm$ in the following.

The eigenvalue problem $\mathscr{H}Z = \lambda Z$ now reads, with $Z(x) = (\hat{Z}_1(x), \hat{Z}_2(x))^T$,

$$T\hat{Z}_2 + \beta(\hat{Z}_1)_x + \lambda\hat{Z}_1 = 0, \quad \hat{Z}_1 - \beta(\hat{Z}_2)_x - \lambda\hat{Z}_2 = 0. \tag{27}$$

From the fact that T is defined with the soliton $Q_\beta(x) = Q(\gamma x)$, it is better to use functions depending on the coordinate $s = \gamma x$. Let

$$Z_1(s) = \hat{Z}_1(x), \quad Z_2(s) = \hat{Z}_2(x).$$

Replacing Z_1 in the first equation in (27), we get, in the variable s,

$$-\gamma^2 Z_2'' + Z_2 - f'(Q)Z_2 + \beta\gamma(\lambda Z_2' + \beta\gamma Z_2'') + \lambda(\beta\gamma Z_2' + \lambda Z_2) = 0,$$

namely

$$-Z_2'' + Z_2 - f'(Q)Z_2 + 2\beta\gamma\lambda Z_2' = -\lambda^2 Z_2. \tag{28}$$

Performing the transformation $Z_2(s) := \tilde{Z}_2(s)e^{\beta\gamma\lambda s}$, where $s \in \mathbb{R}$, we get

$$-\tilde{Z}_2'' + \tilde{Z}_2 - f'(Q)\tilde{Z}_2 = -(\beta^2\gamma^2 + 1)\lambda^2\tilde{Z}_2 = -\lambda^2\gamma^2\tilde{Z}_2.$$

Therefore, by virtue of Lemma 1, we can take $\tilde{Z}_2 = Q^-(s)$ and $\lambda_\pm\gamma = \pm\sqrt{\lambda_0}$, where $-\lambda_0 < 0$ is the first eigenvalue of the standard Schrödinger operator L^+, defined in (10). Thus,

$$Z_{\pm,2}(s) = Q^-(s)e^{\pm\beta\sqrt{\lambda_0}s}.$$

Note that, from (11), $Z_{\pm,2}$ decreases exponentially at both sides of the origin, since $|\beta| < 1$ and $\beta\sqrt{\lambda_0} - \sqrt{1+\lambda_0} < 0$.

From (27), we have

$$\begin{aligned} Z_{\pm,1}(s) &= \beta\gamma Z_{\pm,2}'(s) + \lambda_\pm Z_{\pm,2}(s) \\ &= \left[\beta\gamma(Q^-)_s \pm \beta^2\gamma\sqrt{\lambda_0}Q^- \pm \frac{\sqrt{\lambda_0}}{\gamma}Q^-\right]e^{\pm\beta\sqrt{\lambda_0}s} \\ &= \gamma[\beta(Q^-)_s \pm \sqrt{\lambda_0}Q^-]e^{\pm\beta\sqrt{\lambda_0}s}. \end{aligned}$$

By the same reasons as above, $Z_{\pm,1}$ is an exponentially decreasing function. From these identities, we have

$$\begin{aligned} Z_\pm(x) &= \begin{pmatrix} \gamma\beta(Q^-)_s(\gamma x) \pm \gamma\sqrt{\lambda_0}Q^-(\gamma x) \\ Q^-(\gamma x) \end{pmatrix} e^{\pm\beta\sqrt{\lambda_0}\gamma x} \\ &= \begin{pmatrix} \beta(Q_\beta^-)_x(x) \pm \gamma\sqrt{\lambda_0}Q_\beta^-(x) \\ Q_\beta^-(x) \end{pmatrix} e^{\pm\beta\sqrt{\lambda_0}\gamma x}. \end{aligned} \tag{29}$$

Now, we consider the computation of Z_0. Replacing $\lambda = 0$ in (28), we can choose

$$Z_{0,2}(s) = Q'(s), \quad \text{and} \quad Z_{0,1} = \beta\gamma Q''(s),$$

from which we get

$$Z_0(x) = \gamma \begin{pmatrix} Z_{0,1}(x) \\ Z_{0,2}(x) \end{pmatrix} = \gamma \begin{pmatrix} \beta\gamma Q''(\gamma x) \\ Q'(\gamma x) \end{pmatrix} = \begin{pmatrix} \beta Q''_\beta(x) \\ Q'_\beta(x) \end{pmatrix}. \tag{30}$$

It is clear that $\mathscr{H} Z_0 = 0$. Similarly, we have $\mathscr{H} Z_\pm = \pm(\sqrt{\lambda_0}/\gamma) Z_\pm$, which proves (26). ☐

In order to prove Proposition 2, we need to prove the existence of two additional functions, both associated to $Z_\pm$.

LEMMA 3. *There exist unique functions $Y_\pm$, with components exponentially decreasing in space, such that*

$$HY_\pm = Z_\pm, \quad \langle \Phi_0 | Y_\pm \rangle = 0.$$

Moreover, $Y_\pm$ satisfy the additional orthogonality conditions $\langle Y_\pm | HY_\pm \rangle = 0$.

Proof. Let us prove the existence of $Y_\pm$. It is well known that a necessary and sufficient condition for existence is the following condition: it suffices to check that $Z_\pm$ are orthogonal to Φ_0, the generator of the kernel of H. Indeed, we have from (26), (25), the self-adjointedness of H, and Proposition 1, that

$$\langle \Phi_0 | Z_\pm \rangle = \pm \frac{\gamma}{\sqrt{\lambda_0}} \langle \Phi_0 | \mathscr{H} Z_\pm \rangle = \mp \frac{\gamma}{\sqrt{\lambda_0}} \langle \Phi_0 | HJZ_\pm \rangle = 0.$$

However, we need some additional estimates on $Y_\pm$. In what follows, we write down explicitly the equation $HY_\pm = Z_\pm$. It is not difficult to check that $Y_\pm = (Y_{\pm,1}, Y_{\pm,2})^T$ satisfies the equations

$$TY_{\pm,1} - \beta(Y_{\pm,2})_x = Z_{\pm,1}, \quad \beta(Y_{\pm,1})_x + Y_{\pm,2} = Z_{\pm,2}.$$

Replacing the second equation in the first one, we get (see (10))

$$L_\beta^+ Y_{\pm,1} = \beta(Z_{\pm,2})_x + Z_{\pm,1}.$$

Note that $(\beta(Z_{\pm,2})_x + Z_{\pm,1} | \partial_x Q_\beta) = 0$. Therefore, $Y_{\pm,1}$ exists and it is exponentially decreasing, with the same rate as $Z_{\pm,1}$ and $Z_{\pm,2}$. A similar conclusion follows for $Y_{\pm,2}$.

Since $Y_\pm$ is unique modulo the addition of a constant times Φ_0, it is clear that we can choose $Y_\pm$ such that $\langle \Phi_0 | Y_\pm \rangle = 0$. On the other hand, from Lemma 2,

$$\begin{aligned} \langle Y_\pm | HY_\pm \rangle &= \langle Y_\pm | Z_\pm \rangle = \pm \frac{\gamma}{\sqrt{\lambda_0}} \langle Y_\pm | \mathscr{H} Z_\pm \rangle = \mp \frac{\gamma}{\sqrt{\lambda_0}} \langle HY_\pm | JZ_\pm \rangle \\ &= \mp \frac{\gamma}{\sqrt{\lambda_0}} \langle Z_\pm | JZ_\pm \rangle = 0. \end{aligned}$$ ☐

The main result of this section is the following alternative to Proposition 1.

PROPOSITION 2. *There exists* $\mu_0 > 0$ *such that the following holds. Let* $V \in H^1 \times L^2$ *such that* $\langle \Phi_0 | V\rangle = 0$. *Then*

$$\langle HV|V\rangle \geqslant \mu_0 \|V\|^2 - \frac{1}{\mu_0}[\langle Z_+|V\rangle^2 + \langle Z_-|V\rangle^2].$$

Proof. It is enough to prove that the orthogonalities $\langle \Phi_0|V\rangle = \langle Z_+|V\rangle = \langle Z_-|V\rangle = 0$ imply that

$$\langle HV|V\rangle \geqslant \mu_0 \|V\|^2,$$

for some $\mu_0 > 0$, independently of V. In order to prove this assertion, we first assume that $\beta \neq 0$ and decompose orthogonally V and $Y_\pm$ (cf. the previous Lemma) as follows:

$$V = \tilde{V} + \alpha_- \Phi_- + \alpha_0 \Phi_0, \qquad Y_\pm = \tilde{Y}_\pm + \delta_0 \Phi_0 + \delta_\pm \Phi_-, \tag{31}$$

with

$$\langle \tilde{V}|\Phi_0\rangle = \langle \tilde{Y}_\pm|\Phi_0\rangle = \langle \tilde{V}|H\Phi_-\rangle = \langle \tilde{Y}_\pm|H\Phi_-\rangle = 0. \tag{32}$$

Since $\langle \Phi_0|\Phi_-\rangle = \langle \Phi_0|V\rangle = \langle \Phi_0|Y_\pm\rangle = 0$ and $\langle \Phi_-|H\Phi_-\rangle < 0$, it is clear that $\alpha_0 = \delta_0 = 0$ and α_-, $\delta_\pm$ are well defined. Moreover, we have the following.

CLAIM. *For all* $\beta \in (-1, 1)\backslash\{0\}$, $\tilde{Y}_+$ *and* $\tilde{Y}_-$ *are linearly independent as* $L^2(\mathbb{R})^2$ *vector-valued functions with real coefficients.*

Indeed, to see this, assume that there is $\tilde{\lambda} \neq 0$ such that $\tilde{Y}_+ = \tilde{\lambda}\tilde{Y}_-$. Then, from the previous decomposition and Lemma 3,

$$Z_+ - \tilde{\lambda} Z_- = H(Y_+ - \tilde{\lambda} Y_-) = (\delta_+ - \tilde{\lambda}\delta_-)H\Phi_-. \tag{33}$$

This identity contradicts (29) and (20), which establish that Z_+ and Z_- have essentially different rates of decay at infinity, different to that of $H\Phi_-$, for all $\beta \neq 0$, which makes (33) impossible.

The analysis is now similar to that in [7, Lemma 5.2]. We have, from (31),

$$\langle HV|V\rangle = \langle H\tilde{V} + \alpha_- H\Phi_-|\tilde{V} + \alpha_-\Phi_-\rangle = \langle H\tilde{V}|\tilde{V}\rangle + \alpha_-^2\langle H\Phi_-|\Phi_-\rangle. \tag{34}$$

On the other hand, since $\langle Z_\pm|V\rangle = 0$, we have, from Lemma 3,

$$0 = \langle Y_\pm|HV\rangle = \langle \tilde{Y}_\pm + \delta_\pm\Phi_-|H\tilde{V} + \alpha_- H\Phi_-\rangle = \langle \tilde{Y}_\pm|H\tilde{V}\rangle + \alpha_-\delta_\pm\langle H\Phi_-|\Phi_-\rangle.$$

Similarly,

$$0 = \langle HY_\pm|Y_\pm\rangle = \langle H\tilde{Y}_\pm|\tilde{Y}_\pm\rangle + \delta_\pm^2\langle H\Phi_-|\Phi_-\rangle.$$

We then get

$$\langle HV|V\rangle = \langle H\tilde{V}|\tilde{V}\rangle - \frac{\langle \tilde{Y}_-|H\tilde{V}\rangle\langle \tilde{Y}_+|H\tilde{V}\rangle}{\sqrt{\langle H\tilde{Y}_+|\tilde{Y}_+\rangle\langle H\tilde{Y}_-|\tilde{Y}_-\rangle}}. \tag{35}$$

Consider

$$a := \sup_{W\in \mathrm{Span}(\tilde{Y}_+,\tilde{Y}_-)\setminus\{\vec{0}\}} \left| \frac{\langle \tilde{Y}_+|HW\rangle}{\sqrt{\langle H\tilde{Y}_+|\tilde{Y}_+\rangle\langle HW|W\rangle}} \cdot \frac{\langle \tilde{Y}_-|HW\rangle}{\sqrt{\langle H\tilde{Y}_-|\tilde{Y}_-\rangle\langle HW|W\rangle}} \right|.$$

Recall that $\langle H\cdot|\cdot\rangle$ is positive definite on $\mathrm{Span}(\Phi_0, H\Phi_-)^\perp$. Hence apply Cauchy–Schwarz's inequality to both terms of the product: it transpires that $a \leqslant 1$. Furthermore, if $a = 1$ (as $\mathrm{Span}(\tilde{Y}_+, \tilde{Y}_-)$ is finite dimensional), there exists W of norm 1 such that both terms are in the equality case in the Cauchy–Schwarz inequality; that is, W and $\tilde{Y}_+$ are linearly dependent, and W and $\tilde{Y}_-$ are also linearly dependent. But it would then follow that $\tilde{Y}_+$ and $\tilde{Y}_-$ are linearly dependent, a contradiction to the above claim. This proves that $a < 1$.

Now, using H-orthogonal decomposition on $\mathrm{Span}(\Phi_0, H\Phi_-)^\perp$, we deduce that

$$\forall W \in \mathrm{Span}(\Phi_0, H\Phi_-)^\perp, \qquad \left| \frac{\langle \tilde{Y}_-|HW\rangle\langle \tilde{Y}_+|HW\rangle}{\sqrt{\langle H\tilde{Y}_+|\tilde{Y}_+\rangle\langle H\tilde{Y}_-|\tilde{Y}_-\rangle}} \right| \leqslant a\langle HW|W\rangle.$$

By (35), (32) and (18), we get

$$\langle HV|V\rangle \geqslant (1-a)\langle H\tilde{V}|\tilde{V}\rangle \geqslant \alpha_0(1-a)\|\tilde{V}\|^2 \geqslant 0,$$

and so (34) implies that $\langle H\tilde{V}|\tilde{V}\rangle \geqslant \alpha_-^2|\langle H\Phi_-|\Phi_-\rangle|$.

We then conclude that, for $C = 4/((1-a))\max(1/(\alpha_0), \|\Phi_-\|^2/(|\langle H\Phi_-|\Phi_-\rangle|))$,

$$\begin{aligned} C\langle HV|V\rangle &\geqslant C(1-a)\langle H\tilde{V}|\tilde{V}\rangle \\ &\geqslant \frac{C(1-a)}{2}(\langle H\tilde{V}|\tilde{V}\rangle + \alpha_-^2|\langle H\Phi_-|\Phi_-\rangle|) \\ &\geqslant 2\|\tilde{V}\|^2 + 2\alpha_-^2\|\Phi_-\|^2 \geqslant \|\tilde{V} + \alpha_-\Phi_-\|^2 = \|V\|^2. \end{aligned}$$

Finally, if $\beta = 0$, we proceed as follows. First of all, we have, from (29) and (20),

$$Z_\pm = Q^- \begin{pmatrix} \pm\sqrt{\lambda_0} \\ 1 \end{pmatrix}, \qquad \Phi_0 = Q' \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so that $\langle \Phi_0|V\rangle = \langle Z_\pm|V\rangle = 0$ imply $(v_1|Q') = (v_1|Q^-) = (v_2|Q^-) = 0$, where $V = (v_1, v_2)^T$. Therefore,

$$\langle HV|V\rangle = (L^+v_1|v_1) + (v_2|v_2) \geqslant \nu_0\|V\|^2.$$

□

2.3. Extension to higher dimensions. The equivalent of Lemma 1 (and therefore assumption (iv) of the Introduction) in dimension $d \geqslant 2$ has the following form.

LEMMA 4. *Assume that $d \geqslant 2$ and that assumption* (B) *holds. L^+ has exactly one negative eigenvalue, and its kernel is spanned by $(\partial_{x_i} Q)_{i=1,\dots,d}$. Its continuous spectrum is* $[1, +\infty)$.

Proof. See Maris [15] and McLeod [17]. ☐

As mentioned in the Introduction, this result is open for general nonlinearity f. In that case, we need to assume that it holds, that is, we need to assume assumption (v).

The null directions for H are now the d-dimensional vector space spanned by the functions $\Phi_{0,i} = \binom{\partial_i Q}{\beta.\nabla\partial_i Q}$. In the proof of Lemma 2, one should rather perform the transformation $\tilde{Z}_2 = Z_2 e^{-\gamma\lambda\beta\cdot x}$. The rest of the argument is dimension insensitive.

3. Construction of approximate N-solitons

In this section, we prove Theorem 1. Again, we will give a detailed proof in the one-dimensional case $d = 1$, and point out how to extend the proof in higher dimension, which is done in a similar fashion as in [16].

3.1. The topological argument. We continue with the same notation as in the previous section, in particular for $\beta \in (-1, 1)$, $Q_\beta(x) = Q(\gamma x)$. We suppose that we are given N different velocities $\beta_1, \dots, \beta_N \in (-1, 1)$, already arranged in such a way that

$$-1 < \beta_1 < \beta_2 < \cdots < \beta_N < 1, \tag{36}$$

and N translation parameters $x_1, \dots, x_N \in \mathbb{R}$, and we define the solitons and their sum

$$\begin{aligned} R_j(t,x) = R_{\beta_j,x_j}(t,x) &= \begin{pmatrix} Q_{\beta_j}(x-\beta_j t - x_j) \\ -\beta_j(\partial_x Q_{\beta_j})(x-\beta_j t - x_j) \end{pmatrix}, \\ R(t,x) &= \sum_{j=1}^N R_j(t,x), \end{aligned} \tag{37}$$

and their center

$$y_j(t) := \beta_j t + x_j, \quad j = 1, \dots, N, \tag{38}$$

Finally, given $\mathscr{B}$ a real Banach space, $x \in B$, and $r \geqslant 0$, we denote

$$B_{\mathscr{B}}(x,r) = \{y \in \mathscr{B} \mid \|x-y\|_{\mathscr{B}} \leqslant r\}$$

the *closed* ball in $\mathscr{B}$ centered at x of radius r, and $\|\cdot\|_{\mathscr{B}}$ is the associated Banach norm on $\mathscr{B}$.

LEMMA 5 (Modulation). *There exist $L_0 > 0$ and $\varepsilon_0 > 0$ such that the following holds for some $C > 0$. For any $L \geqslant L_0$ and $0 < \varepsilon < \varepsilon_0$, if $U \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ is sufficiently near a sum of solitons whose centers $\hat{y}_j$ are sufficiently far apart,*

$$\left\| U - \sum_{j=1}^N \begin{pmatrix} Q_{\beta_j} \\ -\beta_j \partial_x Q_{\beta_j} \end{pmatrix} (\cdot - \hat{y}_j) \right\| \leqslant \varepsilon, \quad \min\{|\hat{y}_j - \hat{y}_i| \mid i \neq j\} \geqslant L,$$

then there exist shifts $\tilde{y}_j = \tilde{y}_j((\beta_k, \hat{y}_k)_k)$ such that, if we define

$$V(x) = U(x) - \sum_{j=1}^N \begin{pmatrix} Q_{\beta_j} \\ -\beta_j \partial_x Q_{\beta_j} \end{pmatrix} (x - \tilde{y}_j), \quad \Phi_{0,j}(x) = \Phi_{0,\beta_j}(x - \tilde{y}_j)$$

(see Proposition 1 for the definition of $\Phi_{0,\beta}$), then

$$\|V\| \leqslant C\varepsilon, \quad \text{and} \quad \langle V | \Phi_{0,j} \rangle = 0. \tag{39}$$

In such a case, we say that U can be modulated into $(V, (\tilde{y}_j)_j)$.

Moreover, the map $U \mapsto (V, (\tilde{y}_j)_j)$ is a $\mathscr{C}^\infty$-diffeomorphism on a neighborhood of $\sum_{j=1}^N \begin{pmatrix} Q_{\beta_j} \\ -\beta_j \partial_x Q_{\beta_j} \end{pmatrix} (\cdot - \hat{y}_j)$.

Proof. This is the classical modulation result, stated as in [3, Lemma 2]. We also refer to [30, 31], and give a proof in Appendix B. □

In what follows, we introduce additional notation. Fix a constant γ_0 given by

$$\gamma_0 := \min\left\{ \frac{1}{4}\sqrt{\lambda_0} \min\left\{ \frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \ldots, \frac{1}{\gamma_N} \right\}, \right.$$
$$\left. \frac{1}{4} \min\{\gamma_1, \gamma_2, \ldots, \gamma_N\} \min\{\beta_1, \beta_2 - \beta_1, \ldots, \beta_N - \beta_{N-1}\} \right\} > 0. \tag{40}$$

Let now $\varepsilon \in (0, \varepsilon_0)$ and $L \geqslant L_0$ be given, where ε_0 and L_0 are obtained by Lemma 5. It is clear that there exists $T_0 \in \mathbb{R}$ such that, for all $t \geqslant T_0$, the center y_j of the solitons R_j satisfies

$$\min\{|y_j - y_i| \mid i \neq j\} \geqslant L.$$

From now on, we fix $t \geqslant T_0$. According to Lemma 5, if $U \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies $\|U - R(t)\| \leqslant \varepsilon$, then U can be modulated into $(V, (\tilde{y}_j)_j)$. Moreover, up to increasing T_0, we can assume that $e^{-\gamma_0 T_0} < \varepsilon_0$.

We can then introduce the modulated solitons and their sum

$$\tilde{R}_j(x) := \begin{pmatrix} Q_{\beta_j} \\ -\beta_j \partial_x Q_{\beta_j} \end{pmatrix} (x - \tilde{y}_j), \quad \tilde{R}(x) := \sum_{j=1}^{N} \tilde{R}_j(x), \tag{41}$$

and, for any $j = 1, \dots, N$ (see Proposition 1 and Lemma 2 for the definitions), let

$$\begin{cases} Z_{\pm,j}(s) := Z_{\pm}(\gamma_j(s - \tilde{y}_j)), \\ \Phi_{0,j}(s) := \Phi_0(\gamma_j(s - \tilde{y}_j)), \quad \Phi_{-,j}(s) := \Phi_{-}(\gamma_j(s - \tilde{y}_j)), \end{cases} \tag{42}$$

where $\gamma_j := (1 - \beta_j^2)^{-1/2}$, and

$$a_{\pm,j} := \langle V | Z_{\mp,j} \rangle, \tag{43}$$

(please mind the signs) along with the vectors

$$\boldsymbol{a}_+ = (a_{+,j})_j, \quad \boldsymbol{a}_- = (a_{-,j})_j, \quad \text{and} \quad \tilde{\boldsymbol{y}} = (\tilde{y}_j)_j. \tag{44}$$

We are now in a position to define our shrinking set.

DEFINITION 1 (Shrinking set $\mathscr{V}(t)$). For $t \geqslant T_0$, we define the set

$$\mathscr{V}(t) \subset B_{H^1 \times L^2}(R(t), \varepsilon_0)$$

in the following way: $U \in \mathscr{V}(t)$ if and only if U can be modulated into $(V, \tilde{\boldsymbol{y}})$ where (see (41) and (44))

$$V = U - \sum_{j=1}^{N} \tilde{R}_j,$$

with

$$\|V\| \leqslant e^{-\gamma_0 t}, \quad |\tilde{y}_j - \beta_j t - x_j| \leqslant e^{-\gamma_0 t}, \tag{45}$$

$$\|\boldsymbol{a}_+\|_{\ell^2} \leqslant e^{-3\gamma_0 t/2}, \quad \|\boldsymbol{a}_-\|_{\ell^2} \leqslant e^{-3\gamma_0 t/2}. \tag{46}$$

DEFINITION 2. We denote by $\varphi = (u, \partial_t u)^T$ the flow of the NLKG equation; that is, given $S_0 \in \mathbb{R}$ and $U_0 \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$,

$$t \mapsto \varphi(S_0, t, U_0) \tag{47}$$

is the solution to (NLKG) with initial data U_0 at time S_0 (with values in $H^1 \times L^2$).

In most of what we do, we will have $t \leqslant S_0$ so that U_0 can be thought of as a final data, and we work backwards in time. The key result of this section is the following construction of an approximate N-soliton.

PROPOSITION 3 (Approximate N-soliton). *There exists $T_0 > 0$ such that the following holds. For any $S_0 \geqslant T_0$, there exists a final data U_0 such that*

$$\forall t \in [T_0, S_0], \quad \varphi(S_0, t, U_0) \in \mathscr{V}(t).$$

At this point, the solution $\varphi(S_0, t, U_0)$ depends on S_0. To prove Theorem 1, we will need to derive such a solution independent of S_0, which we will do via a compactness argument in the next (and last) section, Section 4.

Our goal is now to prove Proposition 3.

Fix $S_0 \geqslant T_0$. Consider an initial data U_0 at time S_0 such that $U_0 \in \mathscr{V}(S_0)$. Due to the blow-up criterion for (NLKG), and the fact that $R(t)$ defined in (37) is bounded in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, we have that $\varphi(S_0, t, U_0)$ is defined at least as long as it belongs to $B_{H^1 \times L^2}(R(t), 1)$. In particular, $\varphi(S_0, t, U_0)$ does not blow up as long as it belongs to $\mathscr{V}(t)$, and we can define the (backward) exit time

$$T^*(U_0) := \inf\{T \in [T_0, S_0] \mid \forall t \in [T, S_0],\ \varphi(S_0, t, U_0) \in \mathscr{V}(t)\}.$$

Notice that we could have $T^*(U_0) = S_0$. Also observe that $\varphi(S_0, t, U_0)$ is defined and can be modulated on a open neighborhood containing the point S_0, due to local well-posedness theory, up to choosing T_0 sufficiently large. Therefore we can perform computations, for example differentiations, even in the case when $T^*(U_0) = S_0$, by a standard limiting procedure. Our goal is to find $U_0 \in \mathscr{V}(T_0)$ such that $T^*(U_0) = T_0$.

In order to show such an assertion, we will only consider some very specific initial data, namely $U_0 \in \mathscr{V}(S_0)$ such that (see (44))

- $U_0 \in R(S_0) + \operatorname{Span}(Z_\pm(\gamma_j(\cdot - \hat{y}_j))_{j=1,\dots,N})$,
- $\boldsymbol{a}_-(S_0) = 0$, and
- $\boldsymbol{a}_+(S_0) \in B_{\mathbb{R}^N}(0, e^{-3\gamma_0 S_0/2})$.

These conditions can be satisfied due to the almost orthogonality of $Z_{\pm,j}$, and this is the content of the following.

LEMMA 6 (Modulated final data). *Let $S_0 \geqslant T_0$ be sufficiently large. There exists a $\mathscr{C}^1$ map $\Theta : B_{\mathbb{R}^N}(0,1) \to \mathscr{V}(S_0)$ as follows. Given $\mathfrak{a}_+ = (\mathfrak{a}_{+,j})_j \in B_{\mathbb{R}^N}(0,1)$, $U_0 =: \Theta(\mathfrak{a}_+) \in \mathscr{V}(S_0)$ such that U_0 can be modulated into $(V_0, \tilde{\boldsymbol{y}})$ and the* *associated parameters* (44) *satisfy*

$$\boldsymbol{a}_+(S_0) = e^{-3\gamma_0 S_0/2}\mathfrak{a}_+, \quad \boldsymbol{a}_-(S_0) = 0. \tag{48}$$

Moreover,

$$\|V_0\| \leqslant Ce^{-3\gamma_0 T_0/2}. \tag{49}$$

Proof. The main idea is to consider the map $B_{\mathbb{R}^{2N}}(0,1) \to B_{\mathbb{R}^{2N}}(0,1)$, $\mathfrak{b}_\pm \mapsto \mathfrak{a}_\pm$, where $\mathfrak{a}_\pm$ corresponds to the data $U_0 = R(S_0) + \sum_{\pm,j} \mathfrak{b}_{\pm,j} Z_\pm(\gamma_j(\cdot - \hat{y}_j))$, and to invoke the implicit mapping theorem. We refer to [3, Lemma 3], and provide a fully detailed proof in Appendix B for the convenience of the reader. ☐

If $T^* := T^*(U_0) > T_0$, by maximality, we also have that, for the function $\varphi(S_0, T^*, U_0)$, at least one of the inequalities in the definition of $\mathscr{V}(T^*)$ is actually an equality. It turns out that the equality is achieved by $\boldsymbol{a}_+(T_0)$ only, and that the rescaled quantity $e^{3\gamma_0 T^*/2}\boldsymbol{a}_+(t)$ is *transverse* to the sphere at $t = T^*$. This is at the heart of the proof and is the content of the following.

Proposition 4. *Let* $\mathfrak{a}_+ \in B_{\mathbb{R}^N}(0,1)$, *and assume that its maximal exit time is (strictly) greater that* T_0*:*

$$T^* = T^*(\Theta(\mathfrak{a}_+)) > T_0.$$

Denote, for all $t \in [T^*, S_0]$, *the associated modulation* $(V(t), \tilde{\mathbf{y}}(t))$ *of* $\varphi(S_0, t, \Theta(\mathfrak{a}_+))$, *defined in* (47). *Then, for all* $t \in [T^*, S_0]$,

$$\|V(t)\| \leqslant \tfrac{1}{2}e^{-\gamma_0 t}, \quad |\tilde{y}_j(t) - \beta_j t - x_j| \leqslant \tfrac{1}{2}e^{-\gamma_0 t}, \tag{50}$$

$$\|\boldsymbol{a}_-(t)\|_{\ell^2} \leqslant \tfrac{1}{2}e^{-3\gamma_0 t/2}, \tag{51}$$

and

$$\|\boldsymbol{a}_+(T^*)\|_{\ell^2} = e^{-3\gamma_0 T^*/2}. \tag{52}$$

Furthermore, $\boldsymbol{a}_+(T^*)$ *is transverse to the sphere; that is,*

$$\frac{d}{dt}(e^{3\gamma_0 t}\|\boldsymbol{a}_+(t)\|_{\ell^2}^2)\bigg|_{t=T^*} < 0.$$

(Recall that we consider the flow backwards in time.)

For the sake of continuity, we postpone the proof of Proposition 4 until later, and conclude the proof of Proposition 3 here, assuming Proposition 4.

Let us state a few direct consequences of Proposition 4 (their proofs will also be given later, in Section 3.2).

COROLLARY 2. *We have the following properties.*

(1) *The set of final data which give rise to solutions which exit strictly after* T_0

$$\Omega := \{\mathfrak{a}_+ \in B_{\mathbb{R}^N}(0,1) \mid T^*(\Theta(\mathfrak{a}_+)) > T_0\}$$

is open (in $B_{\mathbb{R}^N}(0,1)$*).*

(2) *The map* $\Omega \to \mathbb{R}$, $\mathfrak{a}_+ \mapsto T^*(\Theta(\mathfrak{a}_+)) \in \mathbb{R}$ *is continuous (we emphasize that the data belong to* Ω*).*

(3) *The exit is instantaneous on the sphere:*

$$\textit{if} \quad \|\mathfrak{a}_+\|_{\ell^2} = 1, \quad \textit{then } T^*(\mathfrak{a}_+) = S_0. \tag{53}$$

We are now in a position to complete the proof of Proposition 3.

End of the proof of Proposition 3. We argue by contradiction. Assume that all possible $\mathfrak{a}_+ \in B_{\mathbb{R}^N}(0,1)$ give rise to initial data $U_0 = \Theta(\mathfrak{a}_+) \in \mathcal{V}(S_0)$ and corresponding solutions $\varphi(S_0, t, U_0)$ that exit $\mathcal{V}(t)$ strictly after T_0; that is,

$$\text{assume that } \Omega = B_{\mathbb{R}^N}(0,1). \tag{54}$$

Given $U_0 \in \mathcal{V}(S_0)$, we denote $\Phi(U_0)$ the rescaled projection of the *exit spot*,

$$\Phi(U_0) = e^{3\gamma_0 T^*(U_0)/2} \boldsymbol{a}_+(T^*(U_0)),$$

so that $\Phi(U_0) \in B_{\mathbb{R}^N}(0,1)$. Let us finally consider the rescaled projection of the exit spot Ψ, defined as follows:

$$\Psi : B_{\mathbb{R}^N}(0,1) \to B_{\mathbb{R}^N}(0,1), \quad \mathfrak{a}_+ \mapsto \Psi(\mathfrak{a}_+) = \Phi \circ \Theta(\mathfrak{a}_+).$$

Corollary 2 then translates into the following properties for Ψ.

- $\Psi : B_{\mathbb{R}^N}(0,1) \to \mathbb{S}^{N-1}$ is continuous (like T^*, Φ and Θ).
- If $\|\mathfrak{a}_+\|_{\ell^2} = 1$, $\Psi(\mathfrak{a}_+) = \mathfrak{a}_+$ (see (53) and (48)); that is, $\Psi|_{\mathbb{S}^{N-1}} = \mathrm{Id}$.

These two affirmations contradict Brouwer's theorem. Hence our assumption (54) is wrong, and there exists $\mathfrak{a}^+$ such that the solution $U(t) = \varphi(S_0, t, \Theta(\mathfrak{a}_+))$ satisfies $T^*(\Theta(\mathfrak{a}_+)) = T_0$. In particular, $U(t) \in \mathcal{V}(t)$ for all $t \in [T_0, S_0]$, and $U_0 := U(S_0) = \Theta(\mathfrak{a}_+)$ satisfies the conditions of Proposition 3. ☐

3.2. Bootstrap estimates. This section is devoted to the last remaining results needed to complete Proposition 3: Proposition 4 and Corollary 2.

Proof of Proposition 3. **Step 1.** First, we introduce some notation. Consider the flow $\varphi(t) = \varphi(S_0, t, \Theta(\mathfrak{a}_+))$ given by Proposition 4, and valid for all $t \in [T^*, S_0]$. From Lemma 5, we have

$$\varphi(t) = \tilde{R}(t) + V(t), \tag{55}$$

where

$$\tilde{R}(t,x) = \sum_{j=1}^{N} \tilde{R}_j(t,x), \quad \tilde{R}_j(t,x) = (Q_{\beta_j}, \partial_t Q_{\beta_j})^T(x - \tilde{y}_j(t)), \tag{56}$$

$$\tilde{y}_j(t) = \beta_j t + \tilde{x}_j(t), \tag{57}$$

and

$$V(t) = (v_1(t), v_2(t))^T.$$

Additionally, from the equation satisfied by φ, we have

$$\varphi_t = \begin{pmatrix} 0 & \mathrm{Id} \\ \partial_x^2 - \mathrm{Id} & 0 \end{pmatrix} \varphi + \begin{pmatrix} 0 \\ f(u) \end{pmatrix},$$

where $\varphi = (u, u_t)^T$. Replacing the decomposition (55), we have, for

$$Q_{\beta_j} = Q_{\beta_j}(\cdot - \tilde{y}_j),$$

$$V_t = \begin{pmatrix} 0 & \mathrm{Id} \\ \partial_x^2 - \mathrm{Id} + f'(Q_{\beta_j}) & 0 \end{pmatrix} V + Rem(t) = \mathcal{L}_j V + Rem(t), \tag{58}$$

with $\mathcal{L}_j := \mathcal{L}(\beta_j)$ defined in (24) with potential $f'(Q_{\beta_j}(\cdot - \tilde{y}_j))$ (and the same for $\mathcal{H}_j := \mathcal{H}_{\beta_j}$),

$$\begin{aligned} Rem(t) &:= \begin{pmatrix} 0 & \mathrm{Id} \\ \partial_x^2 - \mathrm{Id} & 0 \end{pmatrix} \tilde{R} - \tilde{R}_t + \begin{pmatrix} 0 \\ f(u) - f'(Q_{\beta_j}) v_1 \end{pmatrix} \\ &= \sum_{k=1}^{N} \tilde{x}_k'(t) \partial_x \begin{pmatrix} Q_{\beta_k} \\ -\beta_k \partial_x Q_{\beta_k} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ f\left(\sum_{k=1}^{N} Q_{\beta_k} + v_1 \right) - \sum_{k=1}^{N} f(Q_{\beta_k}) - f'(Q_{\beta_j}) v_1 \end{pmatrix}. \end{aligned}$$

First of all, note that from (20) we have $\binom{\partial_x Q_{\beta_k}}{-\beta_k\partial_{xx}Q_{\beta_k}} = \Phi_{0,k}$. If we take the scalar product of (58) with $(\tilde{R}_j)_x$, then the orthogonality (39) (coming from modulation) leads to the estimate

$$|\tilde{x}_j'(t)| \leqslant C(\|V(t)\| + e^{-3\gamma_0 t}), \tag{59}$$

valid for all $j = 1, \ldots, N$. Indeed, we have

$$\langle(\tilde{R}_j)_x|V_t\rangle = \langle(\tilde{R}_j)_x|\mathcal{L}_j V\rangle + \langle(\tilde{R}_j)_x|Rem(t)\rangle.$$

Note that, from (57),

$$\langle(\tilde{R}_j)_x|V_t\rangle = -\langle(\tilde{R}_j)_{xt}|V\rangle = (\beta_j + \tilde{x}_j'(t))\langle\partial_{xx}\tilde{R}_j|V\rangle.$$

Consequently,

$$|\langle(\tilde{R}_j)_x|V_t\rangle| \leqslant C(1 + |\tilde{x}_j'(t)|)\|V(t)\|.$$

On the other hand,

$$\langle(\tilde{R}_j)_x|\mathcal{L}_j V\rangle = \left\langle\begin{pmatrix}\partial_x Q_{\beta_j}\\ \partial_{xt}Q_{\beta_j}\end{pmatrix}\middle|\begin{pmatrix}v_2\\ -T_{\beta_j}v_1\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix}\partial_x Q_{\beta_j}\\ -T_{\beta_j}\partial_{xt}Q_{\beta_j}\end{pmatrix}\middle|\begin{pmatrix}v_2\\ v_1\end{pmatrix}\right\rangle,$$

so that

$$|\langle(\tilde{R}_j)_x|\mathcal{L}_j V\rangle| \leqslant C\|V(t)\|.$$

Finally, we deal with the term $\langle(\tilde{R}_j)_x|Rem(t)\rangle$. From the definition of $Rem(t)$, we have

$$\begin{aligned}\langle(\tilde{R}_j)_x|Rem(t)\rangle &= \sum_{k=1}^{N}\tilde{x}_k'(t)\langle(\tilde{R}_j)_x|\Phi_{0,k}\rangle\\ &\quad + \left(\partial_{xt}Q_{\beta_j}\middle| f\left(\sum_{k=1}^{N}Q_{\beta_k} + v_1\right) - \sum_{k=1}^{N}f(Q_{\beta_k}) - f'(Q_{\beta_j})v_1\right).\end{aligned}$$

Since $(\tilde{R}_j)_x = \Phi_{0,j}$, we get

$$\langle(\tilde{R}_j)_x|\Phi_{0,j}\rangle = \|\Phi_{0,j}\|^2,$$

and, if $k \neq j$,

$$|\langle(\tilde{R}_j)_x|\Phi_{0,k}\rangle| = |\langle\Phi_{0,j}|\Phi_{0,k}\rangle| \leqslant Ce^{-3\gamma_0 t}.$$

We introduce the parameters

$$m_j := \tfrac{1}{2}(\beta_j + \beta_{j-1}),$$

with $j = 2, \ldots, N-1$, and $m_1 := -\infty$, $m_N = +\infty$. If $x \in [m_j t, m_{j+1} t]$, then, for all $p \neq j$ (see (40)),

$$|Q_{\beta_p}(t,x)| \leqslant Ce^{-3\gamma_0 t}.$$

Therefore, inside this region (note that if $d \geqslant 2$ then f is a pure power nonlinearity)

$$\left\| f\left(\sum_{k=1}^{N} Q_{\beta_k} + v_1\right) - \sum_{k=1}^{N} f(Q_{\beta_k}) - f'(Q_{\beta_j})v_1 \right\|_{L^2} \leqslant Ce^{-3\gamma_0 t} + C\|V(t)\|^2.$$

On the other hand, if $x \notin [m_j t, m_{j+1} t]$,

$$|\partial_{xt} Q_{\beta_j}| \leqslant Ce^{-3\gamma_0 t}.$$

In conclusion, we have

$$\left(\partial_{xt} Q_{\beta_j} \middle| f\left(\sum_{k=1}^{N} Q_{\beta_k} + v_1\right) - \sum_{k=1}^{N} f(Q_{\beta_k}) - f'(Q_{\beta_j})v_1 \right) \leqslant Ce^{-3\gamma_0 t} + C\|V(t)\|^2. \tag{60}$$

Collecting the preceding estimates, we get (59).

Step 2. Control of degenerate directions. The next step of the proof is to consider the dynamics of the associated scalar products $a_{\pm,j}(t)$ and $a_{0,j}(t)$ introduced in (43).

LEMMA 7. *Let $a_{\pm,j}(t)$ be as defined in (43).* There is a constant $C > 0$, independent of S_0 and $T^* \geqslant T_0$, such that, for all $t \in [T^*, S_0]$,

$$\left| a'_{\pm,j}(t) \pm \frac{\sqrt{\lambda_0}}{\gamma_j} a_{\pm,j}(t) \right| \leqslant C\|V(t)\|^2 + Ce^{-3\gamma_0 t}. \tag{61}$$

Proof. We prove the case of $a_{-,j}(t)$. The other case is similar. We compute the time derivative of $a_{-,j}$ using (56) and (58), and we choose $\gamma_0 > 0$ as small as needed, but fixed.

$$\begin{aligned} a'_{-,j}(t) &= -\tilde{y}'_j(t)\langle (Z_{+,j})_x | V(t)\rangle + \langle Z_{+,j} | V_t(t)\rangle \\ &= -\tilde{x}'_j \langle (Z_{+,j})_x | V(t)\rangle + \langle (\mathscr{L}_j^* - \beta_j \partial_x) Z_{+,j} | V(t)\rangle + \sum_{k=1}^{N} x'_k \langle \Phi_{0,k} | Z_{+,j}\rangle \\ &\quad + O(\|V(t)\|^2 + e^{-3\gamma_0 t}). \end{aligned}$$

From Lemma 3, we have $\langle \Phi_{0,j} | Z_{+,j}\rangle = 0$. Therefore, since $\mathscr{L}_j^* - \beta_j \partial_x = \mathscr{H}_j$, where $\mathscr{H}_j := \mathscr{H}(Q_{\beta_j})$ (see (25)), we have, from Lemma 2 and (59),

$$a'_{-,j}(t) = \frac{\sqrt{\lambda_0}}{\gamma_j} a_{-,j}(t) + O(|x'_j| \|V(t)\| + \|V(t)\|^2 + e^{-3\gamma_0 t})$$

$$= \frac{\sqrt{\lambda_0}}{\gamma_j} a_{-,j}(t) + O(\|V(t)\|^2 + e^{-3\gamma_0 t}).$$

Step 3. Lyapunov functional. Let $L_0 > 0$ be a large constant to be chosen later. Let $(\phi_j)_{j=1,\ldots,N}$ be a partition of the unity of $\mathbb{R}$ placed at the midpoint between two solitons. More precisely, let

$$\phi \in C^\infty(\mathbb{R}), \quad \phi' > 0, \quad \lim_{-\infty} \phi = 0, \quad \lim_{+\infty} \phi = 1. \tag{62}$$

We have (do not confuse the constant L in (63) with the operator L in (14)), for all $L > L_0$,

$$\sum_{j=1}^{N} \phi_j(t,x) \equiv 1, \qquad \phi_j(t,x) = \phi\left(\frac{x - m_j t}{L}\right) - \phi\left(\frac{x - m_{j+1} t}{L}\right), \tag{63}$$

where $m_j := \frac{1}{2}(\beta_j + \beta_{j-1})$, with $j = 2, \ldots, N-1$, and $m_1 := -\infty$, $m_N = +\infty$. We introduce the jth portion of momentum,

$$P_j[\varphi](t) := \frac{1}{2} \int \phi_j u_t u_x \, dx, \quad \varphi = (u, u_t)^T, \tag{64}$$

and the modified *Lyapunov functional*,

$$\mathcal{F}[\varphi](t) := E[\varphi](t) + 2 \sum_{j=1}^{N} \beta_j P_j[\varphi](t), \tag{65}$$

with $E[\varphi]$ being the energy defined in (2). Our first result is a suitable decomposition of $\mathcal{F}[u]$ around the multi-soliton solution.

LEMMA 8. *Let $V(t) = (v_1(t), v_2(t))^T$ be the error function defined in Proposition 4. There is a positive constant $C > 0$ such that*

$$\left| \mathcal{F}[\varphi](t) - \frac{1}{2} \int [Q_x^2 + Q^2 - 2F(Q)] \sum_{j=1}^{N} \frac{1}{\gamma_j} - \sum_{j=1}^{N} \langle H_j V | V \rangle \right| \\ \leqslant C\|V(t)\|^3 + \frac{C}{L} e^{-2\gamma_0 t}, \tag{66}$$

where

$$\langle H_j V | V \rangle := \int \phi_j (v_2^2 + (v_1)_x^2 + v_1^2 - f'(Q_{\beta_j}) v^2 + 2\beta_j v_2 (v_1)_x). \tag{67}$$

Proof. From the decomposition

$$\varphi(t) = (u, u_t)(t) = (\tilde{R}_1, \tilde{R}_2)^T + (v_1, v_2)^T(t), \tag{68}$$

we have

$$\begin{aligned}
\mathcal{F}[\varphi](t) &= \frac{1}{2}\int (u_t^2 + u_x^2 + u^2 - F(u)) + \sum_{j=1}^{N} \beta_j \int \phi_j u_t u_x \\
&= \frac{1}{2}\int (\tilde{R}_2^2 + (\tilde{R}_1)_x^2 + \tilde{R}_1^2 - 2F(\tilde{R}_1)) + \sum_{j=1}^{N} \beta_j \int \tilde{R}_2 (\tilde{R}_1)_x\, \phi_j \\
&\quad + \int \Bigg[\tilde{R}_2 v_2 + (\tilde{R}_1)_x (v_1)_x + \tilde{R}_1 v_1 - f(\tilde{R}_1) v_1 \\
&\quad + \sum_{j=1}^{N} \beta_j (\tilde{R}_2 (v_1)_x + v_2 (\tilde{R}_1)_x)\phi_j \Bigg] \\
&\quad + \frac{1}{2}\int (v_2^2 + (v_1)_x^2 + v_1^2 - f'(\tilde{R}_1) v_1^2) + \sum_{j=1}^{N} \beta_j \int v_2 (v_1)_x\, \phi_j \\
&\quad - \int (F(\tilde{R}_1 + v_1) - F(\tilde{R}_1) - f(\tilde{R}_1) v_1 - \tfrac{1}{2} f'(\tilde{R}_1) v_1^2) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us consider the term I_1. Since $\tilde{R}_2 = -\sum_{j=1}^{N} \beta_j (Q_{\beta_j})_x$ and $(\tilde{R}_1)_x = \sum_{j=1}^{N} (Q_{\beta_j})_x$, one has

$$\begin{aligned}
I_1 &= \frac{1}{2}\sum_{j=1}^{N} \int [\beta_j^2 (Q_{\beta_j})_x^2 + (Q_{\beta_j})_x^2 + Q_{\beta_j}^2 - 2F(Q_{\beta_j}) - 2\beta_j^2 (Q_{\beta_j})_x^2] + O(e^{-3\gamma_0 t}) \\
&= \frac{1}{2}\int [Q_x^2 + Q^2 - 2F(Q)] \sum_{j=1}^{N} \frac{1}{\gamma_j} + O(e^{-3\gamma_0 t}).
\end{aligned}$$

Now, we consider I_2. Integrating by parts, we have

$$\begin{aligned}
I_2 &= \int v_2 \Bigg[\tilde{R}_2 + (\tilde{R}_1)_x \sum_{j=1}^{N} \beta_j \phi_j \Bigg] \\
&\quad - \int v_1 \Bigg[(\tilde{R}_1)_{xx} - \tilde{R}_1 + f(\tilde{R}_1) + (\tilde{R}_2)_x \sum_{j=1}^{N} \beta_j \phi_j \Bigg] - \sum_{j=1}^{N} \beta_j \int v_1 \tilde{R}_2 (\phi_j)_x.
\end{aligned}$$

Note that

$$\begin{aligned}\tilde{R}_2 + (\tilde{R}_1)_x \sum_{j=1}^N \beta_j \phi_j &= \sum_{k=1}^N \left[-\beta_k (Q_{\beta_k})_x + (Q_{\beta_k})_x \sum_{j=1}^N \beta_j \phi_j \right] \\ &= \sum_{k=1}^N (Q_{\beta_k})_x \sum_{j \neq k} \beta_j \phi_j .\end{aligned}$$

Hence,

$$\int v_2 \left[\tilde{R}_2 + (\tilde{R}_1)_x \sum_{j=1}^N \beta_j \phi_j \right] = O(e^{-3\gamma_0 t}).$$

On the other hand,

$$\int v_1 \left[(\tilde{R}_1)_{xx} - \tilde{R}_1 + f(\tilde{R}_1) + (\tilde{R}_2)_x \sum_{j=1}^N \beta_j \phi_j \right] = O(e^{-3\gamma_0 t}).$$

Finally,

$$\left| \sum_{j=1}^N \beta_j \int v_1 \tilde{R}_2 (\phi_j)_x \right| \leqslant C \|v_1\|_{L^2(\mathbb{R})} e^{-2\gamma_0 t}.$$

Gathering the above estimates, we get

$$|I_2| \leqslant C e^{-3\gamma_0 t}.$$

Let us consider the integral I_3. Since $\sum_j \phi_j = 1$, we have

$$\begin{aligned} I_3 &= \frac{1}{2} \sum_{j=1}^N \int \phi_j (v_2^2 + (v_1)_x^2 + v_1^2 - f'(\tilde{R}_1) v_1^2 + 2\beta_j v_2 (v_1)_x) \\ &= \frac{1}{2} \sum_{j=1}^N \int \phi_j (v_2^2 + (v_1)_x^2 + v_1^2 - f'(Q_{\beta_j}) v_1^2 + 2\beta_j v_2 (v_1)_x) \\ &\quad - \frac{1}{2} \sum_{k \neq j} \int \phi_j f'(Q_k) v_1^2 \\ &\quad - \frac{1}{2} \sum_{j=1}^N \int \phi_j \left(f' \left(\sum_{k=1}^N Q_{\beta_k} \right) - \sum_{k=1}^N f'(Q_{\beta_k}) \right) v_1^2 .\end{aligned}$$

Fix $\ell \in \{1, \dots, N-1\}$. If $x \in [m_\ell t, m_{\ell+1} t]$, then, for all $p \neq \ell$,

$$|Q_{\beta_p}(t,x)| \leqslant C e^{-2\gamma_0 t}.$$

Therefore, for all $x \in [m_\ell t, m_{\ell+1} t]$,

$$\left| f'\left(\sum_{k=1}^{N} Q_{\beta_k}(t,x)\right) - \sum_{k=1}^{N} f'(Q_{\beta_k}(t,x))\right| \leqslant Ce^{-\gamma_0 t}.$$

Repeating the same argument for each ℓ, and using (50), we get

$$I_3 = \tfrac{1}{2}\langle H_j V | V\rangle + O(e^{-3\gamma_0 t}).$$

Finally, we consider I_4. It is not difficult to check that

$$|I_4| \leqslant C\|V\|^3.$$

Collecting the above results, we get finally (66).

Our next result describes the variation of the momentum P_j.

LEMMA 9. *There exists $C > 0$ independent of time and L such that, for all $t \in [T^*, S_0]$,*

$$\big|P_j[\varphi](t) - P_j[\varphi](S_0)\big| \leqslant \frac{C}{L} e^{-2\gamma_0 t}. \tag{69}$$

Proof. A simple computation using (NLKG) shows that

$$\begin{aligned} \partial_t P_j[\varphi](t) \;=\; & -\frac{1}{4}\int u_t^2(\phi_j)_x - \frac{1}{4}\int u_x^2(\phi_j)_x + \frac{1}{4}\int u^2(\phi_j)_x \\ & -\frac{1}{2}\int F(u)(\phi_j)_x + \frac{1}{2}\int u_t u_x(\phi_j)_t. \end{aligned} \tag{70}$$

Indeed, one has

$$\begin{aligned} \partial_t P_j[\varphi](t) &= \frac{1}{2}\int u_t u_x(\phi_j)_t + \frac{1}{2}\int u_t u_{tx}\phi_j + \frac{1}{2}\int u_{tt}u_x\phi_j \\ &= \frac{1}{2}\int u_t u_x(\phi_j)_t + \frac{1}{4}\int (u_t^2)_x\phi_j + \frac{1}{2}\int (u_{xx} - u + f(u))u_x\phi_j \\ &= \frac{1}{2}\int u_t u_x(\phi_j)_t - \frac{1}{4}\int u_t^2(\phi_j)_x - \frac{1}{4}\int (u_x^2 - u^2 + 2F(u))(\phi_j)_x, \end{aligned}$$

as desired. Now, from the decomposition (68), we replace above to obtain (compare with (60))

$$\begin{aligned} &|\partial_t P_j[\varphi](t)| \\ &\leqslant C\left(\frac{e^{-3\gamma_0 t}}{L} + \int v_2^2(\phi_j)_x + \int (v_1)_x^2(\phi_j)_x + \int v_1^2(\phi_j)_x + \int F(v_1)(\phi_j)_x\right). \end{aligned}$$

From the smallness condition of v, we get finally

$$|\partial_t P_j[\varphi](t)| \leqslant \frac{C}{L} e^{-2\gamma_0 t},$$

as desired. The conclusion follows after integration in time. ☐

The previous lemma and the energy conservation law imply the following.

COROLLARY 3. *There exists $C > 0$ independent of time and $L > 0$ such that, for all $t \in [T^*, S_0]$,*

$$|\mathcal{F}[\varphi](t) - \mathcal{F}[\varphi](S_0)| \leqslant \frac{C}{L} e^{-2\gamma_0 t}. \tag{71}$$

Now, we use the coercivity associated to H_j. A standard localization argument (see for example [**21**]), Proposition 2, and (43) give

$$\sum_{j=1}^{N} \langle H_j V | V \rangle \geqslant \nu_0 \|V(t)\|^2 - \frac{1}{\nu_0}\big(\|\boldsymbol{a}_+\|_{\ell^2}^2 + \|\boldsymbol{a}_-\|_{\ell^2}^2\big),$$

for an independent constant $\nu_0 > 0$. From this coercivity estimate, using (71) and (66), the initial bound (49), and bounding the terms in $\boldsymbol{a}_\pm$ by (46), we get that, for some $C > 0$,

$$\forall t \in [T^*, S_0], \quad \|V(t)\| \leqslant \frac{C}{\sqrt{L}} e^{-\gamma_0 t} + C e^{-3/2\gamma_0 t}.$$

Therefore, for $L \geqslant 4C^2$, we improve the first condition in (45), to get (50). We can now integrate the modulation equation (59) for $\tilde{x}_j'(t)$ to get the second estimates in (50) (by increasing L if necessary).

Now, using (61) on $\boldsymbol{a}_-'(t)$ and integrating in time, we improve in a similar way the conditions in (46), to obtain (51). In conclusion, (52) must be satisfied.

Step 4. Transversality. For notation, let $\mathcal{N}(\mathfrak{a}_+, t) := e^{3\gamma_0 t}\|\boldsymbol{a}_+(t)\|_{\ell^2}^2$. Using the expansion (61), we compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{N}(\mathfrak{a}_+, t) &= 2\sum_{j=1}^{N} e^{3\gamma_0 t} a_{+,j}'(t) a_{+,j}(t) + 3\gamma_0 \mathcal{N}(\mathfrak{a}_+, t) \\
&= -2e^{3\gamma_0 t}\sqrt{\lambda_0}\sum_{j=1}^{N} \frac{|a_{+,j}(t)|^2}{\gamma_j} \\
&\quad + O(e^{3\gamma_0 t}(\|V(t)\|_{L^2}^2 + e^{-2\gamma_0 t})\|\boldsymbol{a}_+(t)\|_{\ell^2}) + 3\gamma_0 \mathcal{N}(\mathfrak{a}_+, t) \\
&\leqslant -(2c_0 - 3\gamma_0)\mathcal{N}(\mathfrak{a}_+, t) \\
&\quad - O(e^{3\gamma_0 t}(\|V(t)\|_{L^2}^2 + e^{-2\gamma_0 t})\|\boldsymbol{a}_+(t)\|_{\ell^2}),
\end{aligned}$$

where $c_0 = \sqrt{\lambda_0}\,\min_i\{1/\gamma_i\} > 0$. Note that from (40) we have $2c_0 - 3\gamma_0 > \gamma_0 > 0$.

Now, at time $T^* = T^*(\mathfrak{a}_+)$, $\|V(T^*)\|_{L^2} = O(e^{-\gamma_0 T^*})$, whereas $\|\boldsymbol{a}_+(T^*)\|_{\ell^2} = e^{-3\gamma_0 T^*/2}$; that is, $\mathscr{N}(\mathfrak{a}_+, T^*) = 1$. Hence,

$$\frac{d}{dt}\mathscr{N}(\mathfrak{a}_+, t)\bigg|_{t=T^*(\mathfrak{a}_+)} \leqslant -(2c_0 - 3\gamma_0) + O(e^{-\gamma_0 T^*/2}).$$

Choosing T_0 larger if necessary, and as $T^* \geqslant T_0$ for all $\mathfrak{a}_+$, we get

$$\frac{d}{dt}\mathscr{N}(\mathfrak{a}_+, t)\bigg|_{t=T^*(\mathfrak{a}_+)} \leqslant -\frac{1}{2}\gamma_0 < 0. \tag{72}$$

This concludes the proof of Proposition 4. □

We end this section with the proof of Corollary 2.

Proof of Corollary 2. Let us now show that Ω is open and that the mapping $\mathfrak{a}_+ \mapsto T^*(\mathfrak{a}_+)$ is continuous. Let $\mathfrak{a}_+ \in \Omega$. We recall that $\mathscr{N}(\mathfrak{a}_+, t) = e^{3\gamma_0 t}\|\boldsymbol{a}_+(t)\|_{\ell^2}^2$. By (72), for all $\varepsilon > 0$ small, there exists $\delta > 0$ such that

- $\mathscr{N}(\mathfrak{a}_+, T^*(\mathfrak{a}_+) - \varepsilon) > 1 + \delta$, and
- for all $t \in [T^*(\mathfrak{a}_+) + \varepsilon, S_0]$ (possibly empty), $\mathscr{N}(\mathfrak{a}_+, t) < 1 - \delta$.

By continuity of the flow of the NLKG equation, it follows that there exists $\eta > 0$ such that the following holds. For all $\tilde{\mathfrak{a}}_+ \in B_{\mathbb{R}^N}(0, 1)$ such that $\|\tilde{\mathfrak{a}}_+ - \mathfrak{a}_+\| \leqslant \eta$, then $|\mathscr{N}(\tilde{\mathfrak{a}}_+, t) - \mathscr{N}(\mathfrak{a}_+, t)| \leqslant \delta/2$ for all $t \in [T^*(\mathfrak{a}_+) - \varepsilon, S_0]$. In particular, $\tilde{\mathfrak{a}}_+ \in \Omega$ and

$$T^*(\mathfrak{a}_+) - \varepsilon \leqslant T^*(\tilde{\mathfrak{a}}_+) \leqslant T^*(\mathfrak{a}_+) + \varepsilon.$$

This exactly means that Ω contains a neighborhood of $\mathfrak{a}_+$, and hence is open, and that $\mathfrak{a}_+ \mapsto T^*(\mathfrak{a}_+)$ is continuous.

Finally, let us show that the exit is instantaneous on the sphere. If $\|\mathfrak{a}_+\|_{\ell^2} = 1$, then $\mathscr{N}(\mathfrak{a}_+, S_0) = 1$, and hence, by (72), $\mathscr{N}(\mathfrak{a}_+, t) > 1$ for all $t < S_0$ in a neighborhood of S_0. This means that $T^*(\mathfrak{a}_+) = S_0$. □

3.3. Extension to higher dimension. The main part of the proof remains unchanged. One has to work only for a good definition of the Lyapunov functional. The key point is to notice that one can find a suitable direction as in [**16**]. The set

$$\mathscr{M} = \{\beta \in \mathbb{R}^d \mid \forall j,\ \beta \cdot \beta_j = 0\},$$

is of zero measure: let $\bar{\beta} \notin \mathscr{M}$; up to rescaling, we can assume that $|\bar{\beta}| = 1$. Without loss of generality, we can assume that the indices j satisfy

$$-1 < (\bar{\beta} \cdot \beta_1) < (\bar{\beta} \cdot \beta_2) < \cdots < (\bar{\beta} \cdot \beta_N) < 1.$$

We use again the 1d cut-off function ϕ defined in Step 3 (see (62)) to define the new cut-off functions

$$\psi_j(x) = \phi\left(\frac{\bar\beta\cdot x - m_j t}{L}\right) - \phi\left(\frac{\bar\beta\cdot x - m_{j+1} t}{L}\right),$$
$$\text{where}\quad m_j = \tfrac{1}{2}(\beta_j + \beta_{j-1})\cdot\bar\beta.$$

Then all the computations of Step 3 of Section 3.2 follow unchanged. We refer to [16] (Claim 1 and what follows) for further details.

4. Proof of Theorem 1

The proof of Theorem 1 follows from Proposition 3 in a standard fashion; see for example [16]. The main point is continuity of the flow for the weak $H^1 \times L^2$ topology. More precisely, we have the following.

LEMMA 10. *The NLKG flow is continuous for the weak $H^1 \times L^2$ topology. More precisely, let $U_n \in \mathscr{C}([0,T], H^1 \times L^2)$ be a sequence of solutions to (NLKG), and assume that, for some $M > 0$,*

$$U_n(0) \rightharpoonup U^* \quad \text{in } H^1 \times L^2 - \text{weak}, \quad \text{and} \quad \forall n, \quad \|U_n(t)\|_{\mathscr{C}([0,T],H^1\times L^2)} \leqslant M.$$

Define $U \in \mathscr{C}([0, T^+(U)), H^1 \times L^2)$ to be the solution to (NLKG) with initial data $U(0) = U^*$. Then $T^+(U) > T$ and

$$\forall t \in [0,T], \quad U_n(t) \rightharpoonup U(t) \quad \text{in } H^1 \times L^2 - \text{weak}.$$

Proof. This is a simple consequence of the local well-posedness of (NLKG) in $H^s \times \dot H^{s-1}$ for some $s < 1$. More precisely, we have the following.

THEOREM (Local wellposedness). *There exist $0 \leqslant s_{f,d} < 1$ such that, for all $s \geqslant s_{f,g}$, the following holds. Given any data $U_0 = (u_0, u_1) \in H^s \times \dot H^{s-1}$, there exists a unique solution $U \in \mathscr{C}([0, T^+(U)), H^s \times \dot H^{s-1})$ to (NLKG) such that $U(0) = U_0$. Furthermore, we have the following.*

1. *The maximal time of existence $T^+(U)$ is the same in all $H^\sigma \times \dot H^{\sigma-1}$ for $\sigma \in [s_{f,d}, s]$. If finite, it is characterized by*

$$\lim_{t\to T^+(U)} \|U(t)\|_{H^s\times \dot H^{s-1}} = +\infty.$$

2. *The flow is continuous in the sense that, if U_n is a sequence of solutions to (NLKG) such that $U_n(0) \to U(0)$ in $H^s \times \dot H^{s-1}$, then $T^+(U) \geqslant \liminf_n T^+(U_n)$ and*

$$\forall t \in [0, T^+(U)), \quad \|U_n - U\|_{\mathscr{C}([0,t],H^s\times\dot H^{s-1})} \to 0 \quad \text{as } n \to +\infty.$$

We refer to [29, Theorem 1.2] and the Remark following it for a proof and the precise value of $s_{f,d}$ (which is not important for us).

Fix $0 < s < 1$ be such that the theorem holds. Let $t \in [0, T]$ such that $t < T^+(U)$.

Let $V \in (\mathscr{D}(\mathbb{R}^d))^2$ and $R > 0$ such that Supp $V \subset B_{\mathbb{R}^d}(0, R)$.

As $U_n(0) \rightharpoonup U(0)$ weakly in $H^1 \times L^2(\mathbb{R}^d)$, it holds by Sobolev compact embedding that

$$\|U_n(0) - U(0)\|_{H^s \times \dot{H}^{s-1}(B_{\mathbb{R}^d}(0,R+t))} \to 0.$$

It follows by finite speed of propagation and the continuity of the flow in the local well-posedness theorem that

$$\|U_n(t) - U(t)\|_{H^s \times \dot{H}^{s-1}(B_{\mathbb{R}^d}(0,R))} \to 0.$$

Hence, denoting $U_n = (u_n, \partial_t u_n)$ and $V = (v_0, v_1)$,

$$\begin{aligned} |\langle U_n(t) - U(t), V\rangle| &= \left| \int_{|x|\leqslant R} \Big((\partial_t u_n(t,x) - \partial_t u_n(t,x))v_1(x) \right. \\ &\qquad \left. + \nabla(u_n - u)\cdot\nabla v_0 + (u_n(t,x) - u(t,x))v_0(x) \Big)\, dx \right| \\ &\leqslant \|U_n(t) - U(t)\|_{H^s \times \dot{H}^{s-1}(B_{\mathbb{R}^d}(0,R))} \|V\|_{H^{2-s}\times \dot{H}^{1-s}} \to 0. \end{aligned}$$

Therefore, $U_n(t) \rightharpoonup U(t)$ in $\mathscr{D}'$, and, by the $H^1 \times L^2$ bound, $U_n(t) \rightharpoonup U(t)$ weakly in $H^1 \times L^2$.

In particular, $\|U(t)\|_{H^1\times L^2} \leqslant \liminf_{n\to\infty} \|U_n(t)\| \leqslant M$. From there, a continuity argument shows that $T^+(U) > T$. □

We can now prove Theorem 1. Let $(S_n)_{n\geqslant 1} \subset \mathbb{R}$ be a sequence that satisfies $S_n > S_0$, S_n increasing and $S_n \to +\infty$. From Proposition 3, there exists a sequence of final data functions $U_{0,n} \in H^1 \times L^2$ such that

$$\forall t \in [T_0, S_n], \quad U_n(t) := \varphi(S_n, t, U_{0,n}) \in \mathcal{V}(t). \tag{73}$$

(We recall that φ denotes the flow and is defined in (47).) Note that T_0 does not depend on S_n, and observe that there exists M independent of n such that

$$\forall t \in [T_0, S_n], \quad \|U_n(t) - R(t)\|_{H^1\times L^2} \leqslant Me^{-\gamma_0 t}. \tag{74}$$

Let U_0^* be a weak limit in $H^1 \times L^2$ of the bounded sequence $U_n(T_0)$, and define

$$U^*(t) = \varphi(t, T_0, U_0^*).$$

Fix $t \geqslant T_0$. Then the previous lemma applies on $[T_0, t]$ and shows that $T^+(U^*) > t$ and $U_n(t) \rightharpoonup U^*(t)$ weakly in $H^1 \times L^2$. Hence (74) yields

$$\|U^*(t) - R(t)\|_{H^1 \times L^2} \leqslant \liminf_n \|U_n(t) - R(t)\|_{H^1 \times L^2} \leqslant Me^{-\gamma_0 t}.$$

Therefore, $T^+(U^*) = +\infty$ and U^* is the desired multi-soliton.

Appendix A. The orbit of Q under general Lorentz transformations

In this appendix, we prove that the orbit of Q under the group generated by space and time translations, and general Lorentz transformations, is

$$\mathscr{F} := \{(t, x) \mapsto Q_\beta(x - \beta t - x_0) \mid \beta, x_0 \in \mathbb{R}^d,\ |\beta| < 1\}.$$

We recall that we consider Q as a function of time with the slight abuse of notation $Q(t, x) = Q(x)$.

The map $\beta \mapsto \Lambda_\beta$ (see (5)) is a group homomorphism from $(B_{\mathbb{R}^d}(0, 1), \oplus)$ to $(M_{d+1}(\mathbb{R}), \circ)$, where $\oplus$ denotes Einstein's velocity addition

$$x \oplus y = \frac{1}{1 + x \cdot y}\left(y + \frac{x \cdot y}{|y|^2} y + \sqrt{1 - |y|^2}\left(x - \frac{x \cdot y}{|y|^2} y\right)\right).$$

In particular, $\Lambda_{-\beta}\Lambda_\beta = \mathrm{Id}_{d+1}$.

A general Lorentz transformation is an element of $O(1, d) \simeq \mathbb{R}^d \rtimes O(d)$, and hence can be written in the form

$$\Lambda_{U,\beta} := \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & U \\ 0 & \end{pmatrix} \Lambda_\beta, \quad \text{where} \quad U \in SO(d),\ \text{that is } UU^T = \mathrm{Id}_d\,.$$

As $Q(x)$ is radially symmetric, it follows that

$$Q\left(\Lambda_{U,\beta}\begin{pmatrix} t \\ x \end{pmatrix}\right) = Q\left(\Lambda_\beta\begin{pmatrix} t \\ x \end{pmatrix}\right) = Q_\beta(x - \beta t),$$

and hence the orbit of $\{Q\}$ under general Lorentz transformations is simply $\{(t, x) \mapsto Q_\beta(x - \beta t) | \beta \in \mathbb{R}^d\}$. We now want to parameterize the other invariances of (NLKG), that is, time and space shifts. Fortunately, the former reduce to the latter. Indeed, notice that

$$\beta\beta^T \sim \begin{pmatrix} |\beta|^2 & 0 \cdots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix}, \quad \text{so that} \quad \mathrm{Id}_d + \frac{\gamma - 1}{|\beta|^2}\beta\beta^T \sim \begin{pmatrix} \gamma & 0 \cdots 0 \\ 0 & \\ \vdots & \mathrm{Id}_{d-1} \\ 0 & \end{pmatrix}.$$

(Here, $\sim$ indicates similarity of matrices.) In particular, $\mathrm{Id}_d + \frac{\gamma-1}{|\beta|^2}\beta\beta^T$ is invertible. Then, time translations for Q_β can be thought of as an adequate space shift:

$$\begin{aligned} Q_\beta(t+t_0,x) &= Q\left(\Lambda_\beta\begin{pmatrix} t+t_0 \\ x \end{pmatrix}\right) = Q\left(\Lambda_\beta\begin{pmatrix} t \\ x \end{pmatrix} + t_0\begin{pmatrix} \gamma \\ -\beta \end{pmatrix}\right) \\ &= Q\left(\Lambda_\beta\begin{pmatrix} t \\ x \end{pmatrix} - t_0\begin{pmatrix} 0 \\ \beta \end{pmatrix}\right) \\ &= Q_\beta\left(x - t_0\left(\mathrm{Id}_d + \frac{\gamma-1}{\beta|^2}\beta\beta^T\right)^{-1}(\beta) - \beta t\right). \end{aligned}$$

It follows that $\mathscr{F}$ is stable through all general Lorentz transformations and time and space shifts, and hence it is the orbit of Q through the group generated by these transformations.

Appendix B. Proof of Lemmas 5 and 6

Proof of Lemma 5. We use the following notation: $\hat{\boldsymbol{y}} = (\hat{y}_j)_{j=1,\dots,N}$, with the slight change (with respect to (37)) for convenience

$$R_j(x) = \begin{pmatrix} Q_{\beta_j}(x-\hat{y}_j) \\ -\beta_j(\partial_x Q_{\beta_j})(x-\hat{y}_j) \end{pmatrix}, \qquad R(x) = \sum_{j=1}^N R_j(x),$$

and, if $\hat{y}_j = \hat{y}_j(t)$, we define

$$R(t,x) := \sum_{j=1}^N R_j(x).$$

Similarly, we consider $\tilde{R}_j$ and $\tilde{R}$ as in (41). For convenience also, we work with $W := U - R$ small in $H^1 \times L^2$. Consider (see (42))

$$\begin{aligned} \Phi\ :\ (H^1\times L^2)\times\mathbb{R}^N &\to \mathbb{R}^N, \\ (W,(\tilde{y}_j)_j) &\mapsto \left(\langle W + R - \tilde{R}|\Phi_{0,j}\rangle\right)_{j=1,\dots,N}. \end{aligned}$$

Let $\boldsymbol{z} = (z_j)_{j=1,\dots,N}$. By the decay properties of $\tilde{R}_j$,

$$\begin{aligned} &(d_{\tilde{\boldsymbol{y}}}\Phi(W,\tilde{\boldsymbol{y}}).\boldsymbol{z})_j \\ &\quad = \sum_{k=1}^N z_k\langle(\Phi_{0,k}|\Phi_{0,j}\rangle - z_j\langle W + R - \tilde{R}|(\Phi_{0,j})_x\rangle \end{aligned}$$

$$= z_j \|\Phi_{0,j}\|_{L^2}^2 + O\left(e^{-\gamma_0 L}|z_k|\right) + O(|z_j|\|W\|_{L^2}) + O(|z_j|\|\tilde{\mathbf{y}} - \hat{\mathbf{y}}\|).$$

Observe that $\|\Phi_{0,j}\|_{L^2}^2 = \|\partial_x Q_{\beta_j}\|_{L^2}^2 + \beta_j^2 \|\partial_{xx} Q_{\beta_j}\|_{L^2}^2$ does not depend on $\tilde{\mathbf{y}}$. Hence,

$$d_{\tilde{\mathbf{y}}} \Phi(W, \tilde{\mathbf{y}}) = \operatorname{diag}((\|\Phi_{0,j}\|_{L^2}^2)_j) + O(e^{-\gamma_0 L}) + O(\|W\|_{L^2}) + O(\|\tilde{\mathbf{y}} - \hat{\mathbf{y}}\|). \quad (75)$$

Therefore, if L is sufficiently large, then $d_{\tilde{\mathbf{y}}} \Phi(0, \tilde{\mathbf{y}})$ is invertible. Since $\Phi(0, \hat{\mathbf{y}}) = 0$, by the implicit function theorem, it follows that there exist $\varepsilon > 0$, $\varepsilon \leqslant \eta$, and a $\mathscr{C}^1$ function $\phi : B_{L^2}(0, \varepsilon) \to B_{\mathbb{R}^N}(\hat{\mathbf{y}}, \eta)$ such that $\Phi(W, \tilde{\mathbf{y}}) = 0$ in $B_{L^2}(0, \varepsilon) \times \phi(B_{L^2}(0, \varepsilon))$ is equivalent to $\tilde{\mathbf{y}} = \phi(W)$. Finally, we set

$$V = V(W) = W + R - \sum_{j=1}^{N} R_j(\cdot - \phi(W)_j).$$

Proof of Lemma 6. Consider the following maps:

$$\begin{array}{rrcl}
\mathscr{I} : & \mathbb{R}^{2N} & \to & H^1 \times L^2 \\
& \mathfrak{b} & \mapsto & \sum_{j,\pm} \mathfrak{b}_j^{\pm} Z_{\pm}(\gamma_j(\cdot - \hat{y}_j)), \\
\Theta : & \mathscr{V} & \to & (H^1 \times L^2) \times \mathbb{R}^N \\
& W & \mapsto & (V, \tilde{\mathbf{y}}), \\
\mathscr{S} : & (H^1 \times L^2) \times \mathbb{R}^N & \to & \mathbb{R}^{2N} \\
& (V, \tilde{\mathbf{y}}) & \mapsto & \left(\langle V | Z_{\pm,j} \rangle\right)_{j,\pm},
\end{array}$$

where, in the definition of Θ, $(V, \tilde{\mathbf{y}})$ represents the modulation of $U = W + \sum_j R_j(\cdot - \hat{y}_j(S_0))$ and $\mathscr{V} = B_{H^1}(0, \varepsilon)$ (ε being defined in the proof Lemma 5); and in the definition of $\mathscr{S}$, $Z_{\pm,j}$ is as in (42).

Then $\mathscr{I}(0) = 0$, $\Theta(0) = (0, \hat{\mathbf{y}})$, and $\mathscr{S}(0, \hat{\mathbf{y}}) = 0$. Recall also from Lemma 5 and (41) that

$$\|V\|_{L^2} + \|\tilde{\mathbf{y}} - \hat{\mathbf{y}}\| + \|R_j(S_0) - \tilde{R}_j(S_0)\|_{H^1} \leqslant C\|W\|_{L^2}.$$

Lemma 6 is the statement that

$$\Psi := \mathscr{S} \circ \Theta \circ \mathscr{I}$$

is a diffeomorphism (depending on S_0) on a fixed neighborhood of $0 \in \mathbb{R}^{2N}$ (not depending on S_0). This follows from computing $d\Psi = d\mathscr{S} \circ d\Theta \circ d\mathscr{I}$. Indeed, we claim the following.

Claim.

$$d\Psi(\mathfrak{b}) = \begin{pmatrix} A & \langle Z_+ | Z_- \rangle A \\ \langle Z_+ | Z_- \rangle A & A \end{pmatrix} + O(e^{-3/2\gamma_0 S_0} + \|\mathfrak{b}\|), \quad (76)$$

where $A = \operatorname{diag}((\|Z_{+,j}\|_{L^2}^2)_j)$ *is an* $n \times n$ *square matrix.*

Let us first conclude the proof of Lemma 6 assuming the claim.

Since the leading order in $d\Psi(\mathfrak{b})$ is invertible (Z_+ and Z_- are independent as eigenfunctions with different eigenvalues of a commun operator $\mathscr{H}$; see (26)), and independent of S_0 (and of $\mathfrak{b}$), we deduce that $d\Psi$ is invertible on some ball $B_{\mathbb{R}^{2N}}(0,\eta)$ ($\eta > 0$ independent of S_0 sufficiently large). As a consequence, Ψ is a diffeomorphism from $B_{\mathbb{R}^{2N}}(0,\eta)$ to some neighborhood $\mathscr{W}$ of $0 \in \mathbb{R}^{2N}$. Let $\delta > 0$ be such that $B_{\mathbb{R}^{2N}}(0,\delta) \subset \mathscr{W}$. For any $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(0,\delta)$, there exists a unique $\mathfrak{b} = \mathfrak{b}(\mathfrak{a}^+) \in B_{\mathbb{R}^{2N}}(0,\eta)$ such that $\Psi(\mathfrak{b}(\mathfrak{a}^+)) = (\mathfrak{a}^+, 0)$ and $\|\mathfrak{b}(\mathfrak{a}^+)\| \leqslant C\|\mathfrak{a}^+\|$. Take $\mathfrak{a}^+ := e^{-3\gamma_0 S_0/2}\boldsymbol{\mathfrak{a}}_+$ from (48). Then, for all S_0 large, we have $\mathfrak{a}^+ \in B_{\mathbb{R}^N}(0,\delta)$ (note that δ does not depend on S_0), and (48) is satisfied.

We now conclude by proving the claim, (76). We start with the computation of the differentials of $\mathscr{I}$, Θ, and $\mathscr{S}$. First, $\mathscr{I}$ is linear, so that $d\mathscr{I}(\mathfrak{b}) = \mathscr{I}$ for all $\mathfrak{b}$. Second, for $H \in H^1 \times L^2$, $z \in \mathbb{R}^N$,

$$(d\mathscr{S}(V,\tilde{\mathbf{y}}).(H,z))_{j,\pm} = -z_j\langle V|(Z_{\pm,j})_x\rangle + \langle H|Z_{\pm,j}\rangle.$$

Finally, we consider Θ. Let Φ and ϕ be defined as in the proof of the Lemma 5 above for $R(S_0)$. Then, by (75), $d_{\tilde{\mathbf{y}}}\Phi(W,\tilde{\mathbf{y}})$ is a diagonally dominant matrix, and thus it is invertible. Denoting by M its inverse, it follows from (75) that

$$M = \operatorname{diag}((\|\Phi_{0,j}\|_{L^2}^{-2})_j) + O(\|W\|_{L^2} + \|\tilde{\mathbf{y}} - \hat{\mathbf{y}}\| + e^{-\gamma_0 S_0}).$$

Differentiating $\Phi(W,\phi(W)) = 0$ with respect to W, we find that $d\phi(W) = -M \circ d_W\Phi(W,\phi(W))$. Since $(d_W\Phi(W,\tilde{\mathbf{y}}).H)_j = \langle H|(\tilde{R}_j)_x(S_0)\rangle$ and

$$\Theta(W) = \left(W + R(S_0) - \sum_{j=1}^{N} R_j(S_0, \cdot - \phi(W)_j), \phi(W)\right),$$

we obtain

$$\begin{aligned} d\Theta(W).H &= \left(H - \sum_j ((M \circ d_W\Phi).H)_j \Phi_{0,j}, -M \circ d_W\Phi(W,\phi(W)).H\right) \\ &= \left(H + \sum_{j=1}^{N} \frac{\langle H|\Phi_{0,j}\rangle}{\|\Phi_{0,j}\|_{L^2}^2}\Phi_{0,j}, \left(-\frac{\langle H|\Phi_{0,j}\rangle}{\|\Phi_{0,j}\|_{L^2}^2}\right)_{j=1,\dots,N}\right) \\ &\quad + O(\|H\|_{L^2}(e^{-\gamma_0 S_0} + \|W\|_{L^2})). \end{aligned}$$

(Here, $\Phi_{0,j}$ depends on S_0 through the modulated shift $\tilde{y}_j$.) Let $\tilde{\mathfrak{b}} \in \mathbb{R}^{2N}$. We have

$$d\Psi(\mathfrak{b}).\tilde{\mathfrak{b}} = d\mathscr{S}(\Theta(\mathscr{I}(\mathfrak{b}))).(d\Theta(\mathscr{I}(\mathfrak{b})).\mathscr{I}(\tilde{\mathfrak{b}})).$$

By the previous computations, we derive

$$d\Theta(\mathscr{I}(\mathfrak{b})).\mathscr{I}(\tilde{\mathfrak{b}}) = \left(\mathscr{I}(\tilde{\mathfrak{b}}) + \sum_{j=1}^{N} \frac{\langle \mathscr{I}(\tilde{\mathfrak{b}})|\Phi_{0,j}\rangle}{\|\Phi_{0,j}\|_{L^2}^2}\Phi_{0,j}, \left(-\frac{\langle \mathscr{I}(\tilde{\mathfrak{b}})|\Phi_{0,j}\rangle}{\|\Phi_{0,j}\|_{L^2}^2}\right)_{j=1,\dots,N}\right) + O(\|\tilde{\mathfrak{b}}\|_{L^2}(e^{-\gamma_0 S_0} + \|\mathfrak{b}\|_{L^2})).$$

Inserting the expression of $\mathscr{I}(\tilde{\mathfrak{b}})$, using $\|\tilde{\mathbf{y}} - \hat{\mathbf{y}}\| \leqslant C\|\mathfrak{b}\|$, $\langle Z_\pm|\Phi_0\rangle = 0$, and the decay properties of the functions Q and Z, we get

$$d\Theta(\mathscr{I}(\mathfrak{b})).\mathscr{I}(\tilde{\mathfrak{b}}) = (\mathscr{I}(\tilde{\mathfrak{b}}), 0) + O(\|\tilde{\mathfrak{b}}\|(e^{-\gamma_0 S_0} + \|\mathfrak{b}\|)).$$

Therefore, using the expression of $d\mathscr{S}$, we finally obtain

$$d\Psi(\mathfrak{b}) = \mathrm{Gramm}((Z_{\pm,j})_{j,\pm}) + O(e^{-\gamma_0 S_0} + \|\mathfrak{b}\|) = P + O(e^{-\gamma_0} + \|\mathfrak{b}\|),$$

where $\mathrm{Gramm}((Z_{\pm,j})_{j,\pm})$ is the Gramm matrix of the family $(Z_{\pm,j})_{j,\pm}$,

$$\mathrm{Gramm}((Z_{\pm,j})_{j,\pm})_{(j_1,\pm_1),(j_2,\pm_2)} = \langle Z_{\pm_1,j_1}|Z_{\pm_2,j_2}\rangle,$$

so that

$$\mathrm{Gramm}((Z_{\pm,j})_{j,\pm}) = \begin{pmatrix} A & \langle Z_+|Z_-\rangle A \\ \langle Z_+|Z_-\rangle A & A \end{pmatrix},$$

where $A = \mathrm{diag}((\|Z_{+,j}\|_{L^2}^2)_j) = \mathrm{diag}((\|Z_{-,j}\|_{L^2}^2)_j)$ (recall $\|Z_+\| = \|Z_-\| = 1$).

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